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Introduction

As the title announces, the main purpose of this paper is to give a list of all smooth, projective varieties defined over $\mathbb{C}$, of degree 5 and 6.**) We tried to obtain much of the abstract structure and their embedded properties (Hilbert polynomial, number and degrees of generators of their ideals, Hilbert schemes,...) Our results are summarized in the table below. The main technical tool of our investigation is the adjunction mapping, studied in modern literature by Sommese [27] and Van de Ven [29]. It allowed us to give modern proofs of two theorems originally due to Castelnuovo [4], [5], from which one deduces the list of surfaces of degree $\leqslant 6$ (see Semple-Roth [26] p.218). We must emphasize that to complete the list of (smooth) varieties of degree 5 and 6, one needs the difficult classification of Del Pezzo varieties, recently obtained by Fujita [9], [10] (the case of 3-folds was previously done by Iskovskih [17]). For such varieties the adjunction mapping reduces to the constant map, so one needs completely different techniques. We recall that one knows all (not necessarily smooth) varieties of degree $\leqslant 4$, see the anonymous note [30] and Swinnerton-Dyer [28]. For reader's convenience we gave at the end of our paper a uniform way of obtaining the smooth

*) Partial results of this work were the subject of a conference at the "Week of Algebraic Geometry" held in Bucharest, 1981, June 22-28.

**) See Hartshorne [15] for a survey article concerning the classification problem for varieties of small degree.

ones, which we have also included in the table. We are aware that many of our statements were familiar to classical algebraic geometers. However we included proofs for those results for which we couldn't find an adequate reference. We also mentioned several known facts for the sake of completeness.

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§0. Preliminaries

We shall work over the field of complex numbers $\mathbb{C}$. The word variety will mean projective, smooth and connected (if not otherwise stated) algebraic variety. A curve (resp. surface) is a variety of dimension 1 (resp. 2).

We shall denote by $\sigma_{P_1,\dots,P_k}: X' \longrightarrow X$ the blowing-up morphism between surfaces X', X, with center $P_1,\dots,P_k$, E_i the exceptional divisors. For a complete linear system $|D|$ on X we shall write $|D - a_1P_1 - \dots - a_kP_k|$ for the complete linear system $|\sigma^*_{P_1,\dots,P_k}(D) - a_1E_1 - \dots - a_kE_k|$ on X', where a_i are positive integers (see [16] p. 394). By a geometrically ruled surface we mean a surface X isomorphic to $\mathbb{P}(E)$ for some rank-2 locally free sheaf on a curve C. For everything concerning such surfaces we refer the reader to [16] Ch. V§2. In particular if $\pi: X \longrightarrow C$ denotes the natural projection, we shall frequently use the normalisation of E giving a section C_o of π such that : $\mathrm{Pic}(X) \cong \mathbb{Z}[C_o] \oplus \pi^*\mathrm{Pic}(C)$, $\mathrm{Num}(X) \cong \mathbb{Z}[C_o] \oplus \mathbb{Z}[F]$, where F is a fibre of π and $(F^2)=o$, $(F\cdot C_o)=1$, $(C_o^2)=-e$. As in [16] we let $\mathbb{F}_e = \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-e))$, $e \geq o$. By a scroll we mean a variety $X = \mathbb{P}(E)$, where E is a locally free sheaf of rank ≥ 2 over a curve C, embedded in $\mathbb{P}^n$ in such a way that the fibers of the natural projection $\pi: X \longrightarrow C$ are linear varieties. A rational (resp. elliptic) scroll is a scroll over $\mathbb{P}^1$ (resp. over an elliptic curve).

We shall freely use intersection theory (see [18]), the adjunction formula (see [15] p. 243) and the Riemann-Roch theorem (R-R for short) for curves, surfaces and 3-folds ([16] p. 295, 362, 437). We also need Kodaira's vanishing theorem, (see for instance [12] p. 154). For a study of "linkage" in codimension 2 see [25]. For the rest notation and terminology are standard.

We need the following results:

o.1. Lemma Let $(X, \mathcal{O}_X(H))$ be a (smooth) projective variety of dimension ≥ 2 and $\mathcal{L}$ an invertible sheaf on X. Suppose for any smooth hyperplane section H of X, $\mathcal{L}|_H$ is generated by its global sections and $H^1(X, \mathcal{L}\otimes\mathcal{O}_X(-H))=0$. Then $\mathcal{L}$ is generated by global sections.

For a proof apply Bertini's theorem and the standard exact sequence :

$$0 \longrightarrow \mathcal{L}\otimes\mathcal{O}_X(-H) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}|_H \longrightarrow 0.$$

o.2. Lemma Let $X\subset\mathbb{P}^n$ be a (smooth) projective variety of dimension ≥ 2. Suppose its generic hyperplane section $H=X\cap\mathbb{P}^{n-1}$ is arithmetically Cohen-Macaulay (resp. a complete intersection of type $d_1,\dots,d_s$) in $\mathbb{P}^{n-1}$. Then the same holds for X in $\mathbb{P}^n$. If X is arithmetically Cohen-Macaulay and the homogenous ideal of H in $\mathbb{P}^{n-1}$ is generated by forms of degree $\leq k$, the same is true for the ideal of X in $\mathbb{P}^n$.

The proof is standard and we omit it.

The following result is Exercise 2.12 (b) in [16] Ch. V, §2.

o.3. Lemma Let X be a geometrically ruled surface with invariant e over an elliptic curve C and b an integer. Let $H\equiv C_0+bF$. Then $|H|$ is very ample if and only if $b\geq e+3$.

For any nondegenerated variety $X\subset\mathbb{P}^n$ of dimension r and degree d, one has the following elementary inequality (for instance [17], lemma 2.1):

o.4. $d \geq n-r+1$

If r = 1, recall Castelnuovo's bound for the genus of X (see [12] p. 252)

o.5. $g \leq \left[\frac{d-2}{n-1}\right]\left(d-n-\left[\frac{d-n-1}{n-1}\right]\left(\frac{n-1}{2}\right)\right)$, where [ ] denotes least integer function.

o.6. Rational scrolls. If $(X,\mathcal{O}_X(H))$ is a rational scroll of degree d and dimension r we can write $X\cong\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus\ldots\oplus\mathcal{O}_{\mathbb{P}^1}(a_r))$, with $a_i\geq 1$, $i=1,\ldots,r$, $\sum_{i=1}^{r} a_i=d$ and $\mathcal{O}_X(H)$ is the tautological sheaf (see [7] th. 3.8). If $a_i=1$, $i=1,\ldots,r$, this is just the Segre embedding of $\mathbb{P}^1\times\mathbb{P}^{d-1}$ in $\mathbb{P}^{2d-1}$; any other rational scroll of degree d is a linear section of this Segre embedding. Indeed, if say a_1 is >1 we can constuct an exact sequence of the form:

$$0\longrightarrow\mathcal{O}_{\mathbb{P}^1}\longrightarrow\mathcal{O}_{\mathbb{P}^1}(b)\oplus\mathcal{O}_{\mathbb{P}^1}(c)\oplus\mathcal{O}_{\mathbb{P}^1}(a_2)\oplus\ldots\oplus\mathcal{O}_{\mathbb{P}^1}(a_r)\xrightarrow{f}\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus\ldots\oplus\mathcal{O}_{\mathbb{P}^1}(a_r)\longrightarrow 0$$

with $b\geq 1$, $c\geq 1$, $b+c=a_1$ (see [1] p. 19). It is easy to see that f makes $H\cong\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus\ldots+\mathcal{O}_{\mathbb{P}^1}(a_r))$ a hyperplane section of $X\cong\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(b)\oplus\mathcal{O}_{\mathbb{P}^1}(c)\oplus\mathcal{O}_{\mathbb{P}^1}(a_2)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}^1}(a_r))$.

The following table presents the list of all nondegenerated, linearly normal, smooth varieties of degree ≤ 6 and dimension r, embedded in $\mathbb{P}^n$. We let $s=n-r$.

Except for the elliptic scrolls and some curves of genus 3 and degree 6, all the varieties in the table are arithmetically Cohen-Macaulay.

* -See Fujita [10] for various abstract descriptions of such varieties. They are projectively equivalent if they have the same dimension.

** - f_1^3, f_2^3, f_3^3, f_4^3 if the curve is not on a quadric.

DEGREE 1

Projective space $\mathbb{P}^r$.

DEGREE 2

Hyperquadric Q^r

DEGREE 3

s	r	Abstract structure	$H\ (\mathcal{O}_X(H))$	Number and degrees of generators of I(X)	Coef. of the Hilbert pol. in the base $\left\{\binom{z+i-1}{i}\right\}_{i=0,\dots,r}$	Hilbert scheme
2	1	$\mathbb{P}^1$	$\mathcal{O}_{\mathbb{P}^1}(3)$	f_1^2, f_2^2, f_3^2	$(3, 1)$	
	2	$\mathbb{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2))$	tautological sheaf	— " —	$(3, 1, 1)$	
	3	$\mathbb{P}^1 \times \mathbb{P}^2 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$	— " —	— " —	$(3, 1, 1, 1)$	
1	$r \geq 1$			f_1^3	$(3, 0, 1, \dots, 1)$	$\mathbb{P}^{\binom{r+4}{3}-1}$

DEGREE 4

s	r	Abstract structure	$H\ (\mathcal{O}_X(H))$	Number and degrees of generators of I(X)	Coef. of the Hilbert pol.	Hilbert scheme
3	1	$\mathbb{P}^1$	$\mathcal{O}_{\mathbb{P}^1}(4)$	$f_1^2, f_2^2, \dots, f_6^2$	$(4, 1)$	
	2	– $\mathbb{F}_0 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2))$; $\mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3))$	tautological sheaf	— " —	$(4, 1, 1)$	
		– $\mathbb{P}^2$ (Veronese surface)	$\mathcal{O}_{\mathbb{P}^2}(2)$	— " —	$(4, 1, 1)$	
	3	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2))$	tautological sheaf	— " —	$(4, 1, 1, 1)$	
	4	$\mathbb{P}^1 \times \mathbb{P}^3 \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$	— " —	— " —	$(4, 1, 1, 1, 1)$	
2	$r \geq 1$			$f_1^2,\ f_2^2$	$(4, 0, 1, \dots, 1)$	irred., smooth, dim. r^2+7r+8
1	$r \geq 1$			f_1^4	$(4, -2, 2, 1, \dots, 1)$	$\mathbb{P}^{\binom{r+5}{4}-1}$

DEGREE 5

4	1	$\mathbb{P}^1$	$\mathcal{O}_{\mathbb{P}^1}(5)$	$f_1^2, f_2^2, \dots, f_{10}^2$	$(5,1)$	
	2-5	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r))$, $\sum_{i=1}^{r} a_i = 5$, $a_i \geq 1$, $i=1,\dots,r$	tautological sheaf	— " —	$(5,1,\dots,1)$	
3	1	$g=1$		$f_1^2, f_2^2, \dots f_5^2$	$(5,0)$	smooth, dim. 25
	2	$\sigma_{P_1,\dots,P_4}: X \longrightarrow \mathbb{P}^2$ (Del Pezzo surface)	$3L - P_1 - \dots - P_4$	— " —	$(5,0,1)$	
	3	*	*	— " —	$(5,0,1,1)$	
	4	*	*	— " —	$(5,0,1,1,1)$	
	5	*	*	— " —	$(5,0,1,1,1,1)$	
	6	$G(1,4)$ - Grasmmann variety of lines in $\mathbb{P}^4$	Plücker embedding	— " —	$(5,0,1,1,1,1,1)$	
2	1	$g=2$		f_1^2, f_2^3, f_3^3	$(5,-1)$	irred, smooth, dim 20
	2	- $\sigma_{P_0,\dots,P_7}: X \longrightarrow \mathbb{P}^2$ (Castelnuovo surfaces)	$4L - 2P_0 - P_1 - \dots - P_7$	— " —	$(5,-1,1)$	— " — , — " — , — " — 32
		- Elliptic scrolls, $e=-1$	$H \equiv C_0 + 2F$	?	$(5,0,0)$	smooth, dim 25
	3	$b_1 = 0$, $b_2 = 2$, $b_3 = 6$; $Pic(X) \simeq \mathbb{Z} \times \mathbb{Z}$ with base H, Q; $(H)^3 = 5$, $(H^2 Q) = 2$, $(H \cdot Q^2) = (Q^3) = 0$.		f_1^2, f_2^3, f_3^3	$(5,-1,1,1)$	irred, smooth, dim. 46
1	$r \geq 1$			f_1^5	$(5,-5,5,0,1,\dots,1)$	$\mathbb{P}^{\binom{r+6}{5}-1}$

DEGREE 6

5	1	$\mathbb{P}^1$	$\mathcal{O}_{\mathbb{P}^1}(6)$	$f_1^2, f_2^2, \dots, f_{15}^2$	$(6,1)$	
	2 - 6	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r))$, $\sum_{i=1}^{r} a_i = 6$, $a_i \geq 1$, $i = 1, \dots, r$	tautological sheaf	— " —	$(6,1,\dots,1)$	
4	1	$g = 1$		$f_1^2, f_2^2, \dots, f_9^2$	$(6,0)$	smooth, dim. 36
	2	$\sigma_{P_1,P_2,P_3} : X \longrightarrow \mathbb{P}^2$ (Del Pezzo surface)	$3L - P_1 - P_2 - P_3$	— " —	$(6,0,1)$	
	3	– $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$p_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes p_3^*\mathcal{O}_{\mathbb{P}^1}(1)$	— " —	$(6,0,1,1)$	
		– $\mathbb{P}(T_{\mathbb{P}^2})$, $T_{\mathbb{P}^2}$ tangent sheaf to $\mathbb{P}^2$	tautological sheaf	— " —	$(6,0,1,1)$	
	4	$\mathbb{P}^2 \times \mathbb{P}^2$	$p_1^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(1)$	— " —	$(6,0,1,1,1)$	
3	1	$g = 2$		$f_1^2, f_2^2, f_3^2, f_4^2$	$(6,-1)$	smooth, dim 29
	2	– $\sigma_{P_0,\dots,P_6} : X \longrightarrow \mathbb{P}^2$ (Castelnuovo surfaces)	$4L - 2P_0 - P_1 - \cdots - P_6$	— " —	$(6,-1,1)$	?
		– Elliptic scrolls, $e = 0$	$H \equiv C_0 + 3F$	?	$(6,0,0)$	smooth, dim 36
	3	$f : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ double covering, ramified $(2,2)$; $b_1 = 0$, $b_2 = 2$, $b_3 = 4$; $\mathrm{Pic}(X) \simeq \mathbb{Z} \times \mathbb{Z}$ with base H, Q; $(H^3) = 6$, $(H^2 \cdot Q) = 2$, $(H \cdot Q^2) = (Q^3) = 0$ (Fano variety)	$f^*(p_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(1))$	$f_1^2, f_2^2, f_3^2, f_4^2$	$(6,-1,1,1)$	?
2	1	$g = 3$		* *	$(6,-2)$	irred., smooth, dim. 24
	2	– $\sigma_{P_1,\dots,P_{10}} : X \longrightarrow \mathbb{P}^2$ (Bordiga surfaces)	$4L - P_1 - \cdots - P_{10}$	$f_1^3, f_2^3, f_3^3, f_4^3$	$(6,-2,1)$	— " —, — " —, — " — 36

2	3	$X = \mathbb{P}(E)$, E stable rank-2 locally free sheaf on $\mathbb{P}^2$, with $c_1(E)=4$, $c_2(E)=10$, $E\|_L \simeq \mathcal{O}_L(2)\oplus\mathcal{O}_L(2)$, given by: $0 \to \mathcal{O}_{\mathbb{P}^2} \to E \to \mathcal{J}_Y(4) \to 0$, $Y=\{P_1,\dots,P_{10}\}$	tautological sheaf	$f_1^3, f_2^3, f_3^3, f_4^3$	$(6,-2,1,1)$	irred., smooth, dim. 48
	$r\geq 1$			f_1^2, f_2^3	$(6,-3,2,1,\dots,1)$	irred., smooth, dim. $\frac{1}{6}(r+3)(r+4)(r+8)-(r+5$
1	$r\geq 1$			f_1^6	$(6,-9,11,-4,2,1,\dots,1)$	$\mathbb{P}^{\binom{r+7}{6}-1}$

§1. Remarks on the adjunction mapping

Let X be a (smooth, connected, projective) algebraic variety defined over $\mathbb{C}$ of dimension $r \geq 2$, and $\mathcal{L} = O_X(H)$ a very ample invertible sheaf on it. We shall always denote by H a smooth hyperplane section of X. Let $P = P_{X,\mathcal{L}}(z) = \sum_{i=0}^{r} \chi_i \binom{z+i-1}{i}$ be the Hilbert polynomial of the pair $(X, \mathcal{L})$ and denote by $\delta = \delta(X,\mathcal{L}) = \chi_{r-2} - \chi_{r-1}$ and $g = g(X,\mathcal{L}) = 1 - \chi_{r-1}$ the sectional genus. Let $q = q(X) = h^1(X, O_X)$ be the irregularity and $\delta' = \delta'(X,\mathcal{L}) = g - q$.

Remark 1.1. a) $q(X) = q(H)$ if $r \geq 3$

b) $\delta \geq \delta' \geq 0$

The exact sequence $o \to O_X(-H) \to O_X \to O_H \to o$ and Kodaira vanishing give a). To prove b) we can assume $r=2$; $\delta' \geq o$ by the same exact sequence and $\delta = \chi O_X - 1 + g = 1 - q + p_g - 1 + g = \delta' + p_g$, where $p_g = p_g(X)$ is the geometric genus.

Denote by A_r any of the following pairs $(X, \mathcal{L})$: $\mathbb{P}^r$, $O_{\mathbb{P}^r}(1)$; Q^r, $O_{Q^r}(1)$ - smooth hyperquadric; $\mathbb{P}^2$, $O_{\mathbb{P}^2}(2)$; a scroll.

Now, the adjunction mapping is the rational map associated to the complete linear system $|K_X + (r-1)H|$. We have the following result generalising the Sommese-Van de Ven theorem ([27] prop. 1.5. , [29] th. II.) to arbitrary dimensions

Theorem 1.1. The following are equivalent:

i) $(X, \mathcal{L}) \in A_r$

ii) $\delta = 0$

iii) $\delta' = 0$

iv) $h^0(X, O_X(K_X + (r-1)H)) = o$

v) $\chi O_X(K_X + (r-1)H)) = o$

vi) $O_X(K_X + (r-1)H)$ is not generated by global sections.

Proof. The equivalence of iv) and v) follows from Kodaira vanishing. iv) $\Rightarrow$ vi) is trivial and ii) $\Rightarrow$ iii) follows from remark 1.1. iii) $\Leftrightarrow$ vi) is proved in

[27] prop. 4.1. We shall show vi) $\Longrightarrow$ i), i) $\Longrightarrow$ ii) and i) $\Longrightarrow$ iv).

vi) $\Longrightarrow$ i) is by induction on r, the case r=2 being the key result of [27] and [29]. But by [1] th 5. and [2] ths 1,2,3, we have that for $r \geq 3$, $(X,\mathcal{L}) \in A_r$ $\Longleftrightarrow (H, \mathcal{L}|_H) \in A_{r-1}$; on the other hand, by duality and Kodaira vanishing, $H^1(X, O_X(K_X+(r-2)H)) \cong H^{r-1}(X, O_X((2-r)H))=o$, so by o.l. $O_X(K_X+(r-1)H))$ is generated by global sections if $O_H(K_H+(r-2)H)$ is (we used the adjunction formula and induction).

i) $\Longrightarrow$ ii) We can assume r=2 so $\delta = \chi O_X - 1 + g = g-q+p_g = o$ since g=q and $p_g = o$.

i) $\Longrightarrow$ iv) is by induction on r. If r=2 we have by Kodaira vanishing, R-R and the adjunction formula: $h^o(X, O_X(K_X+H)) = \chi(O_X(K_X+H)) = \chi O_X + g - 1 = o$ since $p_g = o$ and g=q. For $r \geq 3$, by adjunction formula $h^o(H, O_H(K_H+(r-2)H))=o$ implies $h^o(X, O_X(K_X+(r-1)H))=o$ since $|H|$ is very ample. q.e.d.

<u>Remark 1.2.</u> We understood that Sommese recently obtained the same theorem by a different method.

As a consequence we have, for instance, the following classical result, due to Enriques-Del Pezzo (see [7] ths. 2.1,2.2,3.8, for a direct proof):

<u>Corollary 1.1.</u> <u>Let g=o. Then $(X,\mathcal{L})$ is one of the following: $\mathbb{P}^r, O_{\mathbb{P}^r}(1)$; $Q^r, O_{Q^r}(1)$; $\mathbb{P}^2$, $O_{\mathbb{P}^2}(2)$; a rational scroll.</u>

<u>Corollary 1.2.</u> (Compare with [9], th. 1.9) <u>The following are equivalent:</u>

i) g=1, q=o

ii) $\delta = 1$

iii) $h^o(X, O_X(K_X+(r-1)H))=1$

iv) $\chi O_X(K_X+(r-1)H)=1$

v) $-K_X = (r-1)H$

<u>Remark 1.3.</u> The pairs $(X,\mathcal{L})$ satisfaying v) are known as Del Pezzo varieties and were recently completely classified by T. Fujita [9], [10] (see [17] for the case of 3-folds).

Proof. i) $\Longrightarrow$ ii) By adjunction, supposing $r=2$, we have: $(H^2)+(H\cdot K_X)=o$ so $p_g=o$ and we are done, since $\delta=\delta'+p_g$.

v) $\Longrightarrow$ iii) is trivial and iii) $\Longrightarrow$ v) follows from the theorem.

To prove v) $\Longrightarrow$ i) suppose $r=2$; then $p_g=o$ and by adjunction $g=1$, so that $1=h^o(X,O_X(K_X+H))=g-q$ and $q=o$. It remains to see that ii) $\Longrightarrow$ v). For $r=2$ we have $1=\delta=h^o(X,O_X(K_X+H))$ so the theorem applies. But for $r\geqslant 3$ the restriction map $Pic(X)\longrightarrow Pic(H)$ is injective by Lefschetz's theorem, so adjunction formula and induction yields the result.

We think the following result is also due to Enriques:

Corollary 1.3. Let $g=1$. Then we have one of the following:

i) $q=o$ and $(X,\mathcal{L})$ is a Del Pezzo variety

ii) $q=1$ and $(X,\mathcal{L})$ is an elliptic scroll.

Proof. If $q=o$ apply cor. 1.2, if $q=1$, $\delta'=o$ and we are done by the theorem.

Corollary 1.4. Let $g=2$. Then we have one of the following:

i) $q=o$ and the adjunction mapping φ is a morphism to $\mathbb{P}^1$, such that except for a finite number of points of $\mathbb{P}^1$, the fibers of φ, together with the restriction of $\mathcal{L}$ are smooth hyperquadrics.

ii) $q=2$ and $(X,\mathcal{L})$ is a scroll over a curve of genus 2.

Proof. If $q=o$, suppose first $r=2$. The adjunction formula gives $(H^2)+(H\cdot K_X)=2$, so $p_g=o$ and $h^o(X,O_X(K_X+H))=2$. We have $o=(H+K_X)^2=(H\cdot H+K_X)+(K_X\cdot H+K_X)$ which gives $(K_X\cdot H+K_X)=-2$. By adjunction we obtain $p_a(H+K_X)=o$ so $|H+K_X|$ is a pencil of conics. For $r\geqslant 3$, as in the proof of the theorem i) $\Longrightarrow$ iv), we have inductively $h^o(X,O_X(K_X+(r-2)H))=o$. Kodaira vanishing and the exact sequence:

$$o\longrightarrow O_X(K_X+(r-2)H)\longrightarrow O_X(K_X+(r-1)H)\longrightarrow O_H(K_H+(r-2)H)\longrightarrow o$$

give $h^o(X,O_X(K_X+(r-1)H))=h^o(H,O_H(K_H+(r-2)H))$. $q=1$ is impossible. Indeed, we can assume $r=2$ and since $p_g=o$, we obtain $\delta=1$ a contradiction to cor. 1.2.

If $q=2$, we have $\delta'=o$ and the theorem applies. q.e.d.

Now consider the following:

Problem. (Castelnuovo) Enumerate all varieties with $g \leq 3$. The problem of enumerating varieties of degree ≤ 6 is a special case of this. More generally, to know all varieties with $d \leq d_0$ it is enough to know all varieties with $g \leq f(d_0)$, where $f(d_0)$ is the Castelnuovo bound for the genus of a curve of degree d_0 in $\mathbb{P}^3$ (see [16] p. 351).

§2. Two theorems of Castelnuovo

Our first objective is to know all surfaces of degree ≤ 6 (see [26] p. 218). As we have already remarked it is sufficient to know all surfaces with $g \leq 3$. The cases $g=0,1$ are well known (see [23]) and can be quickly derived by the same method (cor. 1.1 and 1.2)

Theorem 2.1. (Castelnuovo [4], see also [27], lemma 2.2.2) A surface $(X, \mathcal{O}_X(H))$ with $g=2$ is either a scroll over a curve of genus 2 or one of the following rational surfaces: $\mathbb{F}_i$, $H=2C_0+(3+i)F$, $i=0,1,2$ or a blowing-up of $\mathbb{P}^2$ with center $P_0, P_1, \ldots, P_k$, $1 \leq k \leq 7$ and $H=4L-2P_0-P_1-\ldots-P_k$, where L is a line in $\mathbb{P}^2$. They have $5 \leq (H^2) \leq 12$.

Proof. By cor. 1.4 it is enough to study the morphism $\varphi = \varphi_{|K_X+H|} : X \longrightarrow \mathbb{P}^1$. As it is well-known, [3] p. 36, such a map cannot have multiple fibres. So the nonsmooth fibers of φ consist of 2 lines intersecting in one point, and any such line is an exceptional curve. Denote by $\rho : X \longrightarrow X'$ the contraction of one of the lines from each reducible fiber of φ. Then there is a morphism $\varphi' : X' \longrightarrow \mathbb{P}^1$ making X' a geometrically ruled rational surface and such that $\varphi' \circ \rho = \varphi$. Put $H' = \rho_*(H) = aC_0 + bF$ for some $a, b \in \mathbb{Z}$. Since $(H \cdot F)=2$ it follows $a=2$. But if $D=E+E'$ is a reducible fiber of φ, $(H \cdot E)=(H \cdot E')=1$, so H' is smooth and so $g(H)=g(H')=2$. Adjunction on X' yields:

$2=(2C_0+bF)^2+(2C_0+bF\cdot-2C_0+(-2-e)F)=2b-2e-4$ so $b=e+3$. But since H' is ample on

X', $b > 2e$ (see [16] cor 2.18 p. 380) so $o \leq e < 3$ and we obtain $e=o$, $b=3$; $e=1$, $b=4$; $e=2$, $b=5$. On the other side $(H^2)+(H\cdot K_X)=2$ and since $(H+K_X)^2=o$, $(K_X^2)+(H\cdot K_X)=-2$ which implies: $8 \geq (K_X^2) = (H^2)-4 \geq 1$, since a curve of genus 2 has degree ≥ 5. So we see that ρ' is the blowing-up of at most 7 points lying on different fibres of φ . The plane representation follows by considering elementary transformations (see [23]) from $\mathbb{F}_o$ and $\mathbb{F}_2$ to $\mathbb{F}_1$ (note that $\mathbb{F}_1$ is the blowing-up of $\mathbb{P}^2$ with center P_o). The theorem is proved.

<u>Theorem 2.2.</u> (essentially due to Castelnuovo [5]) <u>A surface $(X, O_X(H))$ with $g=3$ is one of the following:</u>

a) <u>A surface of degree 4 in $\mathbb{P}^3$</u>

b) <u>A scroll over a curve of genus 3</u>

c) <u>A geometrically ruled elliptic surface with $e=-1$, $H \equiv 2C_o+F$ so $(H^2)=$</u>

d) $\mathbb{F}_i$, $H_i=2C_o+(4+i)F$, $i=o,1,2,3$ <u>or a surface obtained by blowing-up one of these geometrically ruled surfaces in k points $P_1,\ldots,P_k$, lying on different fibers, $1 \leq k \leq 9$. If $H_i = 2C_o+(4+i)F$ on $\mathbb{F}_i$, $o \leq i \leq 3$, we have</u> $H=H_i-P_1-\ldots-P_k$. <u>They all have $7 \leq (H^2) \leq 16$</u>

e) <u>$\mathbb{P}^2$, $H=4L$ or a blowing-up of it with center $P_1,\ldots,P_k$, $1 \leq k \leq 1$</u>[illegible] <u>and</u> $H=4L-P_1-\ldots-P_k$. <u>They have $6 \leq (H^2) \leq 16$.</u>

f) <u>The Del Pezzo double plane which is the blowing-up of $\mathbb{P}^2$ with center $P_1, P_2,\ldots,P_7$,</u> $H=6L-2P_1-\ldots-2P_7$, <u>so $(H^2)=8$, or the blowing-up of the Del Pezzo double plane in one point P_o</u> , with $\tilde{H}=H-P_o$, <u>so $(\tilde{H})^2=7$.</u>

<u>Proof.</u> A curve of genus 3 has degree ≥ 4 with equality if and only if it is a plane curve. This last case leads to a). Suppose $(H^2) \geq 5$. We have $o \leq q \leq 3$; if $q=3$, $\delta=o$ so this is case b) by th. 1.1. We show $q=2$ is impossible. Indeed, $(H^2) \geq 5$ and adjunction formula gives $(H^2)+(H\cdot K_X)=4$ so $p_g=o$ and $\delta=\delta_o'+p_g=1$ a contradiction to cor. 1.2. Suppose $q=1$. Since $p_g=o$, $h^o(X, O_X(H+K_X))=2$ so we have $\varphi=\varphi_{|H+K_X|}: X \longrightarrow \mathbb{P}^1$; in particular $(H+K_X)^2=o$. It follows $(K_X\cdot H+K_X)=-4$ and

by adjunction $p_a(H+K_X)=-1$. So a generic $D\in|H+K_X|$ consists of 2 nonintersecting smooth conics. Let $\varphi':X\to C$ be the Stein factorisation of φ, so that C is a (smooth) curve which is a degree 2 covering of $\mathbb{P}^1$. We have $g(C)=q=1$. As in the proof of th. 2,1, let $\rho:X\to X'$ be the contraction of one of the 2 lines of each reducible fiber of φ'. We again have a morphism $\varphi'':X'\to C$ making X' a geometrically ruled elliptic surface, such that $\varphi''\circ\rho=\varphi'$. Let $H'=\rho_*(H)\equiv aC_0+bF$ for some $a,b\in\mathbb{Z}$. Since $(H\cdot F)=2$, $a=2$ and as above H' is smooth, so $g(H')=3$. By the genus formula we obtain $b-e=2$. But H' is ample so by [16] props. 2.2o, 2.21, p.382 $e=-1$, $b>-1$ or $e\geqslant o$, $b>2e$, so we have the following possibilities: $e=-1$, $b=1$; $e=o$, $b=2$; $e=1$, $b=3$. Let $\rho^*C_0=\tilde{C}_0+E_1+\dots+E_r$, $r\geqslant o$, where $\tilde{C}_0$ is the proper transform of C_0 and E_i the exceptional divisors. We have: (*) $(\tilde{C}_0\cdot H)+r=$ $=(\tilde{C}_0+E_1+\dots+E_r\cdot H)=(\rho^*C_0\cdot H)=(C_0\cdot H')=(C_0\cdot 2C_0+bF)=-2e+b$. It follows $(\tilde{C}_0\cdot H)\leqslant -2e+b$. But $\tilde{C}_0$ is an elliptic curve, so $(\tilde{C}_0\cdot H)\geqslant 3$ and we must have $e=-1$, $b=1$. In particular ρ must be an isomorphism (otherwise we would obtain at least 2 values of e, corresponding to the contraction of each of the 2 lines). So we are in case c).

Let now $q=o$, so $h^0(X,O_X(H+K_X))=3$ and we have $\varphi:X\to\mathbb{P}^2$.

Case I: $\varphi(X)$ is a (possibly singular) curve C. Then we have: $(H+K_X)^2=o$, $(H+K_X\cdot K_X)=-4$, so $p_a(H+K_X)=-1$. As above let $\varphi':X\to C'$ be the Stein factorisation of $\varphi:X\to C$ and note that $g(C')=q=o$ so $C'\cong\mathbb{P}^1$ and the fibers of φ' are conics. Let $\rho:X\to X'$ be as before and exactly as in the proof of th. 2.1 we find that X' is one of $\mathbb{F}_i$, $i=o,1,2,3$ and $H_i=2C_0+(4+i)F$. From $(H^2)+(H\cdot K_X)=4$ and $(K_X^2)+(H\cdot K_X)=-4$ we obtain $8\geqslant(K_X^2)=(H^2)-8\geqslant-2$ (a nonplanar curve of genus 3 has degree $\geqslant 6$). But the case $(H^2)=6$, $(K_X^2)=-2$ would give X embedded in $\mathbb{P}^4$, which is impossible by the formula in [16] p. 434. So we have $7\leqslant(H^2)\leqslant 16$ and X is the blowing-up of one of $\mathbb{F}_i$, $i=o,1,2,3$ with center k points belonging to distinct fibers, $o\leqslant k\leqslant 9$. This is case d).

Case II: $\varphi:X\to\mathbb{P}^2$ is surjective. We have $2p_a(H+K_X)-2=(H+K_X)^2+(H+K_X\cdot K_X)=$

$=4+2(H+K_X\cdot K_X)$ so that $p_a(H+K_X)=3+(H+K_X\cdot K_X)$. By Bertini's theorem, a generic $D\in|H+K_X|$ is smooth and connected, so that $p_a(D)\geq 0$; since $(H\cdot H+K_X)=4$ we can have $g(D)=0,1,3$.

α) $g(D)=0$ so $(H+K_X\cdot K_X)=-3$, $(H+K_X)^2=1$ and φ is birational. We obtain $9\geq(K_X^2)=$ $=(H^2)-7\geq -1$ and $6\leq(H^2)\leq 16$. So the number of blown-up points is ≤ 10. If E is an (effective) divisor on X contracted by φ, we have: $0=(E\cdot H+K_X)=(E\cdot H)+(E\cdot K_X)$, so that $(E\cdot K_X)<0$. But φ is a composition of blowings-up (with center a point) so we must have $(E\cdot K_X)\geq -1$. This implies $(E\cdot K_X)=-1$ and $(E\cdot H)=1$ so E is an exceptional curve. This shows that the blown-up points are ordinary. Finally, if we put $H'=\varphi_*(H)$, we must have $H'=4L$ since $g(H')=3$. We obtained case e).

β) $g(D)=1$; since $(H\cdot D)=4$, R-R on D gives that D is contained in $\mathbb{P}^3$. It follows that there is $C>0$ such that $H=C+H+K_X$ so that $h^0(O_X(-K_X))>0$. We have $(H+K_X\cdot K_X)=-2$, $(H+K_X)^2=2$ so φ has degree 2. Since $p_a(-K_X)=1$, $-(H\cdot K_X)\geq 3$ and $(K_X^2)=-2-(H\cdot K_X)\geq 1$ (1)

We have by Serre duality: $H^2(O_X(2K+H))\cong H^0(O_X(-H-K))=0$ and (2) $H^1(O_X(2K+H))\cong$ $=H^1(O_X(-H-K))=0$. To prove (2), look at the exact sequence:

$$0\longrightarrow O_X(-D)\longrightarrow O_X\longrightarrow O_D\longrightarrow 0$$

Since D is smooth, connected and $H^1(O_X)=0$ we are done. We obtain by R-R: $h^0(O_X(2K+H))=\chi(O_X(2K+H))=1+1/2(2K+H\cdot K+H)=1$. Now we have again 2 possibilities:

1) $H=-2K_X$, so X is the Del Pezzo double plane or
2) $h^0(O_X(-H-2K))=0$.

In the second case, first remark that $h^2(O_X(-K))=h^0(O_X(2K))=0$, so we obtain by R-R:

(3) $h^0(O_X(-K))\geq\chi O_X(-K)=(K_X^2)+1$. The exact sequence:

$$0\longrightarrow O_X(-H-2K)\longrightarrow O_X(-K)\longrightarrow O_D(-K)\longrightarrow 0 \quad \text{gives:}$$

$0\longrightarrow H^0(O_X(-H-2K))\longrightarrow H^0(O_X(-K))\longrightarrow H^0(D,O_D(-K))$. Since $H^0(O_X(-H-2K))=0$, $h^0(O_X(-K))\leq h^0(O_D(-K))=2$ by R-R on D. So we obtained by (3): $(K_X^2)+1\leq h^0(O_X(-K))\leq 2$. Together with (1) this gives $(K_X^2)=1$, $(H^2)=7$. Remember that $h^0(O_X(H+2K))=1$, so $|H+2K|=E$ with $(H\cdot E)=1$, $(E^2)=-1$. If $\sigma: X\longrightarrow X'$ denotes the contraction of E, clearly X' is the Del Pezzo double plane and $H=\sigma^*(-2K_{X'})-E$. So this leads to case f)

γ) $g(D)=3$ so D is a plane curve. But then there is a $D'\in|-K_X|$ such that $D+D'\in|H|$. Since in this case $(H+K_X\cdot K_X)=0$, $(D\cdot D')=0$ contradicting the fact that any divisor in $|H|$ is connected. The theorem is completely proved.

§ 3. Varieties of degree 5

Let $X\subset\mathbb{P}^n$ be a nondegenerated, linearly normal variety of degree 5. Denote by $s=n-r$ the codimension of X. We shall discuss the possible values of s. We have by o.4 $s\leqslant 4$.

I. If $s=4$, by o.5 $g=0$ and by cor. 1.1 X is the Veronese embedding $v_5(\mathbb{P}^1)$ or a rational scroll $X\cong\mathbb{P}(O_{\mathbb{P}^1}(a_1)\oplus\ldots\oplus O_{\mathbb{P}^1}(a_r))$ with $\sum_{i=1}^{r}a_i=5$, $a_i\geqslant 1$, $i=1,\ldots r$ and $O_X(H)$ is the tautological sheaf, o.6. In particular $2\leqslant r\leqslant 5$ and for $r=5$ we have the Segre embedding of $\mathbb{P}^1\times\mathbb{P}^4$ in $\mathbb{P}^9$; any other is a linear section of this, o.6. By lemma o.2 any such variety is arithmetically Cohen-Macaulay and its homogenous ideal is generated by 10 hyperquadrics.

II. $s=3$. By o.5 $g=0,1$. Since we have supposed X linearly normal it is enough to consider $g=1$. First of all we have elliptic curves. By [21] they are arithmetically Cohen-Macaulay, and their ideal is generated by 5 hyperquadrics. If $C\subset\mathbb{P}^4$ is such a curve, the standard exact sequences:

$$(1)\quad 0\longrightarrow T_C\longrightarrow T_{\mathbb{P}^4}|_C\longrightarrow N_{C|\mathbb{P}^4}\longrightarrow 0 \qquad\text{and}$$

$$(2)\quad 0\longrightarrow O_C\longrightarrow O_C(1)^{\oplus 5}\longrightarrow T_{\mathbb{P}^4}|_C\longrightarrow 0$$

give $h^0(N_{C|\mathbb{P}^4})=25$ and $h^1(N_{C|\mathbb{P}^4})=0$, so by deformation theory [14], the Hilbert scheme parametrising such curves is smooth, of dimension 25.

If $r\geqslant 2$, by cor. 1.3 X is either a Del Pezzo variety or an elliptic scroll. But the Del Pezzo varieties were recently completely classified by Fujita [9], [10] ([17] for 3-folds). In our case we obtain that any such X is a linear section of the Grassmann variety of lines in $\mathbb{P}^4$ with the Plücker embedding in $\mathbb{P}^9$. Any two such varieties of the same dimension are projectively equivalent. For r=2 we have the well-known Del Pezzo surface, obtained by blowing-up $\mathbb{P}^2$ in 4 points, $H=3L-P_1-P_2-P_3-P_4$. If $r\geqslant 3$ one can find various abstract descriptions of such varieties in [10]. Again by lemma 0.2 any such variety is arithmetically Cohen-Macaulay and its homogenous ideal is generated by 5 hyperquadrics.

As we shall see below, the elliptic scrolls do not occur in this case.

III. <u>s=2</u> By 0.5 we can have g=0, 1, 2. For <u>g=1</u> only elliptic scrolls have to be considered. For r=2, if $H\equiv C_0+bF$, we have by 0.3: $5=(C_0+bF)^2=-e+2b$, $b\geqslant e+3$, so $e\leqslant -1$. But for any geometrically ruled elliptic surface $e\geqslant -1$ ([16] th.2.15 p.377) so $e=-1$, $b=2$, $h^0(O_X(H))=5$ (by R-R) and they really exist in $\mathbb{P}^4$ ([16] th.2.15, p. 377). Let $X\cong\mathbb{P}(E)\xrightarrow{\pi}C$ with C elliptic. The standard exact sequences (1), (2) and:

$$(3)\qquad 0\longrightarrow T_{X/C}\longrightarrow T_X\longrightarrow \pi^*T_C\longrightarrow 0$$

$$(4)\qquad 0\longrightarrow O_X\longrightarrow (\pi^*\check{E})(1)\longrightarrow T_{X/C}\longrightarrow 0$$

give $h^0(N_{X/\mathbb{P}^4})=25$ and $h^1(N_{X/\mathbb{P}^4})=0$, so the corresponding Hilbert scheme is smooth, of dimension 25. It should be remarked that these scrolls are not arithmetically Cohen-Macaulay, since their hyperplane section is an elliptic curve of degree 5 in $\mathbb{P}^3$, hence not linearly normal. This suggests the following useful

<u>Lemma 3.1.</u> <u>Let C be an elliptic curve and $Y\cong\mathbb{P}(E)$ with E a locally free sheaf on C, of rank $r\geqslant 2$. Suppose Y embedded as a linearly normal scroll in $\mathbb{P}^m$. Then there is no smooth (nondegenerated) X embedded in $\mathbb{P}^{m+1}$ such that Y is a hyperplane section of X.</u>

<u>Proof.</u> In the context of the proof of th. 3 in [2], suppose the lemma is not true and consider the exact sequence:

$$(1)\qquad 0\longrightarrow O_X\longrightarrow O_X(Y)\longrightarrow O_Y(Y)\longrightarrow 0$$

It follows that X itself has the structure of a $\mathbb{P}^r$-bundle over C, say $q:X\longrightarrow C$.

Apply q_* to the sequence (1) to obtain:

$$0\longrightarrow O_C\longrightarrow F\longrightarrow E'\longrightarrow 0\ ,$$

with F a locally free sheaf on C which is ample and $E'=E\otimes L'$ for some $L'\in \mathrm{Pic}(C)$.

Passing to global sections we have:

$$0 \longrightarrow \mathbb{C} \longrightarrow H^0(F) \longrightarrow H^0(E') \longrightarrow \mathbb{C} \longrightarrow H^1(F)$$

But by duality, $H^1(F)\cong H^0(\check{F})=0$ since F is ample. Now, we obtained:

$n+1=h^0(Y,O_Y(Y))=h^0(E')=h^0(F)=h(X,O_X(Y))$ which is absurd. q.e.d.

So, for g=1 only surfaces can occur.

Consider $\underline{g=2}$. For such curves we have the following classical result (for instance [26] p. 93).

Lemma 3.2. Any curve C of genus 2 and degree 5 in $\mathbb{P}^3$ is "linked" to a line by 2 surfaces of degrees 2 and 3 respectively.

Proof. By R-R it follows that C is contained in exactly one quadric Q (necessarily irreducible). Again by R-R, the family of cubic surfaces containing C has dimension ≥ 5 so C lies on an irreducible cubic S. So there is a line L such that $Q\cap S=C\cup L$. q.e.d.

In particular C is arithmetically Cohen-Macaulay, [25] prop. 1.2. (this follows also from general results in [21] or [8]). By [25] p.281, we can obtain a resolution for O_C from a resolution of O_L, namely:

$$(*)\quad 0\to O_{\mathbb{P}^3}(-4)\oplus O_{\mathbb{P}^3}(-4)\longrightarrow O_{\mathbb{P}^3}(-2)\oplus O_{\mathbb{P}^3}(-3)\oplus O_{\mathbb{P}^3}(-3)\longrightarrow O_{\mathbb{P}^3}\longrightarrow O_C\longrightarrow 0$$

By [6] the Hilbert scheme of such curves is irreducible, smooth, of dimension 20.

Consider now r=2. By th. 2.1. X can be a blowing-up of $\mathbb{P}^2$ with center P_0, $P_1,\dots,P_7$, $H=4L-2P_0-P_1-\dots-P_7$ (which we shall call Castelnuovo surfaces) or a scroll over a curve of genus 2. The scrolls do not occur in virtue of the following:

Lemma 3.3. A scroll over a curve C of genus 2 has degree ≥ 8.

Proof. Since a curve of genus 2 must have degree ≥ 5, we obtain $(C_0\cdot H)=(C_0\cdot C_0+bF)=-e+b\geq 5$, so $b\geq 5+e$ and $(H^2)=(C_0+bF)^2=-e+2b\geq e+10$. By a theorem of Nagata [24] th.1, we have $e\geq -2$ so we obtain $(H^2)\geq 8$. q.e.d.

The Castelnuovo surfaces are easily seen (as in lemma 3.2) to be linked to a plane by a hyperquadric and a hyper cubic. They have the resolution $(*)$ ($\mathbb{P}^4$ instead of $\mathbb{P}^3$) and are arithmetically Cohen-Macaulay. Such surfaces do exist by [25] th.6.2 (or simply taking $P_0,\dots,P_7 \in \mathbb{P}^2$ to be "in general position" and showing that $4L-2P_0-P_1-\dots-P_7$ is very ample - see [3] ex. 17 p. 73). By [6] their Hilbert scheme is irreducible, smooth, of dimension 32. Let $r=3$. As above we have the resolution $(*)$ ($\mathbb{P}^5$ instead of $\mathbb{P}^3$). These are arithmetically Cohen-Macaulay and by [25] th.6.2. they do exist. The Hilbert scheme is irreducible, smooth, of dimension 46 by [6]. It would be nice to have a description of such varieties in terms of some known 3-folds. For the moment we have:

Lemma 3.4. *A (linearly normal, nondegenerated) 3-fold X of degree 5 in $\mathbb{P}^5$ has Betti numbers* $b_1=0$, $b_2=2$, $b_3=6$. *The classes of H and $2H+K_X=Q$ form a base of Pic(X) with* $(H^3)=5$, $(H^2\cdot Q)=2$, $(H\cdot Q^2)=(Q^3)=0$.

Proof. The adjunction mapping $\varphi=\varphi_{|2H+K_X|}$ gives a morphism to $\mathbb{P}^1$ whose general fiber Q is a smooth quadric (cor.1.4). We show that any fiber of φ is integral, so any nonsmooth fiber of φ is an ordinary cone. Suppose F is a nonintegral fiber of φ. Since $(H\cdot H\cdot F)=2$, we can only have $F=H'+H''$ or $F=2H'$, where H', H'' are planes. But remember (proof of th. 2.1) that for H the adjunction mapping gives a morphism to $\mathbb{P}^1$ such that any fiber is either a smooth conic or 2 lines intersecting in a point. It follows that $F=2H'$ is ruled out and if $F=H'+H''$, H' must be an exceptional plane, so it can be contracted to a point. This is absurd, since in this case the curve $H'\cap H''$ would be contracted to a point. Now return to the proof of 3.4. We shall show below that φ has exactly 8 singular fibers. Let F be one of them. Since it is a cone, the restriction of H to F generates Pic(F). So, for any divisor class D on X we have $D|_F=bH|_F$ for some $b\in\mathbb{Z}$. It follows that for any D there are $a,b\in\mathbb{Z}$ such that $D=aQ+bH$, and the intersection numbers are given by: $(H^3)=5$, $(H^2\cdot Q)=2$, $(H\cdot Q^2)=(Q^3)=0$.

Next, we want to compute the topological Euler-Poincaré characteristic.

Using the map φ, the formula in [3] lemma VI.4 p. 95 gives: (o) $\chi_{top}(X)=8-n$, where n is the number of singular fibers of φ. We shall compute $\chi_{top}(X)=c_3$ by R-R, using the method in [16] ex. 4.1.3. p. 433. If $N=N_{X/\mathbb{P}^5}$ denotes the normal bundle of $X\subset\mathbb{P}^5$, from the standard exact sequences:

$$o\longrightarrow T_X\longrightarrow T_{\mathbb{P}^5}|X\longrightarrow N\longrightarrow o \quad \text{and}$$

$$o\longrightarrow O_{\mathbb{P}^5}\longrightarrow O_{\mathbb{P}^5}(1)^{\oplus 6}\longrightarrow T_{\mathbb{P}^5}\longrightarrow o \quad \text{we obtain:}$$

(1) $c_1(N)=6H+K$,

(2) $c_2(N)=15(H^2)+K(6H+K)-c_2$

(3) $(-K\cdot c_2(N))+(c_2\cdot c_1(N))+c_3=20(H^3)$

But we have by the self-intersection formula (Lascu-Mumford-Scott, Math. Proc. Camb. Phil. Soc. 78(1975), 117-123.):

(4) $(c_2(N)\cdot H)=(X\cdot X\cdot H)=25$. To compute intersection numbers write $K=Q-2H$. We obtain from (2),(4):

(5) $(c_2\cdot H)=14$. Substituting (1),(2) in (3) and taking into account (5) we have: (6) $c_3=-2(c_2\cdot K)-48$. By R-R we have:

(7) $\chi O_X=-1/24(c_2\cdot K)$. But $H^i(O_X)=o$, $i=1,2,3$. Indeed, $H^1(O_X)=H^2(O_X)=o$, since X is arithmetically Cohen-Macaulay and $H^3(O_X)\cong H^0(O_X(K))=o$ since $(K\cdot H\cdot H)=-8$. So (7) gives $(c_2\cdot K)=-24$ and from (6) $c_3=o$. Thus in (o) we have n=8 as claimed. Since we have seen that $b_1=o$, $b_2=2$ and $\chi_{top}=o$, by Poincaré duality $b_3=6$. q.e.d.

$r\geqslant 4$ is not possible. Indeed, by lemma o.2 such varieties must be arithmetically Cohen-Macaulay in $\mathbb{P}^n$, with $n\geqslant 6$, so by a result originally due to Hartshorne (see [25] th.5.1) these must be complete intersections, which is not our case.

IV. <u>s=1.</u> These are just hypersurfaces of degree 5.

§4. <u>Varieties of degree 6</u>

Let $X\subset\mathbb{P}^n$ be a <u>linearly normal</u>, <u>nondegenerated</u> variety of degree 6. As before, we discuss the possible values of the codimension s=n-r. By o.4 we have $s\leqslant 5$.

I. If <u>s=5,</u> g=o so as above X is either the Veronese embedding $v_6(\mathbb{P}^1)$ or a rational scroll $X\cong\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus\ldots\oplus\mathcal{O}_{\mathbb{P}^1}(a_r))$, $\sum_{i=1}^{r}a_i=6$, $a_i\geq 1, i=1,\ldots,r$ and $\mathcal{O}_X(H)$ is the tautological sheaf. We have $2\leq r\leq 6$ and for $r=6$ X is the Segre embedding of $\mathbb{P}^1\times\mathbb{P}^5$ in $\mathbb{P}^{11}$. Any other is a linear section of this one; they are arithmetically Cohen-Macaulay and their homogenous ideal is generated by 15 hyperquadrics(o.6,o.2).

II. <u>s=4.</u> As above, for <u>g=1</u> we first have elliptic curves. By [21] they are arithmetically Cohen-Macaulay and their homogenous ideal is generated by 9 hyperquadrics. The corresponding Hilbert scheme is smooth, of dimension 36.

If $r\geq 2$, by cor. 1.3, X is either a Del Pezzo variety or an elliptic scroll. By the work of Fujita [9] (Iskovskih [17] for 3-folds) the Del Pezzo varieties of degree 6 are: - the Del Pezzo surface, which is the blowing-up of $\mathbb{P}^2$ with center 3 points, $H=3L-P_1-P_2-P_3$; - the Segre embedding of $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ in $\mathbb{P}^7$; - $\mathbb{P}(T_{\mathbb{P}^2})$ projectivised tangent sheaf to $\mathbb{P}^2$, $\mathcal{O}_X(H)$ being the tautological sheaf; - $\mathbb{P}^2\times\mathbb{P}^2$ embedded Segre in $\mathbb{P}^8$. Again by o.2 any such variety is arithmetically Cohen-Macaulay and the corresponding homogenous ideal is generated by 9 hyperquadrics. As we shall see in a moment, elliptic scrolls do not occur here.

III. <u>s=3.</u> In virtue of o.5 we can have g=o,1,2. For <u>g=1</u> only elliptic scrolls are in question. For r=2, we have by o.3:

$6=(C_o+bF)^2=-e+2b$ and $b\geq e+3$ so $e\leq o$. [16] th.2.15 p.377, gives $e\geq -1$; but $e=-1$ implies $2b=5$ which is absurd, so e=o, b=3. R-R gives $h^0(\mathcal{O}_X(H))=6$ and they do exist by [16] loc.cit. As in the case of degree 5, we obtain $h^0(N_{X/\mathbb{P}^5})=36$, $h^1(N_{X/\mathbb{P}^5})=o$ so the Hilbert scheme is smooth, of dimension 36. Again they are not arithmetically Cohen-Macaulay. By lemma 3.1, for $r\geq 3$ such scrolls do not exist.

Consider <u>g=2.</u> We first have curves of genus 2. By [8] cor.1.11,1.14, they are arithmetically Cohen-Macaulay and their homogenous ideal is generated by 4 hyperquadrics. As above, the corresponding Hilbert scheme is smooth, of dimension 29.

Let now r=2 and apply th. 2.1. It follows that X is either a blowing-up of $\mathbb{P}^2$ with center $P_o,\ldots,P_6$ $H=4L-2P_o-P_1-\ldots-P_6$ (which we shall also call <u>Castelnuovo</u>

surfaces) or a scroll over a curve of genus 2. By lemma 3.3 scrolls do not occur. The Castelnuovo surfaces are arithmetically Cohen-Macaulay and their homogenous ideal is generated by 4 hyperquadrics.

Let r= 3. We have:

Lemma 4.1. A 3-fold of degree 6 with g=2 is a double covering $f:X\to\mathbb{P}^1\times\mathbb{P}^2$ such that if we put $E=p_1^*(O_{\mathbb{P}^1}(1))$, $F=p_2^*(O_{\mathbb{P}^2}(1))$ we have $O_X(H)=f^*(E\otimes F)$ and the ramification divisor $R\in|H|$. Such varieties do exist in $\mathbb{P}^6$; they have the anticanonical class very ample and Betti numbers $b_1=0$, $b_2=2$, $b_3=4$; $f^*(E)$ and $f^*(F)$ give a base of Pic(X).

Proof. Remember from cor. 1.4 that the adjunction mapping $\varphi=\varphi_{|2H+K_X|}$ gives a morphism to $\mathbb{P}^1$ with general fiber a smooth quadric, say Q. We show that the linear system $|H-Q|$ is basepoints free and maps X onto $\mathbb{P}^2$. From the exact sequence:

$$o\longrightarrow O_X(-Q)\longrightarrow O_X\longrightarrow O_Q\longrightarrow o$$

we infer $H^1(O_X(-Q))=o$ (since $H^1(O_X)=o$). Then consider the exact sequence:

$$o\longrightarrow O_X(-Q)\longrightarrow O_X(H-Q)\longrightarrow O_H(H-Q)\longrightarrow o$$

It follows $h^0(O_X(H-Q))=h^0(O_H(H-Q))$ and it is enough to prove that $O_H(H-Q)$ is spanned by global sections. Doing the same once again we can assume H is a curve (of genus 2). Since the degree of H-Q is 4, $O_H(H-Q)$ is generated by global sections and nonspecial, so by R-R $h^0(O_H(H-Q))=3$. So we have a morphism $\psi=\varphi_{|H-Q|}:X\longrightarrow\mathbb{P}^2$ which is surjective since $(H-Q)^3=1$. Now combine φ and ψ to obtain a morphism $f:X\longrightarrow\mathbb{P}^1\times\mathbb{P}^2$, corresponding to some subsystem of $|H|$. In particular f is finite, of degree $(H-Q\cdot H-Q\cdot Q)=2$. We have $-K=2H-Q=H+H-Q$ so $|-K|$ is very ample since $|H|$ is and $|H-Q|$ is basepoints free. The ramification divisor R is given by the formula: (1) $K_X=f^*K_{\mathbb{P}^1\times\mathbb{P}^2}+R$ and we immediately obtain $O_X(R)=f^*(E\otimes F)$. Since $H^1(O_X)=o$, $b_1=o$. Now apply exactly the same method as in lemma 3.4 to see that any nonsmooth fiber of φ is an ordinary cone and there are exactly 6 such. The same computation via R-R gives $c_3=\chi_{top}=2$

(we leave the details to the reader). As in the proof of 3.4, (the classes of) H and Q form a base for $\mathrm{Pic}(X)$, with intersection numbers $(H^3)=6$, $(H^2\cdot Q)=2$, $(H\cdot Q^2)=(Q^3)=0$. By Poincaré duality we obtain $b_3=4$. Now start with a double covering $f:X \longrightarrow \mathbb{P}^1\times\mathbb{P}^2$, ramified $(2,2)$ (with smooth discriminant divisor) and we shall prove that $\mathcal{O}_X(H)=f^*(E\otimes F)$ is very ample and maps X to $\mathbb{P}^6$ with degree 6. Put $E'=f^*(E)$, $F'=f^*(F)$ and think of them as divisor classes. First of all remark that the ramification divisor R belongs to $|E'+F'|$ since f_*R belongs to $|2E+2F|$. Formula (1) gives $K_X=-E'-2F'$, so $-K_X$ is ample (X is a Fano variety, see [17]). In particular $H^i(\mathcal{O}_X)=0$ for $i=1,2,3$ by duality and Kodaira vanishing, so we have: (2) $\chi\mathcal{O}_X=1$. We want to prove first that $h^0(\mathcal{O}_X(H))=7$. We claim that a smooth $Q\in|E'|$ is a quadric and $\mathcal{O}_X(F')|_Q=\mathcal{O}_Q(1)$. Indeed, adjunction formula gives $\mathcal{O}_Q(K_Q)=\mathcal{O}_X(K_X+Q)|_Q=\mathcal{O}_X(-2F')|_Q$ and $\mathcal{O}_X(F')|_Q=\mathcal{O}_X(E'+F')|_Q$ is ample. From [7] th. 2.2, it follows Q is a (smooth) quadric and $\mathcal{O}_X(F')|_Q=\mathcal{O}_Q(1)$ as claimed. Now consider the exact sequence:

$$(*)\qquad 0\longrightarrow \mathcal{O}_X(F')\longrightarrow \mathcal{O}_X(H)\longrightarrow \mathcal{O}_Q(F')\longrightarrow 0$$

By duality and Kodaira vanishing we have: $H^i(\mathcal{O}_X(H))=H^i(\mathcal{O}_X(F'))=0$ for $i=1,2,3$. Let $h^0(\mathcal{O}_X(H))=a$ and $h^0(\mathcal{O}_X(F'))=b$. The sequence $(*)$ gives (3) $a-b=4$ since $h^0(\mathcal{O}_Q(F'))=4$. By R-R we obtain: (4) $a=\chi(\mathcal{O}_X(H))=1/12(H\cdot H-K\cdot 2H-K)+1/12(H\cdot c_2)+\chi\mathcal{O}_X$ and (5) $b=\chi\mathcal{O}_X(F')=1/12(F'\cdot F'-K\cdot 2F'-K)+1/12(F'\cdot c_2)+\chi\mathcal{O}_X$. By (2) $1=\chi\mathcal{O}_X=1/24(-K\cdot c_2)$. But $-K=H+F'$, so: (6) $24=(-K\cdot c_2)=(H\cdot c_2)+(F'\cdot c_2)$. Intersection numbers are easily computed: $(H\cdot H-K\cdot 2H-K)=2(E+F\cdot 2E+3F\cdot 3E+4F)=58$ and $(F'\cdot F'-K\cdot 2F'-K)=2(F\cdot E+3F\cdot E+4F)=14$. Summing (4) and (5) and taking into account (6) we obtain $a+b=10$, so by (3) we have $a=7$, $b=3$. And clearly $(H^3)=2(E+F)^3=6$. Now we can prove that $\mathcal{O}_X(H)$ is very ample. Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\varphi_{|H|}} & Y & \hookrightarrow & \mathbb{P}^6 \\ & \searrow^{f} & \downarrow{\scriptstyle q} & & \\ & & \mathbb{P}^1\times\mathbb{P}^2 & \xrightarrow{p_1} & \mathbb{P}^1 \end{array}$$

where $\varphi_{|H|}$ is the morphism corresponding to the complete linear system $|H|$ and Y is the image of $\varphi_{|H|}$. Since $(E+F)^3=3$, the resulting morphism q is a double covering of $\mathbb{P}^1\times\mathbb{P}^2$ and $\varphi_{|H|}$ is finite and birational. We want to show that Y is normal, so $\varphi_{|H|}$ is an isomorphism. To prove this we shall see that any fiber of $g=p_1\circ q$ is integral, hence normal since it is a quadric, so Y itself is normal. It is enough to show that any fiber of $g\circ\varphi_{|H|}=p_1\circ f$ is integral. Let D be a fiber of $p_1\circ f$. Since $(D\cdot H\cdot H)=2(E\cdot E+F\cdot E+F)=2$ we can only have $D=D'+D''$ or $D=2D'$, with D', D'' integral. In any case, $1=(D'\cdot H\cdot H)=((E'+F')|_{D'}\cdot(E'+F')|_{D'})_{D'}=$ $=(F'|_{D'}\cdot F'|_{D'})_{D'}$. Since $\varphi_{|H|}(D')$ is 2-dimensional, $h^0(O_X(F')|_{D'})=h^0(O_X(H)|_{D'})\geqslant 3$ and $O_X(F')|_{D'}$ is ample, so by a result of Kobayashi-Ochiai [19] it follows $D'\simeq\mathbb{P}^2$ and $O_X(F')|_{D'}\simeq O_{\mathbb{P}^2}(1)$. But by adjunction we have: $O_{\mathbb{P}^2}(-3)=O_{D'}(K_{D'})=O_X(K_X+D')|_{D'}=$ $=O_X(-2F'+D')|_{D'}$, so $O_X(D')|_{D'}\cong O_{\mathbb{P}^2}(-1)$. But then D' is an exceptional plane and can be contracted. The same applies to D'', so in case $D=D'+D''$ the curve $D'\cap D''$ would be contracted to a point which is impossible. If $D=2D'$, we have a morphism f' making the diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{p_1\circ f} & \mathbb{P}^1 \\ {\scriptstyle cont_{D'}}\downarrow & \nearrow{\scriptstyle f'} & \\ X' & & \end{array}$$

which is clearly absurd. The lemma is completly proved.

Again, the 3-folds such obtained are arithmetically Cohen-Macaulay and their homogenous ideal is generated by 4 hyperquadrics.

Case $r\geqslant 4$ is impossible. Indeed, exactly as above, we would obtain a morphism from X to $\mathbb{P}^1\times\mathbb{P}^2$, corresponding to some subsystem of $|H|$, which is absurd.

IV. <u>s=2.</u> By o.5 we can have g=o,1,2,3,4. First for <u>g=1</u> only elliptic scrolls of dimension $r\geqslant 3$ could possibly occur. But in this case we would have a 2-dimensional elliptic scroll in $\mathbb{P}^4$ which must be the (isomorphic) projection of one of those discussed in the previous case. This is impossible, for instance by a theorem of Severi asserting that the Veronese surface $v_2(\mathbb{P}^2)$ is the only one which projects isomorphically from $\mathbb{P}^5$ to $\mathbb{P}^4$. However in our case the formula in [16] p.434 is sufficient.

Consider now <u>g=2</u>, r=2, X a Castelnuovo surface. By th.2.1 we would have $(H\cdot K_X)=-4$,

$(K_X^2)=2$ contradicting again the formula in [16] p.434 for surfaces in $\mathbb{P}^4$. So $g=2$ is impossible too.

Assume $g=3$. Consider first a curve of degree 6 and genus 3 in $\mathbb{P}^3$. It is not on a quadric cone since this would imply it is a complete intersection, which is not the case. There are 2 types of such curves (see [26] p.93): the first is a curve of type (2,4) on a smooth quadric. They are not arithmetically Cohen-Macaulay. The other type is given by the following:

Lemma 4.2 (see also [6], ex.2, p.430) Any curve of degree 6 and genus 3 in $\mathbb{P}^3$ which is not on a quadric is linked to the twisted cubic by 2 cubic surfaces In particular it is arithmetically Cohen-Macaulay.

Proof. By R-R the family of cubic surfaces containing C has dimension ≥ 3 and by hypothesis they must be irreducible. It follows that C is linked to a -possibly reducible or singular-curve C' of degree 3 by 2 cubic surfaces, say S and S'. We want to prove that for suitable choice of S and S', C' is integral. Denote by $m(P,C)$ or $m(P,S)$ the multiplicity of a point P on the curve C or the surface S. By Bertini's theorem, we have $m(P,C\cup C')\leq 3$. Indeed, the only unpleasant case is when there is a fix point V on C such that C' consists always in 3 lines through V. This implies any S is a cone with vertex V, so their intersection would be a union of lines, a contradiction. Since $m(P,C\cup C')\leq 3$, we have $m(P,S)\cdot m(P,S')\leq m(P,C\cup C')\leq 3$, so there are smooth cubic surfaces containing C (the same argument as in [20]). So we have $C+C'=3H$ on a smooth cubic surface S. If any $C'\in|3H-C|$ is reducible, say $C'=L+Q$, L a line and Q a conic, L must be a fixed component of $|3H-C|$. But $\dim|Q|=1$ and on the other side, the exact sequence:

$$0 \longrightarrow \mathcal{J}_C(3) \longrightarrow \mathcal{O}_S(3) \longrightarrow \mathcal{O}_C(3) \longrightarrow 0$$

gives $h^0(\mathcal{J}_C(3))\geq 3$ so $\dim|3H-C|\geq 2$. This is a contradiction. By the formulas in [25] prop. 3.1 , we must have $p_a(C')=0$, so C' is the twisted cubic. q.e.d.
Using again [25] p. 281, we can write down a resolution for $\mathcal{O}_C$ from the resolu-

tion of the twisted cubic, namely:

$$(**)\quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4)^{\oplus 3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 4} \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

The Hilbert scheme of curves of degree 6 and genus 3 in $\mathbb{P}^3$ is irreducible, smooth, of dimension 24.

Now let r=2. By th. 2.2 X is either a blowing-up of $\mathbb{P}^2$ with center $P_1,\dots,P_{10}$, $H=4L-P_1-\dots-P_{10}$ (these are known as Bordiga surfaces) or a scroll over a curve of genus 3. The scrolls do not occur by the following:

Lemma 4.3. A scroll X over a curve C of genus 3 has degree ≥ 9.

Proof. By 0.5 a curve of genus 3 is either a plane curve of degree 4 or it has degree ≥ 6. Suppose the section C_0 is a plane curve of degree 4. In particular it is not hyperelliptic. A pencil of hyperplanes containing the plane in which C_0 lies will give a pencil of degree 2 (reducible) curves, consisting of 2 fibers of the ruling (a rational curve cannot dominate the base which has genus 3). Consequently the pencil is without basepoints and gives a morphism $f: X \longrightarrow \mathbb{P}^1$. The Stein factorisation of f, say $f': X \longrightarrow C'$ is just the original ruling, so $C' \simeq C_0$. But C' is a double covering of $\mathbb{P}^1$ so it is hyperelliptic - a contradiction. Suppose now degree of $C_0 \geq 6$. This gives: $(C_0 \cdot H) = (C_0 \cdot C_0 + bF) = -e+b \geq 6$ so $b \geq e+6$ and $(H^2) = (C_0 + bF)^2 = -e+2b \geq e+12$. The theorem of Nagata [24] implies $e \geq -3$, so $(H^2) \geq 9$ and we are done. q.e.d.

Consider now the Bordiga surfaces. We have:

Lemma 4.4. Any Bordiga surface X is arithmetically Cohen-Macaulay, having the resolution (**) (with $\mathbb{P}^4$ instead of $\mathbb{P}^3$).

This is a consequence of 0.2, lemma 4.2 and the following:

Lemma 4.5. Let X be a nondegenerated linearly normal surface in $\mathbb{P}^4$ with g=3. Then its generic hyperplane section H is not on any quadric in $\mathbb{P}^3$.

Proof. Assume the contrary and consider the standard exact sequence:

$$0 \longrightarrow \mathcal{J}_X(1) \longrightarrow \mathcal{J}_X(2) \longrightarrow \mathcal{J}_H(2) \longrightarrow 0$$

Since $H^1(\mathbb{P}^4, \mathcal{J}_X(1)) = 0$ and we suppose $H^0(\mathbb{P}^3, \mathcal{J}_H(2)) \neq 0$ it follows $H^0(\mathbb{P}^4, \mathcal{J}_X(2)) \neq 0$, so

X is on a (nondegenerated) hyperquadric, say Q. If Q is smooth, by Klein's theorem X would be a complete intersection, which is absurd. If Q is cone with vertex a point, Bertini's theorem implies that some integral member of $|H|$ is on a cone in $\mathbb{P}^3$. If Q is cone with vertex a line, again H is on a cone in $\mathbb{P}^3$. In any case H would be a complete intersection which is not the case. The lemma is proved. In particular such surfaces do exist, for instance by [25] th. 6.2 (or taking lo points "in general position" in $\mathbb{P}^2$ and showing $4L-P_1-\dots-P_{10}$ is very ample, see [3] ex. 2o p. 73). It follows, [6], that the Hilbert scheme of Bordiga surfaces is irreducible, smooth, of dimension 36.

Suppose r=3. We have:

<u>Lemma 4.6. Let X be a 3-fold of degree 6 with g=3 (they do exist in $\mathbb{P}^5$). There is a rank-2 locally free sheaf E on $\mathbb{P}^2$ such that $X \cong \mathbb{P}(E)$ and $O_X(H)$ is the tautological sheaf. E is given by an extension</u> $(+)\quad o \longrightarrow O_{\mathbb{P}^2} \longrightarrow E \longrightarrow \mathcal{J}_Y(4) \longrightarrow o$ <u>where Y is a subscheme of $\mathbb{P}^2$ consisting of lo distinct points. E is stable and it has $c_1(E)=4$, $c_2(E)=10$. If L is a generic line in $\mathbb{P}^2$,</u> $E|L \simeq O_L(2) \oplus O_L(2)$.

<u>Proof.</u> We shall show that the adjunction mapping $\varphi=\varphi_{|2H+K_X|}$ makes X a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$. Remember that H is a Bordiga surface, so that the adjunction mapping makes it a blowing-up of $\mathbb{P}^2$ with center lo ordinary points. Note also the relations on H: $(H\cdot H+K_H)=4$, $(K_H^2)=-1$, $(K_H\cdot H+K_H)=-3$, $(H+K_H)^2=1$, $(H\cdot K_H)=-2$. We have the exact sequence:

$$o \longrightarrow O_X(H+K_X) \longrightarrow O_X(2H+K_X) \longrightarrow O_H(H+K_H) \longrightarrow o$$

and $H^1(O_X(H+K_X))=H^2(O_X(-H))=o$. Also $H^0(O_X(H+K_X))=o$ because $(H\cdot H\cdot H+K_X)=(H\cdot K_H)_H=-2$. So $h^0(O_X(2H+K_X))= h^0(O_H(H+K_H))=3$. Now any fiber of φ is a line. Indeed, we have: $(2H+K_X\cdot 2H+K_X\cdot H)=(H+K_H)^2_H=1$. If D would be a 2-dimensional fiber of φ, it must be an exceptional plane since its trace on H is an exceptional line. But then we would have a morphism φ' making the diagram commutative

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & \mathbb{P}^2 \\ {\scriptstyle cont_D}\downarrow & \nearrow_{\varphi'} & \\ X' & & \end{array}$$

which is absurd.

So φ is a $\mathbb{P}^1$-bundle.

Now apply φ_* to the exact sequence:

$$0 \longrightarrow O_X \longrightarrow O_X(H) \longrightarrow O_{\bar{H}}(H) \longrightarrow 0 \quad \text{to obtain:}$$

$$0 \longrightarrow O_{\mathbb{P}^2} \longrightarrow \varphi_*(O_X(H)) \longrightarrow \varphi_*(O_{\bar{H}}(H)) \longrightarrow 0 \quad \text{since } R^1\varphi_* O_X = 0$$

and $E=\varphi_*(O_X(H))$ is locally free of rank 2. Remember that on H we have $H=4L-P_1-P_2-\dots-P_{10}$. It follows that $\varphi_*(O_{\bar{H}}(H))$ is just $\mathcal{J}_Y(4)$, with $Y=\{P_1,\dots,P_{10}\}$. So E is given by an extension (+). Restricting to a line L in $\mathbb{P}^2$ not passing through any point P_i, we obtain $c_1(E)=4$. Since $X\cong\mathbb{P}(E)$ we see just by the definition of Chern classes: $(H^3)-c_1(E)(2H+K_X\cdot H\cdot H)+c_2(E)(2H+K_X\cdot 2H+K_X\cdot H)=0$, so that $6-4c_1(E)+c_2(E)=0$ and $c_2(E)=10$. We have $h^0(\mathbb{P}^2,\mathcal{J}_Y(4))=h^0(H,O_{\bar{H}}(H))=5$. This implies $h^0(\mathbb{P}^2,\mathcal{J}_Y(2))=0$ and from the extension (+) follows $h^0(\mathbb{P}^2,E(-2))=0$. In our case this means that E is stable. Then the generic splitting of E must be $O_L(2)\oplus O_L(2)$ by the Grauert-Müllich theorem [11]. The lemma is proved.

Any such 3-fold is arithmetically Cohen-Macaulay by 0.2 and has the same resolution (**)($\mathbb{P}^5$ instead of $\mathbb{P}^3$). In particular [25] th.6.2 ensures they do exist in $\mathbb{P}^5$. Their Hilbert scheme is irreducible, smooth, of dimension 48, [6].

$r\geqslant 4$ is impossible, for instance again by Hartshorne's theorem [25] th.5.1.

Let g=4. A curve of degree 6 and genus 4 in $\mathbb{P}^3$ is the complete intersection of a quadric and a cubic (use R-R). So by 0.2 we have for any $r\geqslant 1$ the complete intersectiоns of type (2,3).

V. s=1. These are just the hypersurfaces of degree 6.

Final remark. For reader's convenience we indicate briefly how to obtain in the same manner the list of nondegenerated, linearly normal, smooth varieties of degree $\leqslant 4$ (in fact one knows all of them, without the smoothness condition see [30] and [28]). For degree 1 we have the projective space itself and for degree

2 a hyperquadric. For degree 3 we have $s \leq 2$. For $s=2$, $g=0$ so by cor.1.1 (or better [7] ths. 2.1, 2.2, 3.8) X is $\mathbb{P}^1 \times \mathbb{P}^2$ embedded Segre, its hyperplane section $\mathbb{F}_1 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ or the twisted cubic. For degree 4, $s \leq 3$. If $s=3$, $g=0$ and we have $\mathbb{P}^1 \times \mathbb{P}^3$ embedded Segre, its hyperplane section $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2))$, two scrollar surfaces $\mathbb{F}_0 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ and $\mathbb{F}_2 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3))$, the Veronese surface $v_2(\mathbb{P}^2)$ and the Veronese curve $v_4(\mathbb{P}^1)$. If $s=2$, $g=1$ remark that any elliptic curve of degree 4 in $\mathbb{P}^3$ is the complete intersection of 2 quadrics. So by o.2 we have for any $r \geq 1$ the complete intersections of type (2,2).

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