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SOME PROPERTIES OF CR STRUCTURES

by

Andrei IORDAN

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Andrei IORDAN *)

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*) *Institut of Construction, Bucharest, Bd. Lacul Tei - Romania*

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I TUBULAR NEIGHBORHOODS FOR CR MANIFOLDS

1. INTRODUCTION

In [3] , F.Reese Harvey and R.O. Wells Jr. prove that a totally real submanifold of a complex manifold has a fundamental neighborhood system consisting of open Stein manifolds This represents a generalization of Grauert's theorem about the complexification of a real - analytic manifold [2].

In [1] A.Andreotti and G. Fredericks prove that real analytic CR manifolds are globally isomorphic to a generic submanifold of a complex manifold (the complexification of a real analytic CR manifold) They also prove that a compact real analytic CR manifold of type (m.1) has in its complexification a fundamental neighborhood system consisting of 1 - complete manifolds and this result is conjectured for a non-compact manifold too.

The purpose of this paper is to give an easy generalisation of the result of Hervey and Wells to CR manifolds and to obtain as a corollary the conjecture of Andreotti and Fredericks . P.Flondor and Al. Tancredi have also announced a proof of the above conjecture [5].

2. Preliminaries

Let M be a smooth locally closed m - dimensional real submanifold of a complex manifold X (all manifolds are assumed to be paracompact). For a point $p \in M$ we denote by $T_p(X)$ the tangent space of M at p Let $TC_p(M)$ be the maximal complex subspace of the real space $T_p(M)$ with the complex structure induced from $T_p(X)$. If $\dim_C TC_p(M) = 1 =$ constant for each $p \in M$ we say that M is a CR manifold of type (m,1)

A smooth function φ of an n- dimensional complex manifold X is q-pseudoconvex if its complex Hessian has at least n-q positive eigenvalues and it is strictly q- pseudoconvex if its complex Hessian has at least n-q strictly positive eigenvalues.

An n- dimensional complex manifold X is 1 - complete if there exists a smooth function $\varphi : X \to R$ strictly 1-pseudoconvex such that for ecery $c \in R$, $\{z \in X \mid \varphi(z) < c\}$

is relatively compact in X.

3. Existence of strictly 1- pseudoconvex function defining M.

Lemma 1 Let M be a CR submanifold of type (m,1) of an open set $D \subset \mathbb{C}^n$, $M = \{z \in D \mid \rho_1(z) = \cdots = \rho_{2n-m}(z) = 0\}$ where $\rho_1, \ldots, \rho_{2n-m}$ are real valued smooth functions on D such that $d\rho_1 \wedge \cdots \wedge d\rho_{2n-m} \neq 0$.

There exists a positive function $\varphi \in C^\infty(D)$ such that

a) $M = \{z \in D \mid \varphi(z) = 0\}$.

b) φ is plurisubharmonic on M (0 - pseudoconvex).

c) φ is strictly 1 - pseudoconvex in a neighborhood of M.

d) all first derivatives of φ are vanishing on M.

PROOF. Let define $\varphi = \sum_{k=1}^{2n-m} \rho_k^2$ and then a) and d) are verified.

We have $$\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} = 2\left(\sum_{k=1}^{2n-m} \frac{\partial \rho_k}{\partial \bar{z}_j} \cdot \frac{\partial \rho_k}{\partial z_i} + \sum_{k=1}^{2n-m} \rho_k \frac{\partial^2 \rho_k}{\partial z_i \partial \bar{z}_j}\right)$$

For $p \in M$ we obtain $$\sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(p) z_i \bar{z}_j = 2 \sum_{k=1}^{2n-m} \left| \sum_{i=1}^{n} \frac{\partial \rho_k}{\partial z_i}(p) z_i \right|^2 \geq 0$$

That means that φ is plurisubharmonic on M

Because M is a CR manifold of type (m,1)

$$\operatorname{rk} \frac{\partial(\rho_1, \ldots, \rho_{2n-m})}{\partial(z_1, \ldots, z_n)}(p) = n-1 \text{ for each } p \in M \quad [1]$$

It follows that the complex Hessian of φ has n-1 strictly positive eigenvalues on M.

Hence in a neighborhood of M, φ is strictly 1-pseudoconvex.

REMARK. If M is a CR submanifold of type (m,1) of an open set $D \subset \mathbb{C}^n$ and of differentiability class C^1 we know that there exists a positive function $\varphi \in C^2(D)$ which has for $p \in M$ the same first and second derivatives at p as $d(z, T_p(M))^2$

We denote $s = m - 2l$, $r = l - (m-n)$

By making a complex linear change of coordinates in $\mathbb{C}^n$ we may suppose that p=o and

$T_o(M)=\{z \mid y_1=0,\dots,y_s=0, z_{s+1}=0,\dots,z_{s+r}=0\}$ where $z_j=x_j+iy_j$ [4]

Hence $d(z,T_o(M))^2=\sum_{i=1}^{s} y_i^2+\sum_{i=s+1}^{s+r}|z_i|^2$ and the complex Hessian of $d(z,To(M))^2$ has $s+r = m-2l+l-(m-n) = n-l$ strictly positive eighevalues and l eigenvalues vanishing on M.

The point p being arbitrary on M and because φ has the same first and second derivatives on M as $d(z,Tp(M))^2$ we obtain for φ the conclusions of lemma 1.

THEOREM 1 . Let M be a CR submnanifold of type (m,l) of an n-dimensional complex manifold X. Then there exists a smooth-positive function φ on X, plurisubharmonic on M, strictly l-pseudoconvex in a neighborhood of M such that $M=\{z\in X \mid \varphi(z)=0\}$.

Proof

Let $\{U_i\}_{i\in I}$ denote an open locally finite covering of X by coordinate charts and let $\{\psi_i\}_{i\in I}$ denote a partition of unity subordinate to $\{U_i\}_{i\in I}$.

By lemma 1 for U_i, there exists $\varphi_i\in C^\infty(U_i)$, plurisubharmonic on $M\cap U_i$, strictly l - pseudoconvex on a neighborhood of $M\cap U_i$, with all first derivatives vanishing on $M\cap U_i$, such that $M\cap U_i=\{z\in U_i \mid \varphi_i(z)=0\}$

Define $\varphi=\sum_{i\in I}\psi_i\varphi_i$

Because $\psi_i\geq 0$, $\varphi_i\geq 0$, we have $\varphi(z)=0$ iff $\varphi_i(z)=0$ for $z\in \operatorname{supp}\psi_i\subset U_i$. Hence $\varphi(z)=0$ iff $z\in M$.

If $p\in M$ we have $\frac{\partial^2\varphi}{\partial z_j\partial \bar z_k}(p)=\sum_{i\in I}\psi_i(p)\frac{\partial^2\varphi_i}{\partial z_j\partial\bar z_k}(p)$

It is enough to show that on M, φ has the required properties since, by continuity, there exists a neighborhood of M where all strictly positive eigenvalues on M remain strictly positive.

Now a succesive application of the following lemma gives the result.

Lemma 2. If A and B are Hermitian matrixes with all the eigenvalues positive A having at least q strictly positive eigenvalues , then A+ B has at least q strictly positive eigenvalues.

Proof of the lemma

Let denote $E=\{z\in\mathbb{C}^n \mid \langle Az,z\rangle=0\}$ and $F=\{z\in\mathbb{C}^n \mid \langle (A+B)z,z\rangle=0\}$

where $\langle\ ,\ \rangle$ is the inner product on $\mathbb{C}^n$

Because A has at least q strictly positive eigenvalues $\dim_{\mathbb{C}} E \leq n-q$. Let r be the number of strictly positive eigenvalues of A+B that is $\dim_{\mathbb{C}} F = n-r$

Since $F \subset E$ we obtain $n-r \leq n-q$ i.e. $r \geq q$.

4. <u>Existence of tubular neighborhoods</u>

<u>Theorem 2.</u> Let φ be a smooth nonnegative function strictly 1-pseudoconvex on a complex manifold X and let $M = \{z \in X \mid \varphi(z) = 0\}$

There exists a fundamental neighborhood system of M consisting of 1 - complete open subsets of X.

Proof.

If ε is a smooth function on X with sufficiently small second order derivatives then $\varphi - \varepsilon$ is strictly 1-pseudoconvex on X.

Let $\mathcal{E}$ denote the set of smooth nonnegative functions ε on X such that

a) $\varphi - \varepsilon$ is strictly 1-pseudoconvex

b) ε vanishes at infinity i.e. for any $\eta > 0$ there exists a compact set $K \subset M$ such that $\varepsilon(z) < \eta$ for $z \in X - K$.

We consider $\{U_\varepsilon\}_{\varepsilon \in \mathcal{E}}$ where $U_\varepsilon = \{z \in X \mid \varphi(z) < \varepsilon(z)\}$ and we shall prove that $\{U_\varepsilon\}_{\varepsilon \in \mathcal{E}}$ is a fundamental neighborhood system of M.

Let V be an arbitrary neighborhood of M.

By taking an exhaustion of X with compact sets K_i, $K_i \subset \Omega_i \subset \mathring{K}_{i+1}$, where Ω_i are open subsets of X, we may consider a partition of unity $\{\psi_i\}_{i \in I}$ subordinate to the open locally finite covering of X, $\{U_i\}_{i \in I}$ where $U_i = \Omega_{i+1} - \overline{\Omega}_{i-1}$. Then we can choose succesively positive constants a_i small enough such that $\varphi - \varepsilon$ is strictly 1 - pseudoconvex. If $\{a_i\}$ is sumable for any $\eta > 0$ there exists a finite set $F \subset I$ such that $\sum_{i \notin F} a_i < \eta$ and if K is the compact set $\bigcup_{i \in F} \operatorname{supp} \psi_i$, we have $\varepsilon(z) < \eta$ for $z \in X - K$. Thus supposing that a_i are positive constants small enough, $\varepsilon = \sum_{i \in I} a_i \psi_i$ belongs to $\mathcal{E}$.

Moreover because for each compact set $K \subset X - V$ there exist $c > 0$ such that $\varphi(z) \geq c$ for $z \in K$, it follows that for $a_i > 0$ sufficiently small $\varepsilon(z) \leq \varphi(z)$ on $X - V$

Therefore $U_\varepsilon = \{z \in X \mid \varphi(z) < \varepsilon(z)\} \subset V$,

We shall prove now that each U_ε is 1 - complete

The function $(\varepsilon - \varphi)^{-1}$ is a positive strictly 1 - pseudoconvex function on U_ε.

Indeed $\dfrac{\partial^2(\varepsilon-\varphi)^{-1}}{\partial z_i\partial\bar z_j} = 2(\varepsilon-\varphi)^{-3}\dfrac{\partial}{\partial\bar z_j}(\varepsilon-\varphi)\dfrac{\partial}{\partial z_i}(\varepsilon-\varphi)-(\varepsilon-\varphi)^{-2}\dfrac{\partial^2}{\partial z_i\partial\bar z_j}(\varepsilon-\varphi)$

and $\displaystyle\sum_{i,j}\frac{\partial^2(\varepsilon-\varphi)^{-1}}{\partial z_i\partial\bar z_j}z_i\bar z_j=(\varepsilon-\varphi)^{-2}\sum_{i,j}\frac{\partial^2\varphi}{\partial z_i\partial\bar z_j}(\varphi-\varepsilon)z_i\bar z_j+2(\varepsilon-\varphi)^{-3}\Big|\sum_i\frac{\partial}{\partial z_i}(\varepsilon-\varphi)z_i\Big|^2$

Because $\varphi-\varepsilon$ is strictly 1 - pseudoconvex and $2(\varepsilon-\varphi)^{-3}\left|\sum_i\frac{\partial}{\partial z_i}(\varepsilon-\varphi)z_i\right|^2\geqslant 0$ from lemma 2 it follows that $(\varepsilon-\varphi)^{-1}$ is strictly 1 pseudoconvex.

If $c>0$ we have $\{z\in U_\varepsilon \mid (\varepsilon-\varphi)^{-1}(z)<c\}\subset\{z\in U_\varepsilon\mid \varphi(z)+\frac{1}{c}\leq\varepsilon(z)\}=$ $=\{z\in X\mid \varphi(z)+\frac{1}{c}\leq\varepsilon(z)\}\subset\{z\in X\mid \frac{1}{c}\leq\varepsilon(z)\}$.

But $\{z\in X\mid\frac{1}{c}\leq\varepsilon(z)\}$ is a compact set because ε vanishes at infinity, and because $\{z\in U_\varepsilon\mid\varphi(z)+\frac{1}{c}\leq\varepsilon(z)\}$ is a compact set in U_ε the theorem is proved.

From theorem 1 and theorem 2 we obtain:

<u>Theorem 3</u> Let M be a real analytic CR manifold of type (m,1) and X a complexification of M. There exists a fundamental neighborhood system of M in X which are open 1-complete subsets of X.

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II. A WEIERSTRASS PREPARATION THEOREM FOR CR FUNCTIONS

Let X be a complex analytic manifold and let M be a locally closed C^∞ differentiable submanifold in X of codimension k in X.

If $p \in M$ let U be a neighborhood of p in C^n such that $M \cap U = \{z \in U \mid \rho_1(z) = \dots \rho_k(z) = 0\}$ where $\rho_1, \dots, \rho_k$ are real valued smooth functions on U and $d\rho_1 \wedge \dots \wedge d\rho_k \neq 0$ on U.

We denote by $\mathcal{J}_{(p,q)}(U) = \{ f \in C^\infty_{(p,q)}(U) \mid f = \sum_{j=1}^{k} \rho_j \alpha_j + \sum_{j=1}^{k} \beta_j \wedge \bar\partial \rho_j$, $\alpha_j \in C^\infty_{(p,q)}(U), \beta_j \in C^\infty_{(p,q-1)}(U) \}$

If $u \in C^\infty(M)$ we define $\bar\partial_M u = 0$ on $M \cap U$ iff there exists a smooth extension $\tilde u$ of $u|_{M\cap U}$ to U such that $\bar\partial \tilde u \in \mathcal{J}_{(0,1)}(U)$.

It is easy to verify that the definition does not depend on the choise of $\tilde u$.

We define $\bar\partial_M u = 0$ on M iff $\bar\partial_M u = 0$ in the neighborhood of each point of M.

The operator $\bar\partial_M$ is the Cauchy-Riemann operator induced from the operator $\bar\partial$ on C^n.

A CR function on M is a function $u \in C^\infty(M)$ which satisfies $\bar\partial_M u = 0$ on M. We denote by CR (M) the C-algebra of CR functions on M.

A function $u \in C^\infty(M)$ is a CR function on M iff there exists on extension $\tilde u$ of u in a neighborhood of M such that $\bar\partial \tilde u|_M = 0$

Theorem 1. Let M be a locally closed C^∞-differentiable submanifold of C^n containing the origin and let $P(t,\lambda) = t^p + \sum_{j=1}^{p} \lambda_j t^{p-j}$, $\lambda = (\lambda_1, \dots, \lambda_p) \in C^p$ be a generic monic polynomial in t.

Let f be a complex valued CR function on RxM where we consider RxM as a submanifold of CxC^n.

Then we have $f(t,x) = q(t,x,\lambda)\, P(t,\lambda) + r(t,x,\lambda)$,

where q and r are CR functions in a neighborhood of the origin in RxM of C^∞-differentiability class in the variable λ and r is a polynomial in t of degree less than p.

Proof

We shall denote the variables $(t,x) \in RxM$ and $(w,u) \in CxC^n$

Because every smooth function depending only on the variable t is a CR function on RxM we may suppose that for every fixed $x \in M$, f has compact support in the variable t.

By the Malgrange division theorem we obtain

$f(t,x) = q(t,x,\lambda)\, P(t,\lambda) + r(t,x,\lambda)$ where

$$q(t,x,\lambda) = \frac{1}{2\pi i}\int_{\partial D}\frac{F(z,x,\lambda)}{(z-t)P(z,\lambda)}dz + \frac{1}{2\pi i}\iint_D \frac{F_{\bar z}(z,x,\lambda)}{(z-t)P(z,\lambda)}dz\wedge d\bar z$$

$$r(t,x,\lambda) = \frac{1}{2\pi i}\int_{\partial D}\frac{F(z,x,\lambda)R(t,z,\lambda)}{P(z,\lambda)}dz + \frac{1}{2\pi i}\iint_D \frac{F_{\bar z}(z,x,\lambda)R(t,z,\lambda)}{P(z,\lambda)}dz\wedge d\bar z$$

$R(t,z,\lambda) = \dfrac{P(z,\lambda) - P(t,\lambda)}{z-t}$, D is a disc centered in the origin containing in its interior the point t and all roots z of $P(z,\lambda)$ for $|\lambda|$ small, $F(z,x,\lambda)$ is given by Nirenberg extension lemma and has the following properties

i) F is a C^∞ complex function defined in a neighborhood of the origin for $z\in C$, $x\in M$, $\lambda\in C^p$

ii) $F(t,x,\lambda) = f(t,x)$ for real t

iii) $F_{\bar z}$ vanishes of infinite order on $\operatorname{Im} z = 0$ and on $P(z,\lambda) = 0$ [2].

We shall use the definition of F given in [3]

$$F(z,x,\lambda) = \int_{-\infty}^{\infty}\rho(\xi,\lambda,y)e^{i\xi z}\hat f(\xi,x)\,d\xi \qquad \text{where } y = \operatorname{Im} z$$

$$\hat f(\xi,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t,x)e^{-i\xi t}\,dt$$

and ρ has the properties

i) $\rho : R \times C^p \times R \longrightarrow [0,1]$ is continuous

Let $\delta(y,\lambda) = \inf\{\,|y - \operatorname{Im} z| \mid P(z,\lambda) = 0\}$

ii) $\rho(\xi,\lambda,y) = 1$ in a neighborhood of $y = 0$

iii) $\rho(\xi,\lambda,y) = 0$ if $|\xi y| \geq 1$

iv) $\rho_y(\xi,\lambda,y) = 0$ in a neighbourhood of $\delta(y,\lambda) = 0$

v) ρ is infinitely differentiable with respect to λ and y and its derivatives are continuous with respect to all variables and satisfy

$$\left|\frac{\partial}{\partial\lambda^\alpha}\frac{\partial}{\partial\bar\lambda^\beta}\frac{\partial}{\partial y^\gamma}\rho(\xi,\lambda,y)\right| \leq C\left(1+|\xi|\right)^{(2p+1)(1+|\alpha|+|\beta|+\gamma)}$$

where $C>0$ depends only on the multi-indices α and β and the non-negative integer γ.

Because $f \in CR(R \times M)$ there exists an extension $\tilde f(w,u)$ in a neighbourhood of the origin in $C\times C^n$ such that $\bar\partial \tilde f|_{R\times M} = 0$.

We may suppose that for fixed u, $\tilde f(t,u)$ has compact support in the variable t.

We consider $\hat{\tilde f}(\xi,u) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\tilde f(t,u)e^{-i\xi t}\,dt$

and $\tilde{F}(z,u,\lambda) = \int_{-\infty}^{\infty} \rho(\xi,\lambda,y) e^{i\xi z} \hat{\tilde{f}}(\xi,u)\,d\xi$ where $y = \operatorname{Im} z$

For $x \in M$ we have $\hat{\tilde{f}}(\xi,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(t,x) e^{-i\xi t}\,dt = \hat{f}(\xi,x)$

and $\tilde{F}(z,x,\lambda) = \int_{-\infty}^{\infty} \rho(\xi,\lambda,y) e^{i\xi z} \hat{f}(\xi,x)\,d\xi = F(z,x,\lambda)$

Hence we can define $\tilde{q}(w,u,\lambda) = \frac{1}{2\pi i}\int_{\partial D} \frac{\tilde{F}(z,u,\lambda)}{(z-w)\,P(z,\lambda)}\,dz +$

$+ \frac{1}{2\pi i}\iint_D \frac{\tilde{F}_{\bar{z}}(z,u,\lambda)}{(z-w)\,P(z,\lambda)}\,dz\wedge d\bar{z}$

and $\tilde{r}(w,u,\lambda) = \frac{1}{2\pi i}\int_{\partial D} \frac{\tilde{F}(z,u,\lambda)\tilde{R}(w,z,\lambda)}{P(z,\lambda)}\,dz + \frac{1}{2\pi i}\iint_D \frac{\tilde{F}_{\bar{z}}(z,u,\lambda)\tilde{R}(w,z,\lambda)}{P(z,\lambda)}\,dz\wedge d\bar{z}$

where $\tilde{R}(w,z,\lambda) = \dfrac{P(z,\lambda) - P(w,\lambda)}{z-w}$

Because $\rho(\xi,\lambda,y) = 1$ in a neighborhood of $y = \operatorname{Im} z = 0$ $\tilde{F}_{\bar{z}}$ is vanishing with all its derivatives in a neighbonhood of $y = \operatorname{Im} z = 0$

From this fact and because $\tilde{F}_{\bar{z}}$ vanishes with all its derivatives if $P(z,\lambda) = 0$ it follows that the integrals from the definitions of $\tilde{q}$ and $\tilde{r}$ converge absolutely.

Hence $\tilde{q}$ and $\tilde{r}$ are infinitely differentiable extensions of q and r repsectively.

Differentiating under the integral sign we have

$\frac{\partial \tilde{q}}{\partial \bar{w}} = 0$ and $\frac{\partial \tilde{q}}{\partial \bar{u}_j}(w,u,\lambda) = \frac{1}{2\pi i}\int_{\partial D} \frac{\frac{\partial \tilde{F}}{\partial \bar{u}_j}(z,u,\lambda)}{(z-w)\,P(z,\lambda)}\,dz + \frac{1}{2\pi i}\iint_D \frac{\frac{\partial \tilde{F}_{\bar{z}}}{\partial \bar{u}_j}(z,u,\lambda)}{(z-w)\,P(z,\lambda)}\,dz\wedge d\bar{z}$

$\frac{\partial \tilde{F}}{\partial \bar{u}_j}(z,u,\lambda) = \int_{-\infty}^{\infty} \rho(\xi,\lambda,y) e^{i\xi z} \frac{\partial \hat{\tilde{f}}}{\partial \bar{u}_j}(\xi,u)\,d\xi$

and $\frac{\partial \hat{\tilde{f}}}{\partial \bar{u}_j}(\xi,u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\partial}{\partial \bar{u}_j}\tilde{f}(t,u) e^{-i\xi t}\,dt$

If $x \in M$, $\frac{\partial \tilde{f}}{\partial \bar{u}_j}(t,x) = 0$, hence if $x \in M$, $\frac{\partial \tilde{F}}{\partial \bar{u}_j}(z,x,\lambda) = 0$

We have $\tilde{F}_{\bar{z}}(z,u,\lambda) = \int_{-\infty}^{\infty} i\rho_y(\xi,\lambda,y) e^{i\xi z} \hat{\tilde{f}}(\xi,u)\,d\xi$ and

we obtain $\frac{\partial \tilde{F}_{\bar{z}}}{\partial \bar{u}_j}(z,x,\lambda) = 0$

It follows that $\bar\partial \tilde{q}|_{R\times M} = 0$ and similarly $\bar\partial \tilde{r}|_{R\times M} = 0$.

Theorem 2 . Let M be a locally closed C^∞ differentiable submanifold of C^n containing the origin and $f \in CR\ (R\times M)$ in the neighborhood of the origin

We suppose that $\frac{\partial^p}{\partial t^p} f(0,0) \neq 0$ and $\frac{\partial^k}{\partial t^k} f(0,0) = 0$ for $k<p$

Then there exists $Q \in CR\ (R\times M)$ in the neighborhood of the origin, $Q(0,0) \neq 0$ such that $f(t,x) = Q(t,x)\cdot P(t,x)$ where $P(t,x) = t^p + \sum_{j=1}^{p} \lambda_j(x) t^{p-j}$ with $\lambda_j \in CR(R\times M)$

PROOF

From theorem 1 we have the division

$$f\ (t,x) = q\ (t,x,\lambda)\ P\ (t,\lambda) + \sum_{j=1}^{p} r_j(x,\lambda) t^{p-j} \quad (*)$$

with $q \in CR\ (R\times M)$, $r_j \in CR\ (M)$, $\lambda \in C^p$

Let $\tilde{r}_j\ (u,\lambda)$ be extensions of r_j such that $\bar\partial \tilde{r}_j|_M = 0$

From [2] we know that $\tilde{r}_j\ (0,0) = 0$ for $j = 1,\dots,p$ and $q\ (0,0,0) = 0$ Hence $r_j\ (0,0) = 0$ for $j = 1,\dots,p$

We know also that $\frac{\partial r_j}{\partial \bar\lambda_k}(0,0) = 0$ and $\det\left(\frac{\partial r_j}{\partial \lambda_k}(0,0)\right)_{\substack{1\le j\le p\\ 1\le k\le p}} \neq 0$

But $\frac{\partial r_j}{\partial \bar\lambda_k}(0,0) = \frac{\partial \tilde{r}_j}{\partial \bar\lambda_k}(0,0)$ and $\frac{\partial r_j}{\partial \lambda_k}(0,0) = \frac{\partial \tilde{r}_j}{\partial \lambda_k}(0,0)$

Therefore we can solve the system $\tilde{r}_j\ (u,\lambda) = 0$, $j = 1,\dots,p$ and obtain the C^∞ function $\tilde\lambda\ (u) = (\tilde\lambda_1(u),\dots,\tilde\lambda_p(u))$ such that $r_j(u,\tilde\lambda(u)) = 0$ in the neighborhood of the origin, for $j=1,\dots,p$.

We have $\frac{\partial \tilde{r}_j}{\partial \bar{u}_k}(u,\tilde\lambda(u)) = \frac{\partial \tilde{r}_j}{\partial \bar{u}_k}(u,\tilde\lambda(u)) + \sum_{i=1}^{p} \frac{\partial \tilde{r}_j}{\partial \bar\lambda_i}\cdot\frac{\partial \bar{\tilde\lambda}_i}{\partial \bar{u}_k} +$

$$+ \sum_{i=1}^{p} \frac{\partial \tilde{r}_j}{\partial \lambda_i}\cdot\frac{\partial \tilde\lambda_i}{\partial \bar{u}_k} = 0,\quad k=1,\dots,n,\ j=1,\dots,p.$$

Let $\lambda\ (x) = (\lambda_1(x),\dots,\lambda_p\ (x))$ be the restriction of $\tilde\lambda$ to M

Applying $\frac{\partial}{\partial \bar\lambda_k}$ to $(*)$ we obtain

$$\sum_{j=1}^{p} \frac{\partial r_j}{\partial \bar\lambda_k}(x,\lambda) t^{p-j} = -t^p \frac{\partial}{\partial \bar\lambda_k} q(t,x,\lambda)$$

for every t . It follows that $\frac{\partial r_j}{\partial \bar\lambda_k}(x,\lambda) = 0$ for $j,k = 1,\dots,p$

Hence $\frac{\partial \tilde{r}_j}{\partial u_k}(x,\lambda(x)) = \sum_{i=1}^{p} \frac{\partial \tilde{r}_j}{\partial \lambda_i} \frac{\partial \tilde{\lambda}_i}{\partial u_k}(x) = 0 \quad j=1,\dots,p,\ k=1,\dots,n$

Because $\det\left(\frac{\partial \tilde{r}_j}{\partial \lambda_i}(0,0)\right)_{\substack{1\le i\le p\\ 1\le j\le p}} \neq 0$ it follows that

for x in a neighborhood of the origin we have

$\det\left(\frac{\partial \tilde{r}_j}{\partial \lambda_i}(x,\lambda(x))\right)_{\substack{1\le i\le p\\ 1\le j\le p}} \neq 0$ and we obtain $\frac{\partial \tilde{\lambda}_i}{\partial \bar{u}_k}(x) = 0,\ i=1,\dots,p,\ k=1,\dots,n$

Hence $\bar{\partial}\lambda_i|_M = 0$, $\lambda_i \in CR(M)$ and $f(t,x) = q(t,x,\lambda(x))P(t,x)$.

Remark 1. If M is a totally real submanifold of C^n then every C^∞ function on R x M is a CR function on R x M. In this case the factorization is not unique.

Remark 2. Let $M = R \times C \subset C^2$

Then $R \times M = \{z \in C^3 \mid y_1 = 0,\ y_2 = 0\}$, $z = (z_1, z_2, z_3)$

$z_j = x_j + iy_j$, $j = 1,2,3$.

If $p \in M$ and $W = \sum_{j=1}^{3} w_j\left(\frac{\partial}{\partial z_j}\right)_p \in HT_p(R\times M)$ we have $w_1 = 0$, $w_2 = 0$.

It means that $HT(R\times M)$ is generated by $\frac{\partial}{\partial z_3}$ and every smooth function that does not depend on z_3 is a CR function on R x M.

If $f(t,x_2,z_3) = t + i(x_2 + e^{-\frac{1}{x_2^2}})$ we have $f = 1\cdot f$,

$f = (t+ix_2)\left(1 + \frac{ie^{-\frac{1}{x_2^2}}}{t+ix_2}\right)$ and the factorisation is not unique.

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III THE MAXIMUM MODULUS PRINCIPLE FOR CR FUNCTIONS

Preliminaries

Let M be a smooth manifold embedded as a locally closed real submanifold of C^n. We denote by $\bar\partial_M$ the tangential Cauchy -Riemann operator on M induced by the Cauchy-Riemann operator $\bar\partial$ on C^n and with HTp(M) the holomorphic tangent space to M at a point p M

Let's remember some of the definitions and results of [3]

Definition 1 $\bar\partial_M$ obeys the local maximum modulus principle on M iff given any open connected set U in M and any u differentiable in U such that $\bar\partial_M u=0$ on U them $|u|$ cannot have a (weak) local maximum at any point of U unless u is constant on U.

Definition 2 . We call a point $p\in M$ an extreme point of M iff there exists local holomorphic coordinate system $Z=(Z_1,\dots Z_n)$ in a neighborhood U of p such that $Z(p)=0$ and $M\cap U\subset\{Z|y_1\geq 0\}$ Here we assume that locally near p, M is not contained in any C^k for $k < n$.

Definition 3 . i) For any $p\in M$ and $X\in$ HTp(M) set Z=X-iY where Y= $JX\in$ HTp(M) and J is the multiplication with $(-1)^{1/2}$ which defines the complex structure on R^{2n}.

The Levi form at p assigns to Z the normal vector Lp (Z) defined by Lp(Z)=Bp(X,X)+Bp(Y,Y) where Bp is the second fundamental form of M at p

ii) We denote with Np(M) the normal space of M at p.

For any $\xi\in N_p(M)$ the map L_p^ξ defined by $L_p^\xi(Z)=$ $=\langle L_p(Z),\xi\rangle$ is called the Levi form of M at p in the ξ direction. Here $\langle\ ,\ \rangle$ represents the real inner product in R^{2n}.

We assume that p=o and $\operatorname{codim}_R M=\ell$. Then in a neighborhood U of the origin there are smooth real functions $\rho_1,\dots,\rho_\ell$

such that $d\rho_1\wedge\dots\wedge d\rho_\ell|_0\neq 0$ and
$M\cap U=\{z\in U|\rho_1(z)=\dots=\rho_\ell(z)=0\}$

We may assume that $d\rho_1(0),\dots,d\rho_\ell(0)$. are orthonormal.

If $\xi\in N_0(M)$ then $\xi=\sum_{i=1}^{\ell}\xi_i\, d\rho_i(0)$ and if
$Z=\sum_{j=1}^{n}w_j\left(\frac{\partial}{\partial z_j}\right)_0\in HT_0(M)$ then $\sum_{j=1}^{n}\frac{\partial\rho_i}{\partial z_j}(0)w_j=0$ for $1\le i\le \ell$.

In these conditions
$$L_0^{\xi}(Z)=-4\sum_{i,j,k}\xi_i\frac{\partial^2\rho_i}{\partial z_j\,\partial\bar z_k}(0)w_j\bar w_k \qquad [3]$$

In [3] there are proved the results:

Proposition 1 : If p is an extreme point of M then there exists a normal direction $\xi\in N_p(M)$ such that L_p^{ξ} positive definite.

Proposition 2 If for a point $p\in M$ there exists $\xi\in N_p(M)$ such that L_p^{ξ} is strictly positive definite then p is an extreme point of M.

Theorem 1 If $\bar\partial_M$ obeys the local maximum modulus principle on M than M can contain no extreme point.

Statement of Results

A submanifold M of C^n is called a CR manifold iff $\dim_c HTp(M)$ is constant on M.

We say that a CR submanifold of C^n has CR dimension m iff $\dim_c HTp(M)=m$

A totally real submanifold of C^n is a CR submanifold of C^n with CR dim (M) = 0

A complex valued smooth function on M for which $\bar\partial_M f=0$ on M is called a CR function on M.

Theorem 2 Each point of a totally real submanifold of C^n is an extreme point of M.

Theorem 3 If M is a CR submanifold of C^n with CR dim (M) ≥ 1 and without extreme points, then for any CR function f on M $|f|$ cannot have a strong local maximum at any point of M.

Proof of theorem 2 .

We know from [4] that there exists a nonnegative function $\varphi \in C^2(\mathbb{C}^n)$ strictly plurisubharmonic in a neighborhood D of M such that

$$M = \{z \in D \mid \varphi(z) = 0\} = \{z \in D \mid \operatorname{grad} \varphi = 0\}$$

Let $p \in M$. We assume that p=o and in a neighborhood V of p we have $V \cap M = \{z \in V \mid \rho_1(z) = \cdots = \rho_e(z) = 0\}$ with $d\rho_1 \wedge \cdots \wedge d\rho_e|_0 \neq 0$

Let $\rho = \varphi + \varepsilon \rho_1 \in C^2(V)$ where $\varepsilon > 0$ is chosen small enough such that the complex Hessian of ρ is strictly positive definite at the origin.

We have also $d\rho(0) = \varepsilon d\rho_1(0) \neq 0$ and we may assume that $\frac{\partial \rho_1}{\partial z_1}(0) \neq 0$, so $\frac{\partial \rho}{\partial z_1}(0) \neq 0$.

Because $\rho(0) = 0$ in a neighborhood of the origin we have

$$\rho(z) = 2\operatorname{Re}\sum_{i=1}^{n} \frac{\partial \rho}{\partial z_i}(0) z_i + \operatorname{Re}\Big(\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(0) z_i z_j\Big) + \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(0) z_i \bar{z}_j + O(|z|^3)$$

We make the holomorphic change of coordinates in $\mathbb{C}^n$

$$\begin{cases} z'_1 = 2i \sum_{i=1}^{n} \frac{\partial \rho}{\partial z_i}(0) z_i + i \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(0) z_i z_j \\ z'_i = z_i \ \text{for } 2 \leq i \leq n \end{cases}$$

In the new coordinates we have $\rho(z') = -y'_1 + \sum_{i,j=1}^{n} a_{ij} z'_i \bar{z}'_j + O(|z|^3)$ with $\sum_{i,j=1}^{n} a_{ij} z_i \bar{z}_j$ positive definite.

Define $S = \{z \in V \mid \rho(z) = 0\}$

It follows that $S \subset \{z' \mid y'_1 \geq 0\}$ in the neighborhood of the origin and because

$M \cap V \subset S$ the theorem is proved.

Proof of theorem 3

We shall use the following lemma [1]

Lemma 1. Let Ω be an open subset of $\mathbb{R}^N$ with coordinates $x_1 \ldots x_N$ Let $F \in C^\infty(\Omega)$ and L be a compact subset of Ω. Suppose that $F(x) < \max_L F$ for each $x \in \Omega - L$

Then for any open set Ω_1 with $L\subset\Omega_1\subset\Omega$ there exists a point $y\in\Omega_1$ such that the Hessian of F at y is strictly negative definite.

We suppose that there exists a CR function f on M such that $|f|$ has a point of strong local maximum. Then, there exists a compact set $K\subset M$ such that $\max_K |f| > \max_{\partial K} |f|$. We may assume that K is contained in an open set in $R^{2n-\ell}$ which is part of the atlas what defines M. We may assume also that $\max_K \mathrm{Re} f$ $\max_{\partial K} \mathrm{Re} f$. We denote $\mathrm{Re} f=\varphi$

By the lemma 1 there is $p\in K$ such that $\left(\frac{\partial^2\varphi}{\partial t_i\partial t_j}(p)\right)_{\substack{1\le i\le 2n-\ell\\ 1\le j\le 2n-\ell}}$ is strictly negative definite for any real coordinates $(t_1,\dots,t_{2n-1})$ in a neighborhood of p.

Let CR dim (M) = $m\geq 1$. We denote $s=2n-\ell-2m$, $r=m-(n-\ell)$

After a complex linear change of coordinates in C^n M may be represented in the neighborhood of the point p by the equations:

$$
\begin{aligned}
z_j &= x_j+ig^j(x,w) && j=1,\dots,s\\
z_{j+s} &= h^j(x,w) && j=1,\dots,r\\
z_{j+s+r} &= w_j && j=1,\dots,m
\end{aligned}
$$

Where p corresponds to the origin and $\{g^j\}_{j=1,\dots,s}$ $\{h^j\}_{j=1,\dots,r}$ are real and complex valued functions respectively vanishing to second order at the origin [5].

Therefore, if $\varrho_1=y_1-g^1(x,z),\dots,\varrho_s=y_s-g^s(x,z)$, $\varrho_{s+1}=x_{s+1}-\mathrm{Re}\,h^1(x,z)$, $\varrho_{s+2}=y_{s+1}-\mathrm{Im}\,h^1(x,z)$, $\dots$, $\varrho_\ell=y_{s+r}-\mathrm{Im}\,h^r(x,z)$ (1)

where $X=(x_1,\dots,x_s)$ and $z=(z_{s+r+1},\dots,z_n)$ in a neighborhood of the origin we have

$M=\{z\,|\,\varrho_1(z)=\dots=\varrho_\ell(z)=0\}$, $d\varrho_1\wedge\dots\wedge d\varrho_\ell|_0\neq 0$ and $d\varrho_1(0),\dots,d\varrho_\ell(0)$ are orthonormal.

φ is the real part of a CR function so $(\partial\bar\partial)_b\varphi=0$ [2]

If $\tilde{\varphi}$ is a real extension of φ in a neighborhood U of p in C^n we obtain $\partial\bar{\partial}\tilde{\varphi}=\sum_{k=1}^{\ell} a_k \rho_k+\sum_{k=1}^{\ell} b_k\partial\rho_k+\sum_{k=1}^{\ell} c_k\bar{\partial}\rho_k+\sum_{k=1}^{\ell} d_k\partial\bar{\partial}\rho_k$ with $a_k\in C^\infty_{(1,1)}(U)$, $b_k\in C^\infty_{(0,1)}(U)$, $c_k\in C^\infty_{(1,0)}(U)$, $d_k\in C^\infty(U)$ or

$$\frac{\partial^2\tilde{\varphi}}{\partial z_i\partial\bar{z}_j}=\sum_{k=1}^{\ell} a_{ijk}\rho_k+\sum_{k=1}^{\ell} b_{kj}\frac{\partial\rho_k}{\partial z_i}+\sum_{k=1}^{\ell} c_{ki}\frac{\partial\rho_k}{\partial\bar{z}_j}+\sum_{k=1}^{\ell} d_k\frac{\partial^2\rho_k}{\partial z_i\partial\bar{z}_j} \quad (2)$$

We shall use the next lemma:

Lemma 2. There is an extension $\tilde{\varphi}$ of φ so that

$$\frac{\partial\tilde{\varphi}}{\partial y_i}(0)=0,\ i=1,\dots,s,\ \text{and}\ \frac{\partial\tilde{\varphi}}{\partial x_j}(0)=0,\ \frac{\partial\tilde{\varphi}}{\partial y_j}(0)=0,\ j=s+1,\dots,s+r$$

Proof of the lemma

Let $\tilde{\varphi}_0$ be an extension of φ and $\tilde{\varphi}=\tilde{\varphi}_0+\sum_{k=1}^{\ell}\delta_k\rho_k$ another extension with $\delta_k\in R$. With the functions ρ_k we had chosen we obtain

$$\tilde{\varphi}=\tilde{\varphi}_0+\sum_{k=1}^{s}\alpha_k(y_k-g^k(x,z))+\sum_{k=s+1}^{s+r}\beta_k(x_k-\operatorname{Re}h^k(x,w))+\sum_{k=s+1}^{s+r}\gamma_k(y_k-\operatorname{Im}h^k(x,w))$$

So $\frac{\partial\tilde{\varphi}}{\partial y_j}(0)=\frac{\partial\tilde{\varphi}_0}{\partial y_j}(0)+\alpha_j,\ j=1,\dots,s,\ \frac{\partial\tilde{\varphi}}{\partial x_j}(0)=\frac{\partial\tilde{\varphi}_0}{\partial x_j}(0)+\beta_j,\ \frac{\partial\tilde{\varphi}}{\partial y_j}(0)=\frac{\partial\tilde{\varphi}_0}{\partial y_j}(0)+$ $+\gamma_j,\ j=s+1,\dots,s+r$

If we take $\alpha_j=-\frac{\partial\tilde{\varphi}_0}{\partial x_j}(0),\ j=1,\dots,s;\ \beta_j=-\frac{\partial\tilde{\varphi}_0}{\partial x_j}(0),\ \gamma_j=-\frac{\partial\tilde{\varphi}_0}{\partial y_j}(0)$, $j=s+1,\dots,s+r$ we obtain the needed extension.

In that follows we shall suppose that $\tilde{\varphi}$ is the extension given by lemma 2.

Let $Z\in HT_0(M)$ i.e. $Z=\sum_{i=1}^{n} z_i\left(\frac{\partial}{\partial z_i}\right)_0$ and $\sum_{i=1}^{n} z_i\frac{\partial\rho_k}{\partial z_i}(0)=0$, $k=1,\dots,l$

From (2) we obtain $$\sum_{i,j}\frac{\partial^2\tilde{\varphi}}{\partial z_i\partial\bar{z}_j}(0)z_i\bar{z}_j=\sum_{i,j=1}^{n}\sum_{k=1}^{\ell} d_k(0)\frac{\partial^2\rho_k}{\partial z_i\partial\bar{z}_j}(0)z_i\bar{z}_j \quad (3)$$

But, with the functions ρ_k given by (1) $Z\in HT_0(M)$ iff $z_1=\dots=z_{s+r}=0$

Denote $z_{s+r+1}=w_1,\dots,z_n=w_m$

We shall prove that $$\frac{\partial^2\tilde{\varphi}}{\partial z_{s+r+i}\partial\bar{z}_{s+r+j}}(0)=\frac{\partial^2\varphi}{\partial w_i\partial\bar{w}_j}(0) \quad (4)$$

We know that $\tilde{\varphi}(x_1+ig^1(x,w),\dots,x_s+ig^s(x,w),h^1(x,w),\dots,h^r(x,w),w_1,\dots,w_m)=\varphi(x_1,\dots,x_s,w_1,\dots,w_m)$

It follows that

$$\frac{\partial}{\partial w_i}\varphi(x_1,\dots,x_s,w_1,\dots,w_m)=\frac{\partial}{\partial w_i}\tilde{\varphi}(x_1+ig^1(x,w),\dots,x_s+ig^s(x,w),h^1(x,w),\dots,h^r(x,w),w_1,\dots,w_m)=$$

$$=\sum_{k=1}^{s}\left(\frac{\partial\tilde{\varphi}}{\partial z_k}\cdot\frac{\partial(x_k+ig^k(x,w))}{\partial w_i}+\frac{\partial\tilde{\varphi}}{\partial\bar{z}_k}\cdot\frac{\partial(x_k-ig^k(x,w))}{\partial w_i}\right)+$$

$$+\sum_{k=s+1}^{s+r}\left(\frac{\partial\tilde{\varphi}}{\partial z_k}\cdot\frac{\partial h^k}{\partial w_i}+\frac{\partial\tilde{\varphi}}{\partial\bar{z}_k}\cdot\frac{\partial\bar{h}^k}{\partial w_i}\right)+\frac{\partial\tilde{\varphi}}{\partial w_i}=$$

$$=\sum_{k=1}^{s}\frac{\partial\tilde{\varphi}}{\partial y_k}\cdot\frac{\partial g^k}{\partial w_i}+\sum_{k=s+1}^{s+r}\left(\frac{\partial\tilde{\varphi}}{\partial x_k}\cdot\frac{\partial\operatorname{Re}h^k}{\partial w_i}+\frac{\partial\tilde{\varphi}}{\partial y_k}\cdot\frac{\partial\operatorname{Im}h^k}{\partial w_i}\right)+\frac{\partial\tilde{\varphi}}{\partial w_i}$$

Hence
$$\frac{\partial^2\varphi}{\partial w_i\partial\bar{w}_j}=\sum_{k=1}^{s}\left[\frac{\partial}{\partial\bar{w}_j}\left(\frac{\partial\tilde{\varphi}}{\partial y_k}\right)\frac{\partial g^k}{\partial w_i}+\frac{\partial\tilde{\varphi}}{\partial y_k}\cdot\frac{\partial^2 g^k}{\partial w_i\partial\bar{w}_j}\right]+$$

$$+\sum_{k=s+1}^{s+r}\left[\frac{\partial}{\partial\bar{w}_j}\left(\frac{\partial\tilde{\varphi}}{\partial x_k}\right)\cdot\frac{\partial\operatorname{Re}h^k}{\partial w_i}+\frac{\partial\tilde{\varphi}}{\partial x_k}\cdot\frac{\partial^2\operatorname{Re}h^k}{\partial w_i\partial\bar{w}_j}+\frac{\partial}{\partial\bar{w}_j}\left(\frac{\partial\tilde{\varphi}}{\partial y_k}\right)\frac{\partial\operatorname{Im}h^k}{\partial w_i}+\right.$$

$$\left.+\frac{\partial\tilde{\varphi}}{\partial y_k}\cdot\frac{\partial^2\operatorname{Im}h^k}{\partial w_i\partial\bar{w}_j}\right]+\frac{\partial}{\partial\bar{w}_j}\left(\frac{\partial\tilde{\varphi}}{\partial w_i}\right)$$

and
$$\frac{\partial}{\partial\bar{w}_j}\left(\frac{\partial\tilde{\varphi}}{\partial w_i}\right)=\sum_{k=1}^{s}\left(\frac{\partial^2\tilde{\varphi}}{\partial w_i\partial z_k}i\frac{\partial g^k}{\partial\bar{w}_j}-\frac{\partial^2\tilde{\varphi}}{\partial w_i\partial\bar{z}_k}i\frac{\partial g^k}{\partial\bar{w}_j}\right)+$$

$$+\sum_{k=s+1}^{s+r}\left(\frac{\partial^2\tilde{\varphi}}{\partial w_i\partial z_k}\cdot\frac{\partial h^k}{\partial\bar{w}_j}+\frac{\partial^2\tilde{\varphi}}{\partial w_i\partial\bar{z}_k}\cdot\frac{\partial\bar{h}_k}{\partial\bar{w}_j}\right)+\frac{\partial^2\tilde{\varphi}}{\partial w_i\partial\bar{w}_j}$$

Because $\{g^k\}$ and $\{h^k\}$ are vanishing to second order at the origin and $\frac{\partial\tilde{\varphi}}{\partial y_k}(0)=0$, $k=1,\dots,s$, $\frac{\partial\tilde{\varphi}}{\partial x_k}(0)=0$, $\frac{\partial\tilde{\varphi}}{\partial y_k}(0)=0$ $k= s+1,\dots, s+r$ we obtain (4)

It follows that $\sum_{i,j=1}^{n}\frac{\partial^2\tilde{\varphi}}{\partial z_i\partial\bar{z}_j}(0)z_i\bar{z}_j=\sum_{i,j=1}^{m}\frac{\partial^2\varphi}{\partial w_i\partial\bar{w}_j}(0)w_i\bar{w}_j$

which is strictly negative definite.

Indeed supposing that $\sum_{i,j=1}^{m}\frac{\partial^2\varphi}{\partial w_i\partial\bar{w}_j}(0)w_i\bar{w}_j$ is not strictly negative definite by making a complex linear change of coordinates $w_1,\dots, w_m$, in the new coordinates $w'_1,\dots, w'_m$ we may assume $\frac{\partial^2\varphi}{\partial w'_1\partial\bar{w}'_1}(0)\geq 0$

But $\frac{\partial^2\varphi}{\partial w'_1\partial\bar{w}'_1}(0)=\frac{1}{4}\left[\frac{\partial^2\varphi}{\partial u'^2_1}(0)+\frac{\partial^2\varphi}{\partial v'^2_1}(0)\right]$ where $w'_1 = u'_1+ iv'_1$ and this contradicts the fact that the real Hessian of φ at 0 is strictly negative definite.

Taking the real parts in (3) we obtain

$$\sum_{i,j=1}^{n}\frac{\partial^2\tilde{\varphi}}{\partial z_i\partial\bar{z}_j}(0)z_i\bar{z}_j=\sum_{k=1}^{\ell}\operatorname{Re} d_k(0)\Big(\sum_{i,j=1}^{n}\frac{\partial^2 g_k}{\partial z_i\partial\bar{z}_j}(0)z_i\bar{z}_j\Big)$$

Hence if $\xi=\sum_{k=1}^{\ell}(\operatorname{Re} d_k(0))\,d\varrho_k(0)\in N_0(M)$ if follows that L_p^{ξ} is strictly positive definite and by proposition 2, p is an extreme point of M which contradicts the fact that M has no extreme points.

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