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UNIQUENESS AND CONTINUOUS DEPENDENCE RESULTS IN THE THEORY OF THERMOELASTIC MATERIALS WITH INTERNAL STATE VARIABLES

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1. INTRODUCTION

A general theory of thermoelastic material bodies with internal state variables has been formulated by Coleman and Gurtin [1] and Bowen [2]. In this theory the essential assumption is that the local basic mechanical and thermal variables are to be dependent upon the standard thermodynamic variables: the deformation gradient, the temperature and the temperature gradient as well as an internal state variable (which is a finite-dimensional vector). The evolution of the internal state variable is assumed to be governed by an ordinary differential equation (generally nonlinear) expressing the present rate of change of the internal state variable through a function of the present state. The theory appears to be quite general and suitable for predicting a wide variety of physical phenomena. This theory includes, as a special case, theories of chemically reacting mixtures without diffusion; in this case each internal state variable represents the extent of reactions [2]. Other physical interpretations of the internal state variables are given in [1], [3] - [6].

The theory of wave propagation through elastic materials with internal state variables has been studied extensively (see e.g. [7] - [16]). Most notably, the work by Nachlinger and Nunziato [7] has shown that in one-dimensional theory, the elastic materials with internal state variables have the wave propagation property, that is, all smooth structured waves propagate with bounded velocity. The authors further showed that this boundedness property ensures the uniqueness of solutions satisfying the prescribed boundary and initial conditions for unbounded domains. Bowen and Wang [8] and Bowen and Chen [9] have examined the propagation condition and the growth and decay of acceleration waves in a nonconductor or a definite conductor of heat elastic material with internal state variables.

On the other hand, Gurtin [17, 18] has revealed that the Clausius-Duhem inequality induces Liapunov stability of equilibrium processes. Kosinski [19] has established the uniqueness of the solution to the initial displacement-boundary value problem appropriate to the dynamics of dissipative bodies described by the equations of the purely mechanical theory with internal state variables.

The purpose of this paper is to prove the uniqueness and the continuous dependence of thermodynamic processes upon initial state and supply terms, within the context of thermodynamics with internal state variables. The results are locally and they are established under certain assumptions on material response. In fact, we prove the uniqueness and the continuous dependence results for smooth admissible processes residing in a neighborhood of a positive smooth admissible process, that is, a smooth admissible process for which the elasticity tensor is positive definite and the instantaneous specific heat is strictly positive. When the initial displacement-boundary value problems are considered, the uniqueness and continuous dependence are established under the weaker assumption that the smooth admissible processes reside in a neighborhood of a smooth admissible process where the strong ellipticity condition is satisfied. Our analysis is developed for the elastic materials without heat conduction and for the definite conductor of heat elastic materials.

Within the context of classical nonlinear thermoelasticity these stability statements were recently established by Dafermos [20] for elastic materials without heat conduction and by Chiriţă [21] for definite conductor of heat elastic materials.

We summarize the basic structure and the constitutive relations for a thermoelastic body with internal state variables in Section 2.

The standpoint from which we proceed is an evolutionary identity established in Section 3. Our subsequent development on continuous dependence of thermodynamic processes upon initial state and supply terms require further estimates which are considered in Sections 4 and 5. Further, these estimates when are coupled with the assumptions on the material response and on the thermodynamic process lead to Gronwall type inequalities that demonstrates the continuous dependence and uniqueness results.

2. BASIC EQUATIONS

We consider a body which, at time $t = o$, occupies the properly regular region R of Euclidean three-dimensional space $\mathcal{R}^3$ and it is bounded by the piecewise smooth surface ∂R [22]. The configuration of the body at time $t = o$ is taken as the reference configuration. Let X be a typical particle of the body. We identify the material point X with its position $\underset{\sim}{X}$ in the fixed reference configuration.

The motion of the body is referred to the reference configuration and a fixed system of rectangular Cartesian axes. Let R be at rest relative to the considered system. The location of a typical particle X in the reference configuration is described by the coordinates X_K $(K = 1,2,3)$ relative to these axes. The coordinates of the particle in the position $\underset{\sim}{x}$ at time t are denoted by x_i $(i = 1,2,3)$, and

$$\mathbf{x} = \mathbf{x}(\mathbf{X},t) \quad , \quad (\mathbf{X},t) \in \overline{R} \times [o,t_o] \; . \tag{2.1}$$

If this deformation is to be possible in a real material, then

$$\det(x_{i,K}) > o. \tag{2.2}$$

We assume that there is no diffusion of mass in the body, but the body may deform and conduct heat. Thus, a <u>thermodynamic process</u> for R is described by nine functions, of $\mathbf{X}$ and the time t, whose values have the following physical interpretations:

(1) the spatial position $\mathbf{x} = \mathbf{x}(\mathbf{X},t)$ in the motion;

(2) the first Piola-Kirchhoff stress tensor $\mathbf{T} = \mathbf{T}(\mathbf{X},t)$;

(3) the specific body force $\mathbf{b} = \mathbf{b}(\mathbf{X},t)$ per unit mass (exerted on R at $\mathbf{X}$ by the external world, i.e., by other bodies which do not intersect R);

(4) the Helmholtz free energy $\psi = \psi(\mathbf{X},t)$ per unit mass;

(5) the heat-flux vector $\mathbf{Q} = \mathbf{Q}(\mathbf{X},t)$;

(6) the heat supply $r = r(\mathbf{X},t)$ per unit mass and unit time (absorbed by R at $\mathbf{X}$ and furnished by radiation from the external world);

(7) the specific entropy $\eta = \eta(\mathbf{X},t)$ per unit mass;

(8) the absolute temperature $\theta = \theta(\mathbf{X},t) > o$;

(9) the internal state vector $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{X},t) = (\xi_1,\xi_2,\ldots,\xi_n)$; the scalars $\xi_\alpha = \xi_\alpha(\mathbf{X},t), (\alpha = 1,2,\ldots,n)$ are the internal state variables.

The motion $\mathbf{x} = \mathbf{x}(\mathbf{X},t)$ determines the velocity field $\mathbf{v} = \dot{\mathbf{x}}$ and the deformation gradient field $\mathbf{F} = \mathrm{grad}_{\mathbf{X}}\mathbf{x}$, while the temperature $\theta(\mathbf{X},t)$ of the body determines the temperature gradient $\mathbf{g} = \mathrm{grad}_{\mathbf{X}}\theta$.

Throughout this paper we shall use the following notations:

$\text{grad}_{\underset{\sim}{X}}$ is the gradient operator with respect to the place $\underset{\sim}{X}$ keeping t fixed; a superposed dot denotes the material time derivative; Latin indices have range 1,2,3, while Greek subscripts have range 1,2,...,n; summation over repeated subscripts is implied; subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate; the symbol $|\cdot|$ denotes a norm either in Euclidean vector space or in a tensor space, while $\|\cdot\|$ denotes L_2-norm. In our subsequent development, occasionally it will be convenient to write various expressions in component form and to display vector and tensor fields by their components referred to a fixed system of rectangular Cartesian axes. Thus, for the velocity vector $\underset{\sim}{v}$ we will write v_i, while the components of the deformation gradient field $\underset{\sim}{F}$ and the temperature gradient field $\underset{\sim}{g}$ will be denoted by F_{iK} and g_K, respectively. Similarly, the components of the non-symmetric Piola-Kirchhoff stress tensor $\underset{\sim}{T}$ and the heat flux vector $\underset{\sim}{Q}$, measured relative to the reference configuration, will be denoted by T_{Ki} and Q_K, respectively.

The above set of nine functions, defined for all $\underset{\sim}{X}$ in R, and all t in $[0,t_0]$, is called here a thermodynamic process in R if and only if it is compatible with the law of balance of linear momentum and the law of balance of energy, viz.,

$$\rho \dot{v}_i = T_{Ai,A} + \rho b_i \quad , \tag{2.3}$$

$$\frac{d}{dt}\{\rho(\psi + \theta\eta + \tfrac{1}{2}v_i v_i)\} = (T_{Ai}v_i)_{,A} + \rho b_i v_i + \rho r - Q_{A,A} \quad , \tag{2.4}$$

where $\rho(X)$ is the reference mass density defined on $\bar{R}$. We assume that the density $\rho(X)$ is smooth and strictly positive,

$$\rho(X) \geq \rho_0 > 0 \quad , \quad \underset{\sim}{X} \in \overline{R}. \tag{2.5}$$

In order to specify a thermodynamic process it suffices to prescribe the seven functions $\underset{\sim}{x}$, $\underset{\sim}{T}$, ψ, $\underset{\sim}{Q}$, η, θ and $\underset{\sim}{\xi}$. The remaining functions $\underset{\sim}{b}$ and r are then determined by (2.3) and (2.4).

In theory of thermoelasticity described by internal state variables, the material at the point $\underset{\sim}{X}$ is characterized by five response functions, ψ^*, $\underset{\sim}{T}^*$, η^*, $\underset{\sim}{Q}^*$ and $\underset{\sim}{f}$, which give ψ, $\underset{\sim}{T}$, η, $\underset{\sim}{Q}$ and $\dot{\underset{\sim}{\xi}}$ at $\underset{\sim}{X}$, when $\underset{\sim}{F}$, θ, $\underset{\sim}{g}$ and $\underset{\sim}{\xi}$ are known at $\underset{\sim}{X}$:

$$\psi = \psi^*(\underset{\sim}{F},\theta,\underset{\sim}{g},\underset{\sim}{\xi};\underset{\sim}{X}) \quad ,$$

$$\underset{\sim}{T} = \underset{\sim}{T}^*(\underset{\sim}{F},\theta,\underset{\sim}{g},\underset{\sim}{\xi};\underset{\sim}{X}) \quad ,$$

$$\eta = \eta^*(\underset{\sim}{F},\theta,\underset{\sim}{g},\underset{\sim}{\xi};\underset{\sim}{X}) \quad ,$$

$$\underset{\sim}{Q} = \underset{\sim}{Q}^*(\underset{\sim}{F},\theta,\underset{\sim}{g},\underset{\sim}{\xi};\underset{\sim}{X}) \quad ,$$

$$\dot{\underset{\sim}{\xi}} = \underset{\sim}{f}(\underset{\sim}{F},\theta,\underset{\sim}{g},\underset{\sim}{\xi};\underset{\sim}{X}) \quad . \tag{2.6}$$

We suppose that the functions ψ^*, $\underset{\sim}{T}^*$, η^*, $\underset{\sim}{Q}^*$ and $\underset{\sim}{f}$ are smooth functions defined for $\underset{\sim}{F}$ in the set M^+ of 3×3 matrices with positive determinant, θ in the set of positive real numbers $\mathcal{R}^+$, $\underset{\sim}{g}$ in $\mathcal{R}^3$, $\underset{\sim}{\xi}$ in $\mathcal{R}^n$ and $\underset{\sim}{X}$ in $\overline{R}$. In particular, we assume that the partial derivatives of ψ^*, $\underset{\sim}{T}^*$, η^*, $\underset{\sim}{Q}^*$ and $\underset{\sim}{f}$, at any fixed state $(\underset{\sim}{F},\theta,\underset{\sim}{g},\underset{\sim}{\xi})$ in state space $M^+\times\mathcal{R}^+\times\mathcal{R}^3\times\mathcal{R}^n$, are bounded functions of $(\underset{\sim}{X},t)$ on $\overline{R}\times[0,t_0]$.

We will say that a thermodynamic process is <u>admissible</u> if it is compatible with the constitutive equations (2.6). Therefore, in order to specify an admissible thermodynamic process it suffices to prescribe the functions $\underset{\sim}{x}$, θ and $\underset{\sim}{\xi}$.

We will say that an admissible thermodynamic process $(\underset{\sim}{x},\theta,\underset{\sim}{\xi})(\underset{\sim}{X},t)$ is <u>smooth</u> in R , if it is such that $(\underset{\sim}{v},\underset{\sim}{F},\theta,\underset{\sim}{g},\underset{\sim}{\xi})(\underset{\sim}{X},t)$ are Lipschitz continuous, uniformly on bounded subsets of their domain.

Within the present framework the second law of thermodynamics is given a precise mathematical meaning by the following postulate of positive production of entropy: Every smooth admissible thermodynamic process in R must obey the Clausius-Duhem inequality

$$\rho\dot{\eta} - \rho\frac{r}{\theta} + \left(\frac{Q_A}{\theta}\right)_{,A} \geq 0 \quad , \tag{2.7}$$

at each time $t \in [0, t_0]$ and for all material points $\underset{\sim}{X}$ in R .
In order the Clausius-Duhem inequality (2.7) holds for all smooth admissible thermodynamic processes in R , it is necessary and sufficient that

$$\psi = \psi^*(\underset{\sim}{F},\theta,\underset{\sim}{\xi};\underset{\sim}{X}) \quad ,$$

$$\underset{\sim}{T}^* = \rho\frac{\partial\psi^*}{\partial \underset{\sim}{F}} \quad , \quad \eta^* = -\frac{\partial\psi^*}{\partial\theta} \quad , \tag{2.8}$$

$$\sigma_\alpha(\underset{\sim}{F},\theta,\underset{\sim}{\xi};\underset{\sim}{X})f_\alpha(\underset{\sim}{F},\theta,\underset{\sim}{g},\underset{\sim}{\xi};\underset{\sim}{X}) + \frac{1}{\theta}Q_A(\underset{\sim}{F},\theta,\underset{\sim}{g},\underset{\sim}{\xi};\underset{\sim}{X})g_A \leq 0 \quad , \tag{2.9}$$

where

$$\sigma_\alpha(\underset{\sim}{F},\theta,\underset{\sim}{\xi};\underset{\sim}{X}) = \rho\frac{\partial\psi^*}{\partial\xi_\alpha}(\underset{\sim}{F},\theta,\underset{\sim}{\xi};\underset{\sim}{X}) \quad , \tag{2.10}$$

is called the affinity of $\underset{\sim}{X}$. In theories with chemical reactions, $\sigma_\alpha(\underset{\sim}{F},\theta,\underset{\sim}{\xi};\underset{\sim}{X})$ corresponds to DeDonder's chemical affinity [1 , 2] .

For smooth admissible thermodynamic processes one may write the balance laws in reduced form

$$\rho \dot{v}_i = T_{Ai,A} + \rho b_i \quad , \tag{2.11}$$

$$\rho\theta\dot{\eta} + \sigma_\alpha \dot{f}_\alpha = \rho r - Q_{A,A} \; . \tag{2.12}$$

3. THE FUNDAMENTAL IDENTITY

In this section we establish the evolutionary identity that will be the basis for the subsequent development of our analysis on continuous data dependence and uniqueness of the thermodynamic processes.

Let $(\underset{\sim}{x},\theta,\underset{\sim}{\xi})(\underset{\sim}{X},t)$ and $(\bar{\underset{\sim}{x}},\bar{\theta},\bar{\underset{\sim}{\xi}})(\underset{\sim}{X},t)$ be two smooth admissible thermodynamic processes defined on $R\times[0,t_0]$. We define

$$\begin{aligned} D(t) = \int_R \Big\{ & \tfrac{1}{2}\rho(v_i-\bar{v}_i)(v_i-\bar{v}_i) + \rho\psi^*(\underset{\sim}{F},\theta,\underset{\sim}{\xi};\underset{\sim}{X}) - \rho\psi^*(\bar{\underset{\sim}{F}},\bar{\theta},\bar{\underset{\sim}{\xi}};\underset{\sim}{X}) - \\ & - T^*_{Ai}(\bar{\underset{\sim}{F}},\bar{\theta},\bar{\underset{\sim}{\xi}};\underset{\sim}{X})(F_{iA}-\bar{F}_{iA}) + \rho(\theta-\bar{\theta})\eta^*(\underset{\sim}{F},\theta,\underset{\sim}{\xi};\underset{\sim}{X}) - \\ & - \sigma_\alpha(\bar{\underset{\sim}{F}},\bar{\theta},\bar{\underset{\sim}{\xi}};\underset{\sim}{X})(\xi_\alpha-\bar{\xi}_\alpha)\Big\}(\underset{\sim}{X},t)\,dX \;, \quad t\in[0,t_0] \,. \end{aligned} \tag{3.1}$$

On account of the relations (2.8) and (2.1o) it follows that $D(t)$ is of quadratic order in $\|(\underset{\sim}{v}-\bar{\underset{\sim}{v}},\underset{\sim}{F}-\bar{\underset{\sim}{F}},\theta-\bar{\theta},\underset{\sim}{\xi}-\bar{\underset{\sim}{\xi}})(\cdot,t)\|_{L^2(R)}$.

The evolution in time of this quantity is described by

<u>Theorem 3.1.</u> <u>If</u> $(\underset{\sim}{x},\theta,\underset{\sim}{\xi})(\underset{\sim}{X},t)$ <u>and</u> $(\bar{\underset{\sim}{x}},\bar{\theta},\bar{\underset{\sim}{\xi}})(\underset{\sim}{X},t)$ <u>are two smooth admissible thermodynamic processes for (2.3) and (2.4) , corresponding to the supply terms</u> $(\underset{\sim}{b},r)(\underset{\sim}{X},t)$ <u>and</u> $(\bar{\underset{\sim}{b}},\bar{r})(\underset{\sim}{X},t)$ <u>in</u> $L^\infty(R\times[0,t_0])$, <u>then we have the following evolutionary identity</u>

$$\begin{aligned} \dot{D}(t) = \int_R \Big\{ & \rho(b_i-\bar{b}_i)(v_i-\bar{v}_i) + \frac{\rho}{\theta}(r-\bar{r})(\theta-\bar{\theta}) - \frac{1}{\theta}\dot{\bar{f}}_\alpha(\sigma_\alpha-\bar{\sigma}_\alpha)(\theta-\bar{\theta}) + \\ & + \dot{\bar{F}}_{iA}\Big[T_{Ai}-\bar{T}_{Ai} - \overline{\frac{\partial T^*_{Ai}}{\partial F_{jB}}}(F_{jB}-\bar{F}_{jB}) - \overline{\frac{\partial T^*_{Ai}}{\partial \theta}}(\theta-\bar{\theta}) - \overline{\frac{\partial T^*_{Ai}}{\partial \xi_\alpha}}(\xi_\alpha-\bar{\xi}_\alpha)\Big] - \end{aligned}$$

$$- \rho\dot{\overline{\theta}} \left[\eta - \overline{\eta} - \overline{\frac{\partial \eta^*}{\partial F_{iA}}} (F_{iA} - \overline{F}_{iA}) - \overline{\frac{\partial \eta^*}{\partial \theta}} (\theta - \overline{\theta}) - \overline{\frac{\partial \eta^*}{\partial \xi_\alpha}} (\xi_\alpha - \overline{\xi}_\alpha) \right] +$$

$$+ \dot{\overline{\xi}}_\alpha \left[\sigma_\alpha - \overline{\sigma}_\alpha - \overline{\frac{\partial \sigma_\alpha}{\partial F_{iA}}} (F_{iA} - \overline{F}_{iA}) - \overline{\frac{\partial \sigma_\alpha}{\partial \theta}} (\theta - \overline{\theta}) - \overline{\frac{\partial \sigma_\alpha}{\partial \xi_\beta}} (\xi_\beta - \overline{\xi}_\beta) \right] -$$

$$- \frac{1}{\theta} \rho \dot{\overline{\eta}} (\theta - \overline{\theta})^2 \Big\} (\underset{\sim}{X}, t) dX + \int_{\partial R} \Big\{ \Big[(T_{Ai} - \overline{T}_{Ai})(v_i - \overline{v}_i) -$$

$$- \frac{1}{\theta} (Q_A - \overline{Q}_A)(\theta - \overline{\theta}) \Big] N_A \Big\} (\underset{\sim}{X}, t) dS + \int_R \Big\{ (Q_A - \overline{Q}_A) \left(\frac{\theta - \overline{\theta}}{\theta} \right)_{,A} -$$

$$- \frac{1}{\theta} \sigma_\alpha (\theta - \overline{\theta})(f_\alpha - \overline{f}_\alpha) + (\sigma_\alpha - \overline{\sigma}_\alpha)(f_\alpha - \overline{f}_\alpha) \Big\} (\underset{\sim}{X}, t) dX , \quad t \in [0, t_0] , \qquad (3.2)$$

<u>where N_A are the direction cosines of the outward normal to surface ∂R</u> .

<u>Proof.</u> From (3.1) we obtain

$$\dot{D}(t) = \int_R \Big\{ \frac{d}{dt} \Big[\rho(\psi + \theta\eta + \frac{1}{2} v_i v_i) \Big] - \frac{d}{dt} \Big[\rho(\overline{\psi} + \overline{\theta}\overline{\eta} + \frac{1}{2} \overline{v}_i \overline{v}_i) \Big] - \rho \dot{v}_i \overline{v}_i -$$

$$- \rho v_i \dot{\overline{v}}_i + 2\rho \overline{v}_i \dot{\overline{v}}_i - \dot{\overline{T}}_{Ai} (F_{iA} - \overline{F}_{iA}) - \overline{T}_{Ai} \dot{F}_{iA} + \overline{T}_{Ai} \dot{\overline{F}}_{iA} -$$

$$- \rho \dot{\overline{\theta}} (\eta - \overline{\eta}) - \rho \overline{\theta} (\dot{\eta} - \dot{\overline{\eta}}) - \dot{\overline{\sigma}}_\alpha (\xi_\alpha - \overline{\xi}_\alpha) - \overline{\sigma}_\alpha (\dot{\xi}_\alpha - \dot{\overline{\xi}}_\alpha) \Big\} (\underset{\sim}{X}, t) dX . \qquad (3.3)$$

By using the balance laws (2.3) and (2.4) , as well as the relations (2.8) and (2.1o), we may rewrite (3.3) in the form

$$\dot{D}(t) = \int_{\partial R} \Big[(v_i - \overline{v}_i)(T_{Ai} - \overline{T}_{Ai}) N_A \Big] (\underset{\sim}{X}, t) dS + \int_R \Big\{ \rho (b_i - \overline{b}_i)(v_i - \overline{v}_i) -$$

$$- \dot{\overline{T}}_{Bj} (F_{jB} - \overline{F}_{jB}) + \dot{\overline{F}}_{iA} (T_{Ai} - \overline{T}_{Ai}) + \rho r - Q_{A,A} - \rho \overline{r} + \overline{Q}_{A,A} -$$

$$- \rho\dot{\theta}(\eta-\bar{\eta}) - \rho\bar{\theta}(\dot{\eta}-\dot{\bar{\eta}}) - \dot{\sigma}_\alpha(\xi_\alpha-\bar{\xi}_\alpha) - \bar{\sigma}_\alpha(\dot{\xi}_\alpha-\dot{\bar{\xi}}_\alpha)\}(\underset{\sim}{X},t)dX =$$

$$= \int_{\partial R}\left[(v_i-\bar{v}_i)(T_{Ai}-\bar{T}_{Ai})N_A\right](\underset{\sim}{X},t)dS + \int_R\{\rho(b_i-\bar{b}_i)(v_i-\bar{v}_i) +$$

$$+ \dot{\bar{F}}_{iA}\left[T_{Ai}-\bar{T}_{Ai}-\overline{\frac{\partial T^*_{Ai}}{\partial F_{jB}}}(F_{jB}-\bar{F}_{jB})-\overline{\frac{\partial T^*_{Ai}}{\partial\theta}}(\theta-\bar{\theta})-\overline{\frac{\partial T^*_{Ai}}{\partial\xi_\alpha}}(\xi_\alpha-\bar{\xi}_\alpha)\right] -$$

$$- \rho\dot{\bar{\theta}}\left[\eta-\bar{\eta}-\overline{\frac{\partial\eta^*}{\partial F_{iA}}}(F_{iA}-\bar{F}_{iA})-\overline{\frac{\partial\eta^*}{\partial\theta}}(\theta-\bar{\theta})-\overline{\frac{\partial\eta^*}{\partial\xi_\alpha}}(\xi_\alpha-\bar{\xi}_\alpha)\right] +$$

$$+ \dot{\bar{\xi}}_\alpha\left[\sigma_\alpha-\bar{\sigma}_\alpha-\overline{\frac{\partial\sigma_\alpha}{\partial F_{iA}}}(F_{iA}-\bar{F}_{iA})-\overline{\frac{\partial\sigma_\alpha}{\partial\theta}}(\theta-\bar{\theta})-\overline{\frac{\partial\sigma_\alpha}{\partial\xi_\beta}}(\xi_\beta-\bar{\xi}_\beta)\right]\}(\underset{\sim}{X},t)dX +$$

$$+ \int_R\{\rho r - Q_{A,A} - \rho\bar{r} + \bar{Q}_{A,A} - \bar{\sigma}_\alpha(\dot{\xi}_\alpha-\dot{\bar{\xi}}_\alpha) - \dot{\bar{\xi}}_\alpha(\sigma_\alpha-\bar{\sigma}_\alpha) -$$

$$- \rho\bar{\theta}(\dot{\eta}-\dot{\bar{\eta}}) - \rho(\theta-\bar{\theta})\dot{\bar{\eta}}\}(\underset{\sim}{X},t)dX . \tag{3.4}$$

On the other hand, from the equation (2.12), we deduce

$$\rho r - Q_{A,A} - \rho\bar{r} + \bar{Q}_{A,A} - \bar{\sigma}_\alpha(\dot{\xi}_\alpha-\dot{\bar{\xi}}_\alpha) - \dot{\bar{\xi}}_\alpha(\sigma_\alpha-\bar{\sigma}_\alpha) - \rho\bar{\theta}(\dot{\eta}-\dot{\bar{\eta}}) - \rho(\theta-\bar{\theta})\dot{\bar{\eta}} =$$

$$= (1-\frac{\bar{\theta}}{\theta})\left[(\rho r - Q_{A,A}-\sigma_\alpha\dot{\xi}_\alpha) - (\rho\bar{r}-\bar{Q}_{A,A}-\bar{\sigma}_\alpha\dot{\bar{\xi}}_\alpha)\right] + (\sigma_\alpha-\bar{\sigma}_\alpha)(\dot{\xi}_\alpha-\dot{\bar{\xi}}_\alpha) -$$

$$- (\rho\bar{r}-\bar{Q}_{A,A}-\bar{\sigma}_\alpha\dot{\bar{\xi}}_\alpha)(\frac{\bar{\theta}}{\theta}-2+\frac{\theta}{\bar{\theta}}) = \frac{\rho}{\theta}(r-\bar{r})(\theta-\bar{\theta}) - \frac{\rho}{\theta}\dot{\bar{\eta}}(\theta-\bar{\theta})^2 -$$

$$- \frac{1}{\theta}\dot{\bar{\xi}}_\alpha(\theta-\bar{\theta})(\sigma_\alpha-\bar{\sigma}_\alpha) - \left[\frac{1}{\theta}(Q_A-\bar{Q}_A)(\theta-\bar{\theta})\right]_{,A} + (Q_A-\bar{Q}_A)(\frac{\theta-\bar{\theta}}{\theta})_{,A} -$$

$$- \frac{1}{\theta}\sigma_\alpha(\dot{\xi}_\alpha-\dot{\bar{\xi}}_\alpha)(\theta-\bar{\theta}) + (\sigma_\alpha-\bar{\sigma}_\alpha)(\dot{\xi}_\alpha-\dot{\bar{\xi}}_\alpha) . \tag{3.5}$$

A substitution of the relation (3.5) into (3.4) and an application of the Gauss-Green theorem and the relation $(2.6)_5$ lead to the identity (3.2). The proof is complete.

The fundamental identity (3.2) will be the starting point of our analysis. On the basis of this identity one may establish suitable estimates that demonstrate continuous dependence of thermodynamic processes upon initial data and supply terms by applying Gronwall type inequalities. This program will be implemented in the following sections for elastic materials without heat conduction and for definite conductors of heat.

4. ELASTIC MATERIALS WITHOUT HEAT CONDUCTION

In thermodynamics with internal state variables a nonconductor of heat is a body for which $\underset{\sim}{Q}^*$ and $\underset{\sim}{f}$ are independent of the temperature gradient $\underset{\sim}{g}$. Therefore, a nonconductor of heat elastic material is characterized by the conditions [8]

$$\underset{\sim}{f} = \underset{\sim}{f}(\underset{\sim}{F},\theta,\underset{\sim}{\xi};\underset{\sim}{X}) \quad , \quad \underset{\sim}{Q} = \underset{\sim}{Q}^*(\underset{\sim}{F},\theta,\underset{\sim}{\xi};\underset{\sim}{X}) \ . \tag{4.1}$$

In this case, the inequality (2.9) implies that

$$\underset{\sim}{Q}^*(\underset{\sim}{F},\theta,\underset{\sim}{\xi};\underset{\sim}{X}) = \underset{\sim}{o} \quad , \quad \text{for all} \quad (\underset{\sim}{F},\theta,\underset{\sim}{\xi};\underset{\sim}{X}). \tag{4.2}$$

For a nonconductor of heat elastic material, the evolutionary identity (3.2) takes the form

$$\dot{D}(t) = \int_R \Big\{ \rho(b_i-\bar{b}_i)(v_i-\bar{v}_i) + \frac{\rho}{\theta}(r-\bar{r})(\theta-\bar{\theta}) - \frac{\rho}{\theta}\dot{\eta}(\theta-\bar{\theta})^2 +$$

$$+ \dot{\bar{F}}_{iA}\Big[T_{Ai}-\bar{T}_{Ai}-\overline{\frac{\partial T^*_{Ai}}{\partial F_{jB}}}(F_{jB}-\bar{F}_{jB})-\overline{\frac{\partial T^*_{Ai}}{\partial \theta}}(\theta-\bar{\theta})-\overline{\frac{\partial T^*_{Ai}}{\partial \xi_\alpha}}(\xi_\alpha-\bar{\xi}_\alpha)\Big] -$$

$$- \rho\dot{\bar{\theta}}\Big[\eta-\bar{\eta}-\overline{\frac{\partial \eta^*}{\partial F_{iA}}}(F_{iA}-\bar{F}_{iA})-\overline{\frac{\partial \eta^*}{\partial \theta}}(\theta-\bar{\theta})-\overline{\frac{\partial \eta^*}{\partial \xi_\alpha}}(\xi_\alpha-\bar{\xi}_\alpha)\Big] +$$

$$+ \dot{\bar{\xi}}_\alpha\Big[\sigma_\alpha-\bar{\sigma}_\alpha-\overline{\frac{\partial \sigma_\alpha}{\partial F_{iA}}}(F_{iA}-\bar{F}_{iA})-\overline{\frac{\partial \sigma_\alpha}{\partial \theta}}(\theta-\bar{\theta})-\overline{\frac{\partial \sigma_\alpha}{\partial \xi_\beta}}(\xi_\beta-\bar{\xi}_\beta)\Big] -$$

$$- \frac{1}{\theta}(\theta-\bar{\theta}) \left[\dot{\xi}_\alpha(\sigma_\alpha - \bar{\sigma}_\alpha) + (f_\alpha - \bar{f}_\alpha)\sigma_\alpha \right] + (\sigma_\alpha - \bar{\sigma}_\alpha)(f_\alpha - \bar{f}_\alpha) \Big\}(\underset{\sim}{X},t)dX +$$

$$+ \int_{\partial R} \left[(v_i - \bar{v}_i)(T_{Ai} - \bar{T}_{Ai})N_A \right](\underset{\sim}{X},t)dS \; . \tag{4.3}$$

We are now in a position to establish the following result.

Theorem 4.1. Let $(\bar{\underset{\sim}{x}},\bar{\theta},\bar{\underset{\sim}{\xi}})(\underset{\sim}{X},t)$ be a smooth admissible thermodynamic process corresponding to the supply terms $(\bar{\underset{\sim}{b}},\bar{r})(\underset{\sim}{X},t)$ in $L^\infty(R\times[0,t_0])$. We assume that the elastic material is a nonconductor of heat. Then there are positive constants δ_1, a_1, a_2, a_3 and a_4 with the following property:

If $(\underset{\sim}{x},\theta,\underset{\sim}{\xi})(\underset{\sim}{X},t)$ is any smooth admissible thermodynamic process defined on $R\times[0,t_0]$, with supply terms $(\underset{\sim}{b},r)(\underset{\sim}{X},t)\in L^\infty(R\times[0,t_0])$, such that

$$|\underset{\sim}{F}(\underset{\sim}{X},t)-\bar{\underset{\sim}{F}}(\underset{\sim}{X},t)| + |\theta(\underset{\sim}{X},t)-\bar{\theta}(\underset{\sim}{X},t)| + |\underset{\sim}{\xi}(\underset{\sim}{X},t)-\bar{\underset{\sim}{\xi}}(\underset{\sim}{X},t)| < \delta_1 ,$$

$$(\underset{\sim}{X},t)\in R\times[0,t_0] \; , \tag{4.4}$$

$$(v_i-\bar{v}_i)(T_{Ai}-\bar{T}_{Ai})N_A \leq 0 \quad , \quad \text{on} \quad \partial R\times[0,t_0] \; , \tag{4.5}$$

then we have the following estimate, for all $\tau\in[0,s]$, $s\in[0,t_0]$,

$$\int_R \rho(\underset{\sim}{X})\Big\{(v_i-\bar{v}_i)(v_i-\bar{v}_i) + \frac{\partial\psi^*(\bar{\underset{\sim}{F}},\bar{\theta},\bar{\underset{\sim}{\xi}};\underset{\sim}{X})}{\partial F_{iA}\partial F_{jB}}(F_{iA}-\bar{F}_{iA})(F_{jB}-\bar{F}_{jB}) -$$

$$- \frac{\partial^2\psi^*(\bar{\underset{\sim}{F}},\bar{\theta},\bar{\underset{\sim}{\xi}};\underset{\sim}{X})}{\partial\theta^2}(\theta-\bar{\theta})^2\Big\}(\underset{\sim}{X},\tau)dX + a_1\|(\underset{\sim}{\xi}-\bar{\underset{\sim}{\xi}})(\cdot,\tau)\|^2_{L^2(R)} -$$

$$- \pi_1^2\|(\underset{\sim}{F}-\bar{\underset{\sim}{F}})(\cdot,\tau)\|^2_{L^2(R)} \leq a_2\|(\underset{\sim}{v}-\bar{\underset{\sim}{v}},\underset{\sim}{F}-\bar{\underset{\sim}{F}},\theta-\bar{\theta},\underset{\sim}{\xi}-\bar{\underset{\sim}{\xi}})(\cdot,0)\|^2_{L^2(R)} +$$

$$+ a_3\int_0^\tau \|(\underset{\sim}{v}-\bar{\underset{\sim}{v}},\underset{\sim}{F}-\bar{\underset{\sim}{F}},\theta-\bar{\theta},\underset{\sim}{\xi}-\bar{\underset{\sim}{\xi}})(\cdot,t)\|^2_{L^2(R)}dt +$$

$$+ a_4\int_0^\tau \|(\underset{\sim}{b}-\bar{\underset{\sim}{b}},r-\bar{r})(\cdot,t)\|_{L^2(R)}\|(\underset{\sim}{v}-\bar{\underset{\sim}{v}},\underset{\sim}{F}-\bar{\underset{\sim}{F}},\theta-\bar{\theta},\underset{\sim}{\xi}-\bar{\underset{\sim}{\xi}})(\cdot,t)\|_{L^2(R)}dt \; , \tag{4.6}$$

<u>for arbitrary constant</u> π_1.

<u>Proof.</u> By taking into account the relations (2.8), (2.1o) and $(4.1)_1$ and the boundary condition (4.5) and by using the Schwarz inequality, from (4.3) it follows that one can determine positive constants a_5 and a_6 such that, for all $t \in [0, t_0]$, we have

$$\dot{D}(t) \leq a_5 \|(\underset{\sim}{F}-\overline{\underset{\sim}{F}}, \theta-\overline{\theta}, \underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot, t)\|^2_{L^2(R)} +$$

$$+ a_6 \|(\underset{\sim}{b}-\overline{\underset{\sim}{b}}, r-\overline{r})(\cdot, t)\|_{L^2(R)} \cdot \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}}, \theta-\overline{\theta})(\cdot, t)\|_{L^2(R)}. \tag{4.7}$$

We now fix $s \in [0, t_0]$, integrate (4.7) over $[0, \tau]$, $\tau \in [0, s]$, so that we obtain

$$D(\tau) \leq D(0) + a_5 \int_0^\tau \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}}, \underset{\sim}{F}-\overline{\underset{\sim}{F}}, \theta-\overline{\theta}, \underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot, t)\|^2_{L^2(R)} dt +$$

$$+ a_6 \int_0^\tau \|(\underset{\sim}{b}-\overline{\underset{\sim}{b}}, r-\overline{r})(\cdot, t)\|_{L^2(R)} \cdot \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}}, \underset{\sim}{F}-\overline{\underset{\sim}{F}}, \theta-\overline{\theta}, \underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot, t)\|_{L^2(R)} dt. \tag{4.8}$$

On the other hand, in view of the relations (2.8) and (2.1o), we deduce

$$\rho(\underset{\sim}{X}) \psi^*(\underset{\sim}{F}, \theta, \underset{\sim}{\xi}; \underset{\sim}{X}) - \rho(\underset{\sim}{X}) \psi^*(\overline{\underset{\sim}{F}}, \overline{\theta}, \overline{\underset{\sim}{\xi}}; \underset{\sim}{X}) - T^*_{Ai}(\overline{\underset{\sim}{F}}, \overline{\theta}, \overline{\underset{\sim}{\xi}}; \underset{\sim}{X})(F_{iA}-\overline{F}_{iA}) +$$

$$+ \rho(\underset{\sim}{X})(\theta-\overline{\theta}) \eta^*(\underset{\sim}{F}, \theta, \underset{\sim}{\xi}; \underset{\sim}{X}) - \sigma_\alpha(\overline{\underset{\sim}{F}}, \overline{\theta}, \overline{\underset{\sim}{\xi}}; \underset{\sim}{X})(\xi_\alpha - \overline{\xi}_\alpha) =$$

$$= \frac{1}{2}\rho(\underset{\sim}{X}) \Big\{ \frac{\partial^2 \psi^*(\overline{\underset{\sim}{F}}, \overline{\theta}, \overline{\underset{\sim}{\xi}}; \underset{\sim}{X})}{\partial F_{iA} \partial F_{jB}} (F_{iA}-\overline{F}_{iA})(F_{jB}-\overline{F}_{jB}) - \frac{\partial^2 \psi^*(\overline{\underset{\sim}{F}}, \overline{\theta}, \overline{\underset{\sim}{\xi}}; \underset{\sim}{X})}{\partial \theta^2} (\theta-\overline{\theta})^2 +$$

$$+ \frac{\partial^2 \psi^*(\overline{\underset{\sim}{F}}, \overline{\theta}, \overline{\underset{\sim}{\xi}}; \underset{\sim}{X})}{\partial \xi_\alpha \partial \xi_\beta} (\xi_\alpha - \overline{\xi}_\alpha)(\xi_\beta - \overline{\xi}_\beta) + 2 \frac{\partial^2 \psi^*(\overline{\underset{\sim}{F}}, \overline{\theta}, \overline{\underset{\sim}{\xi}}; \underset{\sim}{X})}{\partial F_{iA} \partial \xi_\alpha} (F_{iA}-\overline{F}_{iA})(\xi_\alpha - \overline{\xi}_\alpha) \Big\} +$$

$$+ o(|\underset{\sim}{F}-\overline{\underset{\sim}{F}}|^2 + |\theta-\overline{\theta}|^2 + |\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}}|^2). \tag{4.9}$$

Combining (4.9) with (3.1), we conclude that there is a positive constant δ_1 with the property that, when (4.4) is satisfied,

$$\int_R \rho(\underset{\sim}{X})\Big\{(v_i-\overline{v}_i)(v_i-\overline{v}_i) + \frac{\partial^2\psi^*(\underset{\sim}{F},\overline{\theta},\overline{\underset{\sim}{\xi}};\underset{\sim}{X})}{\partial F_{iA}\partial F_{jB}}(F_{iA}-\overline{F}_{iA})(F_{jB}-\overline{F}_{jB}) -$$

$$- \frac{\partial^2\psi^*(\underset{\sim}{F},\overline{\theta},\overline{\underset{\sim}{\xi}};\underset{\sim}{X})}{\partial\theta^2}(\theta-\overline{\theta})^2\Big\}(\underset{\sim}{X},\tau)\,dX \leq 2D(o) + 2a_5\int_0^\tau \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}},\underset{\sim}{F}-\overline{\underset{\sim}{F}},\theta-\overline{\theta},\underset{\sim}{\xi}-$$

$$-\overline{\underset{\sim}{\xi}})(\cdot,t)\|^2_{L^2(R)}dt + 2a_6\int_0^\tau \|(\underset{\sim}{b}-\overline{\underset{\sim}{b}},r-\overline{r})(\cdot,t)\|_{L^2(R)}\cdot\|(\underset{\sim}{v}-\overline{\underset{\sim}{v}},\underset{\sim}{F}-\overline{\underset{\sim}{F}},\theta-\overline{\theta},\underset{\sim}{\xi}-$$

$$-\overline{\underset{\sim}{\xi}})(\cdot,t)\|_{L^2(R)}dt - \int_R \rho(\underset{\sim}{X})\Big\{\frac{\partial^2\psi^*(\underset{\sim}{F},\overline{\theta},\overline{\underset{\sim}{\xi}};\underset{\sim}{X})}{\partial\xi_\alpha\partial\xi_\beta}(\xi_\alpha-\overline{\xi}_\alpha)(\xi_\beta-\overline{\xi}_\beta) +$$

$$+ 2\frac{\partial^2\psi^*(\underset{\sim}{F},\overline{\theta},\overline{\underset{\sim}{\xi}};\underset{\sim}{X})}{\partial F_{iA}\partial\xi_\alpha}(F_{iA}-\overline{F}_{iA})(\xi_\alpha-\overline{\xi}_\alpha)\Big\}(\underset{\sim}{X},\tau)\,dX\ . \tag{4.1o}$$

An application of the Schwarz inequality and the inequality

$$ab \leq \frac{1}{2}\Big(\frac{a^2}{\pi^2} + b^2\pi^2\Big)\ , \tag{4.11}$$

to the last term in (4.1o) yields, for arbitrary constant π_1,

$$\int_R \rho(\underset{\sim}{X})\Big\{(v_i-\overline{v}_i)(v_i-\overline{v}_i) + \frac{\partial^2\psi^*(\underset{\sim}{F},\overline{\theta},\overline{\underset{\sim}{\xi}};\underset{\sim}{X})}{\partial F_{iA}\partial F_{jB}}(F_{iA}-\overline{F}_{iA})(F_{jB}-\overline{F}_{jB}) -$$

$$- \frac{\partial^2\psi^*(\underset{\sim}{F},\overline{\theta},\overline{\underset{\sim}{\xi}};\underset{\sim}{X})}{\partial\theta^2}(\theta-\overline{\theta})^2\Big\}(X,\tau)\,dX \leq 2D(o) + 2a_5\int_0^\tau \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}},\underset{\sim}{F}-\overline{\underset{\sim}{F}},\theta-\overline{\theta},\underset{\sim}{\xi}-$$

$$-\overline{\underset{\sim}{\xi}})(\cdot,t)\|^2_{L^2(R)}dt + 2a_6\int_0^\tau \|(\underset{\sim}{b}-\overline{\underset{\sim}{b}},r-\overline{r})(\cdot,t)\|_{L^2(R)}\cdot\|(\underset{\sim}{v}-\overline{\underset{\sim}{v}},\underset{\sim}{F}-\overline{\underset{\sim}{F}},\theta-\overline{\theta},\underset{\sim}{\xi}-$$

$$-\overline{\underset{\sim}{\xi}})(\cdot,t)\|_{L^2(R)}dt + \Big(a_7^2 + \frac{a_8^2}{\pi_1^2}\Big)\|(\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,\tau)\|^2_{L^2(R)} +$$

$$+ \pi_1^2\|(\underset{\sim}{F}-\overline{\underset{\sim}{F}})(\cdot,\tau)\|^2_{L^2(R)}\ , \tag{4.12}$$

where

$$a_7^2 = \max\left|\overline{\frac{\partial \underset{\sim}{\sigma}}{\partial \underset{\sim}{\xi}}}(\underset{\sim}{X},t)\right| \quad , \quad a_8 = \max\left|\overline{\frac{\partial \underset{\sim}{\sigma}}{\partial \underset{\sim}{F}}}(\underset{\sim}{X},t)\right| \quad , \quad \text{on} \quad \overline{R}\times[0,t_0] \, . \tag{4.13}$$

Taking into account the relation $(4.1)_1$ and the relation

$$\begin{aligned} |(\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\underset{\sim}{X},\tau)|^2 &= |(\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\underset{\sim}{X},0)|^2 + \int_0^\tau \frac{d}{dt}\Big[|(\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\underset{\sim}{X},t)|^2\Big]dt = \\ &= |(\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\underset{\sim}{X},0)|^2 + 2\int_0^\tau \Big[(\xi_\alpha-\overline{\xi}_\alpha)(f_\alpha-\overline{f}_\alpha)\Big](\underset{\sim}{X},t)dt \quad , \end{aligned} \tag{4.14}$$

it follows that one can determine the positive constant a_9 so that, we have

$$\begin{aligned} \pi_2\|(\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,\tau)\|^2_{L^2(R)} &\leq \pi_2\|(\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,0)\|^2_{L^2(R)} + \\ &+ \pi_2 a_9\int_0^\tau \|(\underset{\sim}{F}-\overline{\underset{\sim}{F}},\theta-\overline{\theta},\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,t)\|^2_{L^2(R)}dt \quad , \end{aligned} \tag{4.15}$$

for arbitrary positive constant π_2 .

If we use the estimate

$$D(0) \leq a_{10}\|(\underset{\sim}{v}-\overline{\underset{\sim}{v}},\underset{\sim}{F}-\overline{\underset{\sim}{F}},\theta-\overline{\theta},\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,0)\|^2_{L^2(R)} \quad , \tag{4.16}$$

from (4.12) and (4.15), we deduce

$$\begin{aligned} &\int_R \rho(\underset{\sim}{X})\Big\{(v_i-\overline{v}_i)(v_i-\overline{v}_i) + \frac{\partial^2\psi^*(\underset{\sim}{\overline{F}},\overline{\theta},\underset{\sim}{\overline{\xi}};\underset{\sim}{X})}{\partial F_{iA}\partial F_{jB}}(F_{iA}-\overline{F}_{iA})(F_{jB}-\overline{F}_{jB}) - \\ &- \frac{\partial^2\psi^*(\underset{\sim}{\overline{F}},\overline{\theta},\underset{\sim}{\overline{\xi}};\underset{\sim}{X})}{\partial\theta^2}(\theta-\theta)^2\Big\}(\underset{\sim}{X},\tau)dX + \Big[\pi_2-(a_7^2+\frac{a_8^2}{\pi_1^2})\Big]\|(\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,\tau)\|^2_{L^2(R)} - \\ &- \pi_1^2\|(\underset{\sim}{F}-\overline{\underset{\sim}{F}})(\cdot,\tau)\|^2_{L^2(R)} \leq a_2\|(\underset{\sim}{v}-\overline{\underset{\sim}{v}},\underset{\sim}{F}-\overline{\underset{\sim}{F}},\theta-\overline{\theta},\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,0)\|^2_{L^2(R)} + \\ &+ a_3\int_0^\tau \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}},\underset{\sim}{F}-\overline{\underset{\sim}{F}},\theta-\overline{\theta},\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,t)\|^2_{L^2(R)}dt + \\ &+ a_4\int_0^\tau \|(\underset{\sim}{b}-\overline{\underset{\sim}{b}},r-\overline{r})(\cdot,t)\|_{L^2(R)}\cdot\|(\underset{\sim}{v}-\overline{\underset{\sim}{v}},\underset{\sim}{F}-\overline{\underset{\sim}{F}},\theta-\overline{\theta},\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,t)\|_{L^2(R)}dt \quad , \end{aligned} \tag{4.17}$$

where

$$a_2 = \pi_2 + 2a_{10} \quad , \quad a_3 = 2a_5 + \pi_2 a_9 \quad , \quad a_4 = 2a_6 \, . \tag{4.18}$$

We now choose the arbitrary positive constant π_2 so that

$$a_1 = \pi_2 - \left(a_7^2 + \frac{a_8^2}{\pi_1^2}\right) > 0. \tag{4.19}$$

The relation (4.17) implies the estimate (4.6) and the proof is complete.

We now examine two conditions on the material response and the thermodynamic process that allow to obtain, from the estimate (4.6), a Gronwall type inequality that demonstrates the continuous data dependence and the uniqueness.

a) Continuous dependence in a neighborhood of a positive smooth admissible process. We will say that the smooth admissible thermodynamic process $(\bar{\underset{\sim}{x}},\bar{\theta},\bar{\underset{\sim}{\xi}})(\underset{\sim}{X},t)$ is positive if, for each $(\underset{\sim}{X},t)\in R\times[0,t_0]$,

the elasticity tensor, $\frac{\partial \underset{\sim}{T}^*}{\partial \underset{\sim}{F}}(\bar{\underset{\sim}{F}},\bar{\theta},\bar{\underset{\sim}{\xi}};\underset{\sim}{X})$, is positive definite, (4.20)

and

the instantaneous specific heat, $\frac{\partial \eta^*}{\partial \theta}(\bar{\underset{\sim}{F}},\bar{\theta},\bar{\underset{\sim}{\xi}};\underset{\sim}{X})$, is strictly positive. (4.21)

In order to establish our continuous dependence results we shall need the following Gronwall type inequality [20]

Lemma 4.1. Assume that nonnegative functions $y(t)\in L^\infty[0,s]$ and $g(t)\in L^1[0,s]$ satisfy the inequality

$$y^2(\tau) \le M^2 y^2(0) + \int_0^\tau \left[(2\sigma + 4\beta\tau)y^2(t) + 2Ng(t)y(t)\right] dt, \quad \tau\in[0,s] \, , \tag{4.22}$$

with β , σ , M and N nonnegative constants. Then

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$$y(s) \leq M\exp(\alpha s+\beta s^2)y(0) + N\exp(\alpha s+\beta s^2)\int_0^s g(t)dt \ , \quad \alpha = \sigma + \beta/\sigma \ . \quad (4.23)$$

Theorem 4.2. Let $(\bar{x},\bar{\theta},\bar{\xi})(X,t)$ and $(x,\theta,\xi)(X,t)$ be as in Theorem 4.1. We assume that the smooth admissible thermodynamic process $(\bar{x},\bar{\theta},\bar{\xi})(X,t)$ is positive. Then there are positive constants δ_1, α_1, M_1 and N_1 with the property that, whenever (4.4) holds, for any $s\in[0,t_0]$,

$$\|(v-\bar{v},F-\bar{F},\theta-\bar{\theta},\xi-\bar{\xi})(\cdot,s)\|_{L^2(R)} \leq M_1 e^{\alpha_1 s}\|(v-\bar{v},F-\bar{F},\theta-\bar{\theta},\xi-\bar{\xi})(\cdot,0)\|_{L^2(R)} + \\ + N_1 e^{\alpha_1 s}\int_0^s \|(b-\bar{b},r-\bar{r})(\cdot,t)\|_{L^2(R)}dt \ . \quad (4.24)$$

Proof. Since $(\bar{x},\bar{\theta},\bar{\xi})(X,t)$ is a positive smooth admissible thermodynamic process, it follows that there exists positive constant λ such that, for all $\tau\in[0,s]$, $s\in[0,t_0]$,

$$\int_R \rho(X)\Big\{\frac{\partial^2\psi^*(\bar{F},\bar{\theta},\bar{\xi};X)}{\partial F_{iA}\partial F_{jB}}(F_{iA}-\bar{F}_{iA})(F_{jB}-\bar{F}_{jB}) - \\ - \frac{\partial^2\psi^*(\bar{F},\bar{\theta},\bar{\xi};X)}{\partial\theta^2}(\theta-\bar{\theta})^2\Big\}(X,\tau)dX \geq \lambda\|(F-\bar{F},\theta-\bar{\theta})(\cdot,\ )\|^2_{L^2(R)} . \quad (4.25)$$

We now use this relation into estimate (4.6). Thus, we deduce

$$\rho_0\|(v-\bar{v})(\cdot,\tau)\|^2_{L^2(R)} + \lambda_1\|(F-\bar{F})(\cdot,\tau)\|^2_{L^2(R)} + \lambda\|(\theta-\bar{\theta})(\cdot,\tau)\|^2_{L^2(R)} + \\ + a_1\|(\xi-\bar{\xi})(\cdot,\tau)\|^2_{L^2(R)} \leq a_2\|(v-\bar{v},F-\bar{F},\theta-\bar{\theta},\xi-\bar{\xi})(\cdot,0)\|^2_{L^2(R)} + \\ + a_3\int_0^\tau\|(v-\bar{v},F-\bar{F},\theta-\bar{\theta},\xi-\bar{\xi})(\cdot,t)\|^2_{L^2(R)}dt + \\ + a_4\int_0^\tau\|(b-\bar{b},r-\bar{r})(\cdot,t)\|_{L^2(R)}\|(v-\bar{v},F-\bar{F},\theta-\bar{\theta},\xi-\bar{\xi})(\cdot,t)\|_{L^2(R)}dt \ , \quad (4.26)$$

where

$$\lambda_1 = \lambda - \pi_1^2 \, . \tag{4.27}$$

We choose the arbitrary constant π_1 so that

$$\lambda_1 > 0 . \tag{4.28}$$

It is easy to see that the inequality (4.26) lead to an estimate of the form (4.22) with

$$\begin{aligned} y_1(t) &= \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}}, \underset{\sim}{F}-\overline{\underset{\sim}{F}}, \theta-\overline{\theta}, \underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,t)\|_{L^2(R)} \; , \\ g_1(t) &= \|(\underset{\sim}{b}-\overline{\underset{\sim}{b}}, r-\overline{r})(\cdot,t)\|_{L^2(R)} \; , \end{aligned} \tag{4.29}$$

$$M_1^2 = \frac{a_2}{m_1} \; , \quad m_1 = \min(\rho_0, \lambda_1, \lambda, a_1) \; ,$$

$$\beta_1 = 0 \; , \quad \sigma_1 = \frac{a_3}{2m_1} \; , \quad N_1 = \frac{a_4}{2m_1} \; , \quad \alpha_1 = \sigma_1 \, . \tag{4.30}$$

An application of Lemma 4.1 completes the proof.

From the above proposition we get the following uniqueness result.

<u>Theorem 4.3.</u> <u>Let</u> $(\overline{\underset{\sim}{x}}, \overline{\theta}, \overline{\underset{\sim}{\xi}})(\underset{\sim}{X},t)$ <u>and</u> $(\underset{\sim}{x}, \theta, \underset{\sim}{\xi})(\underset{\sim}{X},t)$ <u>be as in Theorem 4.2. We assume that the corresponding supply terms</u> $(\overline{\underset{\sim}{b}}, \overline{r})(\underset{\sim}{X},t)$ <u>and</u> $(\underset{\sim}{b}, r)(\underset{\sim}{X},t)$ <u>coincide on</u> $R \times [0,t_0]$ <u>and that both processes originate from the same state, that is,</u>

$$\begin{aligned} &\underset{\sim}{x}(\underset{\sim}{X},0) = \overline{\underset{\sim}{x}}(\underset{\sim}{X},0) \; , \quad \underset{\sim}{v}(\underset{\sim}{X},0) = \overline{\underset{\sim}{v}}(\underset{\sim}{X},0) \; , \\ &\theta(\underset{\sim}{X},0) = \overline{\theta}(\underset{\sim}{X},0) \; , \quad \underset{\sim}{\xi}(\underset{\sim}{X},0) = \overline{\underset{\sim}{\xi}}(\underset{\sim}{X},0) \; , \quad \underset{\sim}{X} \in \overline{R} . \end{aligned} \tag{4.31}$$

<u>Then</u> $(\overline{\underset{\sim}{x}}, \overline{\theta}, \overline{\underset{\sim}{\xi}})(\underset{\sim}{X},t)$ <u>and</u> $(\underset{\sim}{x}, \theta, \underset{\sim}{\xi})(\underset{\sim}{X},t)$ <u>coincide on</u> $\overline{R} \times [0,t_0]$.

b) <u>Continuous dependence of smooth admissible processes in the strong ellipticity region.</u> We will say that a smooth admissible thermodynamic process $(\overline{\underset{\sim}{x}}, \overline{\theta}, \overline{\underset{\sim}{\xi}})(\underset{\sim}{X},t)$ <u>resides in the strong ellipticity region</u> if there is a positive constant $\varkappa$ with the property that, for any vectors $\underset{\sim}{\zeta}, \underset{\sim}{\chi} \in \mathcal{R}^3$, every $\omega \in \mathcal{R}$ and all $(\underset{\sim}{X},t)$ in the domain of the process,

$$\frac{\partial^2\psi^*(\underset{\sim}{F},\bar\theta,\bar{\underset{\sim}{\xi}};\underset{\sim}{X})}{\partial F_{iA}\partial F_{jB}}\, \zeta_i\zeta_j\chi_A\chi_B - \frac{\partial^2\psi^*(\underset{\sim}{F},\bar\theta,\bar{\underset{\sim}{\xi}};\underset{\sim}{X})}{\partial\theta^2}\,\omega^2 \geqslant$$

$$\geqslant \varkappa(|\underset{\sim}{\zeta}|^2|\underset{\sim}{\chi}|^2 + \omega^2). \qquad (4.32)$$

The strong ellipticity condition in thermodynamics with internal state variables has been studied in connection with wave propagation (see e.g. [3] , [9] , [12]). The implications of strong ellipticity condition upon uniqueness of solutions to the purely mechanical theory with internal state variables have been analyzed by Kosinski [19]. The strong ellipticity condition (4.32) is the same as in nonlinear thermoelasticity [21] and it is related to that used by Bowen and Chen [9] or by Kosinski [19].

<u>Lemma 4.2.</u> <u>Let</u> $(\bar{\underset{\sim}{x}},\bar\theta,\bar{\underset{\sim}{\xi}})(\underset{\sim}{X},t)$ <u>be a smooth admissible thermodynamic process defined on</u> $R\times[0,t_0]$, <u>residing in the strong ellipticity region. Let</u> $(\underset{\sim}{x},\theta,\underset{\sim}{\xi})(\underset{\sim}{X},t)$ <u>be any smooth admissible process defined on</u> $R\times[0,t_0]$ <u>such that</u>

$$\underset{\sim}{x}(\underset{\sim}{X},t) = \bar{\underset{\sim}{x}}(\underset{\sim}{X},t) \quad , \quad (\underset{\sim}{X},t)\in\partial R\times[0,t_0] . \qquad (4.33)$$

<u>Then there are constants</u> $\mu > 0$ <u>and</u> ν <u>with the property that, for any</u> $\tau\in[0,t_0]$,

$$\int_R \rho(\underset{\sim}{X})\Big\{\frac{\partial^2\psi^*(\underset{\sim}{F},\bar\theta,\bar{\underset{\sim}{\xi}};\underset{\sim}{X})}{\partial F_{iA}\partial F_{jB}}(F_{iA}-\bar F_{iA})(F_{jB}-\bar F_{jB}) - \frac{\partial^2\psi^*(\underset{\sim}{F},\bar\theta,\bar{\underset{\sim}{\xi}};\underset{\sim}{X})}{\partial\theta^2}(\theta-\bar\theta)^2\Big\}(\underset{\sim}{X},\tau)\,dX \geqslant$$

$$\geqslant \mu\,\|(\underset{\sim}{F}-\bar{\underset{\sim}{F}},\theta-\bar\theta)(\cdot,\tau)\|^2_{L^2(R)} - \nu\,\|(\underset{\sim}{x}-\bar{\underset{\sim}{x}})(\cdot,\tau)\|^2_{L^2(R)} . \qquad (4.34)$$

The proof of Lemma 4.2 is a straightforward generalization of the proof of the standard Gårding inequality [22].

<u>Theorem 4.4.</u> <u>Let</u> $(\bar{\underset{\sim}{x}},\bar\theta,\bar{\underset{\sim}{\xi}})(\underset{\sim}{X},t)$ <u>and</u> $(\underset{\sim}{x},\theta,\underset{\sim}{\xi})(\underset{\sim}{X},t)$ <u>be as in Theorem 4.1. Moreover, we assume that</u>

$$\underset{\sim}{x}(\underset{\sim}{X},t) = \overline{\underset{\sim}{x}}(\underset{\sim}{X},t) \quad , \quad \text{on} \quad \partial R\times[o,t_o] \ , \tag{4.35}$$

<u>and that the smooth admissible thermodynamic process</u> $(\overline{\underset{\sim}{x}},\overline{\theta},\overline{\underset{\sim}{\xi}})(\underset{\sim}{X},t)$ <u>resides in the strong ellipticity region. Then there are positive constants</u> δ_1 , α_2 , β_2 , M_2 <u>and</u> N_2 <u>with the property that, whenever (4.4) holds, we have, for any</u> $s\in[o,t_o]$,

$$\|(\underset{\sim}{v}-\overline{\underset{\sim}{v}},\underset{\sim}{F}-\overline{\underset{\sim}{F}},\theta-\overline{\theta},\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,s)\|_{L^2(R)} \leq M_2\exp(\alpha_2 s+\beta_2 s^2)\|(\underset{\sim}{v}-\overline{\underset{\sim}{v}},\underset{\sim}{F}-\overline{\underset{\sim}{F}},\theta-\overline{\theta},\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,o)\|_{L^2(R)} + N_2\exp(\alpha_2 s+\beta_2 s^2)\int_0^s\|(\underset{\sim}{b}-\overline{\underset{\sim}{b}},r-\overline{r})(\cdot,t)\|_{L^2(R)}dt. \tag{4.36}$$

<u>Proof.</u> We recall that $(\overline{\underset{\sim}{x}},\overline{\theta},\overline{\underset{\sim}{\xi}})(\underset{\sim}{X},t)$ resides in the strong ellipticity region and $\underset{\sim}{x}-\overline{\underset{\sim}{x}}$ vanishes on $\partial R\times[o,t_o]$. Thus, the inequality (4.34) holds. Combining (4.34) with (4.6), we deduce

$$\rho_o\|(\underset{\sim}{v}-\overline{\underset{\sim}{v}})(\cdot,\tau)\|^2_{L^2(R)} + \mu_1\|(\underset{\sim}{F}-\overline{\underset{\sim}{F}})(\cdot,\tau)\|^2_{L^2(R)} + \mu\|(\theta-\overline{\theta})(\cdot,\tau)\|^2_{L^2(R)} + a_1\|(\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,\tau)\|^2_{L^2(R)} \leq |\nu|\,\|(\underset{\sim}{x}-\overline{\underset{\sim}{x}})(\cdot,\tau)\|^2_{L^2(R)} + a_2\|(\underset{\sim}{v}-\overline{\underset{\sim}{v}},\underset{\sim}{F}-\overline{\underset{\sim}{F}},\theta-\overline{\theta},\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,o)\|^2_{L^2(R)} + a_3\int_0^\tau\|(\underset{\sim}{v}-\overline{\underset{\sim}{v}},\underset{\sim}{F}-\overline{\underset{\sim}{F}},\theta-\overline{\theta},\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,t)\|^2_{L^2(R)}dt + a_4\int_0^\tau\|(\underset{\sim}{b}-\overline{\underset{\sim}{b}},r-\overline{r})(\cdot,t)\|_{L^2(R)}\cdot\|(\underset{\sim}{v}-\overline{\underset{\sim}{v}},\underset{\sim}{F}-\overline{\underset{\sim}{F}},\theta-\overline{\theta},\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,t)\|_{L^2(R)}dt \ , \tag{4.37}$$

where

$$\mu_1 = \mu - \pi_1^2. \tag{4.38}$$

We now choose the arbitrary constant π_1 so that

$$\mu_1 > o. \tag{4.39}$$

Furthermore, upon using Schwarz's inequality and the Poincaré inequality, we find that

$$\int_R |\underset{\sim}{x}(\underset{\sim}{X},\tau)-\overline{\underset{\sim}{x}}(\underset{\sim}{X},\tau)|^2 dX = \int_R \Big|\underset{\sim}{x}(\underset{\sim}{X},o)-\overline{\underset{\sim}{x}}(\underset{\sim}{X},o) + \int_0^\tau [\underset{\sim}{v}(\underset{\sim}{X},t)-\overline{\underset{\sim}{v}}(\underset{\sim}{X},t)]\,dt\Big|^2 dX \leq$$

$$\leq 2a_{11}\int_R |\underset{\sim}{F}(\underset{\sim}{X},o)-\overline{\underset{\sim}{F}}(\underset{\sim}{X},o)|^2 dX + 2\tau\int_0^\tau\int_R |\underset{\sim}{v}(\underset{\sim}{X},t)-\overline{\underset{\sim}{v}}(\underset{\sim}{X},t)|^2 dX dt. \qquad (4.40)$$

From (4.37) and (4.40) we obtain an estimate of the form (4.22) with

$$y_2(t) = \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}},\underset{\sim}{F}-\overline{\underset{\sim}{F}},\theta-\overline{\theta},\underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,t)\|_{L^2(R)} ,$$

$$g_2(t) = \|(\underset{\sim}{b}-\overline{\underset{\sim}{b}},r-\overline{r})(\cdot,t)\|_{L^2(R)} , \qquad (4.41)$$

$$M_2^2 = (a_2 + 2|\vartheta|a_{11})/m_2 \quad , \quad m_2 = \min(\rho_o,\mu_1,\mu,a_1) \quad ,$$

$$\sigma_2 = \frac{a_3}{2m_2} \quad , \quad \beta_2 = \frac{|\vartheta|}{2m_2} \quad , \quad \alpha_2 = \sigma_2 + \frac{\beta_2}{\sigma_2} \quad , \quad N_2 = \frac{a_4}{2m_2} . \qquad (4.42)$$

An application of Lemma 4.1 completes the proof.

An immediate consequence of the Theorem 4.4 is the following uniqueness theorem.

<u>Theorem 4.5. Let</u> $(\overline{\underset{\sim}{x}},\overline{\theta},\overline{\underset{\sim}{\xi}})(\underset{\sim}{X},t)$ <u>and</u> $(\underset{\sim}{x},\theta,\underset{\sim}{\xi})(\underset{\sim}{X},t)$ <u>be as in Theorem 4.4. We assume that the corresponding supply terms</u> $(\overline{\underset{\sim}{b}},\overline{r})(\underset{\sim}{X},t)$ <u>and</u> $(\underset{\sim}{b},r)(\underset{\sim}{X},t)$ <u>coincide on</u> $R\times[o,t_o]$ <u>and that both processes originate from the same state, that is,</u>

$$\underset{\sim}{x}(\underset{\sim}{X},o) = \overline{\underset{\sim}{x}}(\underset{\sim}{X},o) \quad , \quad \underset{\sim}{v}(\underset{\sim}{X},o) = \overline{\underset{\sim}{v}}(\underset{\sim}{X},o) \quad ,$$

$$\theta(\underset{\sim}{X},o) = \overline{\theta}(\underset{\sim}{X},o) \quad , \quad \underset{\sim}{\xi}(\underset{\sim}{X},o) = \overline{\underset{\sim}{\xi}}(\underset{\sim}{X},o) \quad , \quad \underset{\sim}{X}\in\overline{R}. \qquad (4.43)$$

<u>Then</u> $(\overline{\underset{\sim}{x}},\overline{\theta},\overline{\underset{\sim}{\xi}})(\underset{\sim}{X},t)$ <u>and</u> $(\underset{\sim}{x},\theta,\underset{\sim}{\xi})(\underset{\sim}{X},t)$ <u>coincide on</u> $\overline{R}\times[o,t_o]$.

5. HEAT-CONDUCTING ELASTIC MATERIALS

In this section we shall investigate the uniqueness and continuous dependence of smooth admissible thermodynamic processes upon initial data and supply terms for a heat-conducting elastic material. First, we shall establish an estimate similar to (4.6) in the case of a definite conductor elastic material. Further, we will use this estimate in order to establish the uniqueness and continuous dependence results.

For a heat-conducting elastic material, we have

$$Q_A(\mathbf{F},\theta,\mathbf{g},\boldsymbol{\xi};\mathbf{X}) - Q_A(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\boldsymbol{\xi}};\mathbf{X}) = -\overline{K}_{AB}(g_B-\overline{g}_B) - \overline{L}_A(\theta-\overline{\theta}) -$$

$$- \overline{M}_{AiB}(F_{iB}-\overline{F}_{iB}) - \overline{N}_{A\alpha}(\xi_\alpha-\overline{\xi}_\alpha) + o_A(|\mathbf{F}-\overline{\mathbf{F}}|^2+|\theta-\overline{\theta}|^2+|\mathbf{g}-\overline{\mathbf{g}}|^2+|\boldsymbol{\xi}-\overline{\boldsymbol{\xi}}|^2), \tag{5.1}$$

where

$$\overline{K}_{AB} = K_{AB}(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\boldsymbol{\xi}};\mathbf{X}) = -\frac{\partial Q_A^*}{\partial g_B}(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\boldsymbol{\xi}};\mathbf{X}) \quad ,$$

$$\overline{L}_A = L_A(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\boldsymbol{\xi}};\mathbf{X}) = -\frac{\partial Q_A^*}{\partial \theta}(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\boldsymbol{\xi}};\mathbf{X}) \quad ,$$

$$\overline{M}_{AiB} = M_{AiB}(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\boldsymbol{\xi}};\mathbf{X}) = -\frac{\partial Q_A^*}{\partial F_{iB}}(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\boldsymbol{\xi}};\mathbf{X}) \quad ,$$

$$\overline{N}_{A\alpha} = N_{A\alpha}(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\boldsymbol{\xi}};\mathbf{X}) = -\frac{\partial Q_A^*}{\partial \xi_\alpha}(\overline{\mathbf{F}},\overline{\theta},\overline{\mathbf{g}},\overline{\boldsymbol{\xi}};\mathbf{X}). \tag{5.2}$$

We will say that a smooth admissible thermodynamic process $(\overline{\mathbf{x}},\overline{\theta},\overline{\boldsymbol{\xi}})(\mathbf{X},t)$ *resides in the region of state space where the elastic material is a definite conductor of heat if*

$$\overline{k}_{AB} = \frac{1}{2}(\overline{K}_{AB}+\overline{K}_{BA}) \quad \text{is positive definite.} \tag{5.3}$$

The notion of a definite conductor elastic material is an essential ingredient in researches on the propagation of acceleration waves in heat-conducting elastic materials (see e.g. [8] , [24]-[27]), as well as in researches on the uniqueness and continuous data dependence of thermodynamic processes in heat-conducting elastic materials [21] .

For a definite conductor of heat elastic material, we first establish a suitable estimate of the last term in evolutionary identity (3.2). Thus, we have

Theorem 5.1. Let $(\bar{x},\bar{\theta},\bar{\xi})(X,t)$ be a smooth admissible thermodynamic process defined on $R\times[0,t_0]$, residing in the region of state space where the elastic material is a definite conductor of heat. Then there are positive constants δ_2 , c_1 and c_2 with the following property:

If $(x,\theta,\xi)(X,t)$ is any smooth admissible process defined on $R\times[0,t_0]$ such that

$$|F(X,t)-\bar{F}(X,t)| + |\theta(X,t)-\bar{\theta}(X,t)| + |g(X,t)-\bar{g}(X,t)| +$$

$$+ |\dot{\xi}(X,t)-\dot{\bar{\xi}}(X,t)| < \delta_2 , \quad (X,t)\in R\times[0,t_0] , \tag{5.4}$$

then we have, for any $t\in[0,t_0]$,

$$\int_R \left\{(Q_A-\bar{Q}_A)\left(\frac{\theta-\bar{\theta}}{\theta}\right)_{,A} - \frac{1}{\theta}\sigma_\alpha(\theta-\bar{\theta})(f_\alpha-\bar{f}_\alpha) + (\sigma_\alpha-\bar{\sigma}_\alpha)(f_\alpha-\bar{f}_\alpha)\right\}(X,t)\,dX \leq$$

$$\leq - c_1\|(g-\bar{g})(\cdot,t)\|^2_{L^2(R)} + c_2\|(F-\bar{F},\theta-\bar{\theta},\xi-\bar{\xi})(\cdot,t)\|^2_{L^2(R)} . \tag{5.5}$$

Proof. Taking into account the relations $(2.6)_5$ and (2.1o) , we get

$$f_\alpha-\bar{f}_\alpha = \bar{p}_{\alpha iA}(F_{iA}-\bar{F}_{iA}) + \bar{q}_\alpha(\theta-\bar{\theta}) + \bar{r}_{\alpha A}(g_A-\bar{g}_A) + \bar{s}_{\alpha\beta}(\xi_\beta-\bar{\xi}_\beta) +$$

$$+ \sigma_\alpha(|F-\bar{F}|^2 + |\theta-\bar{\theta}|^2 + |g-\bar{g}|^2 + |\xi-\bar{\xi}|^2) ,$$

$$\sigma_\alpha-\bar{\sigma}_\alpha = \bar{\lambda}_{\alpha iA}(F_{iA}-\bar{F}_{iA}) + \bar{\mu}_\alpha(\theta-\bar{\theta}) + \bar{\nu}_{\alpha\beta}(\xi_\beta-\bar{\xi}_\beta) +$$

$$+ \sigma_\alpha(|F-\bar{F}|^2 + |\theta-\bar{\theta}|^2 + |\xi-\bar{\xi}|^2) , \tag{5.6}$$

where

$$\bar{p}_{\alpha iA} = \frac{\partial f_\alpha}{\partial F_{iA}}(\bar{\mathbb{F}},\bar{\theta},\bar{g},\bar{\xi};X) \quad , \quad \bar{q}_\alpha = \frac{\partial f_\alpha}{\partial \theta}(\bar{\mathbb{F}},\bar{\theta},\bar{g},\bar{\xi};X) \quad ,$$

$$\bar{r}_{\alpha A} = \frac{\partial f_\alpha}{\partial g_A}(\bar{\mathbb{F}},\bar{\theta},\bar{g},\bar{\xi};X) \quad , \quad \bar{s}_{\alpha\beta} = \frac{\partial f_\alpha}{\partial \xi_\beta}(\bar{\mathbb{F}},\bar{\theta},\bar{g},\bar{\xi};X) \quad ,$$

$$\bar{\lambda}_{\alpha iA} = \frac{\partial \sigma_\alpha}{\partial F_{iA}}(\bar{\mathbb{F}},\bar{\theta},\bar{\xi};X) \; , \; \bar{\mu}_\alpha = \frac{\partial \sigma_\alpha}{\partial \theta}(\bar{\mathbb{F}},\bar{\theta},\bar{\xi};X) \; , \; \bar{\nu}_{\alpha\beta} = \frac{\partial \sigma_\alpha}{\partial \xi_\beta}(\bar{\mathbb{F}},\bar{\theta},\bar{\xi};X) . \quad (5.7)$$

Since $(\bar{x},\bar{\theta},\bar{\xi})(X,t)$ resides in the region of state space where the elastic material has the behaviour of a definite conductor of heat, it follows that there is a positive constant Ω such that,

$$\int_R \left[\frac{1}{\theta}\bar{K}_{AB}(g_A-\bar{g}_A)(g_B-\bar{g}_B)\right](X,t)dX =$$

$$= \int_R \left[\frac{1}{\bar{\theta}}\bar{K}_{AB}(g_A-\bar{g}_A)(g_B-\bar{g}_B)\right](X,t)dX \geq \Omega \,\|(g-\bar{g})(\cdot,t)\|^2_{L^2(R)} . \quad (5.8)$$

By virtue of the relations (5.1) and (5.6) and the above inequality, we conclude that there is a positive constant δ_2 with the property that, when (5.4) is satisfied,

$$\int_R \left\{(Q_A-\bar{Q}_A)\left(\frac{\theta-\bar{\theta}}{\theta}\right)_{,A} - \frac{1}{\theta}\sigma_\alpha(\theta-\bar{\theta})(f_\alpha-\bar{f}_\alpha) + (\sigma_\alpha-\bar{\sigma}_\alpha)(f_\alpha-\bar{f}_\alpha)\right\}(X,t)dX \leq$$

$$\leq -\Omega\,\|(g-\bar{g})(\cdot,t)\|^2_{L^2(R)} + \int_R \{\mathcal{L}_A(g_A-\bar{g}_A)(\theta-\bar{\theta}) + \mathcal{M}_{AiB}(g_A-\bar{g}_A)(F_{iB}-\bar{F}_{iB}) +$$

$$+ \mathcal{N}_{A\alpha}(g_A-\bar{g}_A)(\xi_\alpha-\bar{\xi}_\alpha)\}(X,t)dX + \int_R \{\mathcal{A}_{iAjB}(F_{iA}-\bar{F}_{iA})(F_{jB}-\bar{F}_{jB}) +$$

$$+ \mathcal{B}(\theta-\bar{\theta})^2 + \mathcal{C}_{\alpha\beta}(\xi_\alpha-\bar{\xi}_\alpha)(\xi_\beta-\bar{\xi}_\beta) + \mathcal{D}_{iA}(F_{iA}-\bar{F}_{iA})(\theta-\bar{\theta}) +$$

$$+ \mathcal{E}_{iA\alpha}(F_{iA}-\bar{F}_{iA})(\xi_\alpha-\bar{\xi}_\alpha) + \mathcal{F}_\alpha(\xi_\alpha-\bar{\xi}_\alpha)(\theta-\bar{\theta})\}(X,t)dX \quad , \quad (5.9)$$

where we have used the notations

$$\mathcal{L}_A = -\frac{1}{\theta}\overline{L}_A + \frac{1}{\theta^2}\overline{K}_{BA}g_B + \overline{r}_{\alpha A}(\overline{\mu}_\alpha - \frac{1}{\theta}\sigma_\alpha) \ , \quad \mathcal{M}_{AiB} = -\frac{1}{\theta}\overline{M}_{AiB} + \overline{\lambda}_{\alpha iB}\overline{r}_{\alpha A} \ ,$$

$$\mathcal{N}_{A\alpha} = -\frac{1}{\theta}\overline{N}_{A\alpha} + \overline{\nu}_{\beta\alpha}\overline{r}_{\beta A} \ , \qquad \mathcal{A}_{iAjB} = \frac{1}{2}(\overline{p}_{\alpha iA}\overline{\lambda}_{\alpha jB} + \overline{p}_{\alpha jB}\overline{\lambda}_{\alpha iA}) \ ,$$

$$\mathcal{B} = \frac{1}{\theta^2}\overline{L}_A g_A + \overline{q}_\alpha(\overline{\mu}_\alpha - \frac{1}{\theta}\sigma_\alpha) \ , \qquad \mathcal{C}_{\alpha\beta} = \frac{1}{2}(\overline{s}_{\gamma\alpha}\overline{\nu}_{\gamma\beta} + \overline{s}_{\gamma\beta}\overline{\nu}_{\gamma\alpha}) \ ,$$

$$\mathcal{D}_{iA} = \frac{1}{\theta^2}\overline{M}_{BiA}g_B + \overline{\lambda}_{\alpha iA}\overline{q}_\alpha + \overline{p}_{\alpha iA}(\overline{\mu}_\alpha - \frac{1}{\theta}\sigma_\alpha) \ ,$$

$$\mathcal{E}_{iA\alpha} = \overline{\nu}_{\beta\alpha}\overline{p}_{\beta iA} + \overline{s}_{\beta\alpha}\overline{\lambda}_{\beta iA} \ , \quad \mathcal{F}_\alpha = \frac{1}{\theta^2}\overline{N}_{A\alpha}g_A + \overline{q}_\beta\overline{\nu}_{\beta\alpha} + \overline{s}_{\beta\alpha}(\overline{\mu}_\beta - \frac{1}{\theta}\sigma_\beta). \qquad (5.10)$$

Application of the Schwarz inequality and the inequality (4.11) to the two last terms in (5.9) yields, for arbitrary constants γ_1, γ_2 and γ_3,

$$2\int_R \{(Q_A - \overline{Q}_A)(\frac{\theta - \overline{\theta}}{\theta})_{,A} - \frac{1}{\theta}\sigma_\alpha(\theta - \overline{\theta})(f_\alpha - \overline{f}_\alpha) + (\sigma_\alpha - \overline{\sigma}_\alpha)(f_\alpha - \overline{f}_\alpha)\}(\underset{\sim}{X}, t)\,dX \leq$$

$$\leq (-2\Omega + \gamma_1^2 + \gamma_2^2 + \gamma_3^2)\|(\underset{\sim}{g} - \overline{\underset{\sim}{g}})(\cdot, t)\|^2_{L^2(R)} + (\frac{c_3^2}{\gamma_1^2} + c_4^2 + 2)\|(\theta - \overline{\theta})(\cdot, t)\|^2_{L^2(R)} +$$

$$+ (\frac{c_5^2}{\gamma_2^2} + c_6^2 + c_7^2 + c_8^2)\|(\underset{\sim}{F} - \overline{\underset{\sim}{F}})(\cdot, t)\|^2_{L^2(R)} +$$

$$+ (\frac{c_9^2}{\gamma_3^2} + c_{10}^2 + c_{11}^2 + 1)\|(\underset{\sim}{\xi} - \overline{\underset{\sim}{\xi}})(\cdot, t)\|^2_{L^2(R)} \ , \qquad (5.11)$$

where

$$c_3 = \max|\mathcal{L}(\underset{\sim}{X}, t)| \ , \quad c_4^2 = 2\max|\mathcal{B}(\underset{\sim}{X}, t)| \ , \quad c_5 = \max|\mathcal{M}(\underset{\sim}{X}, t)| \ ,$$

$$c_6^2 = 2\max|\mathcal{A}(\underset{\sim}{X}, t)| \ , \quad c_7 = \max|\mathcal{D}(\underset{\sim}{X}, t)| \ , \quad c_8 = \max|\mathcal{E}(\underset{\sim}{X}, t)| \ ,$$

$$c_9 = \max|\mathcal{N}(\underset{\sim}{X}, t)| \ , \quad c_{10}^2 = 2\max|\mathcal{C}(\underset{\sim}{X}, t)| \ , \quad c_{11} = \max|\mathcal{F}(\underset{\sim}{X}, t)| \ . \qquad (5.12)$$

From (5.11) we obtain the estimate (5.5) with

$$c_1 = \frac{1}{2}\left[2\Omega - (\gamma_1^2 + \gamma_2^2 + \gamma_3^2)\right] ,$$

$$c_2 = \frac{1}{2} \max\left(\frac{c_3^2}{\gamma_1^2} + c_4^2 + 2 , \frac{c_5^2}{\gamma_2^2} + c_6^2 + c_7^2 + c_8^2 , \frac{c_9^2}{\gamma_3^2} + c_{1o}^2 + c_{11}^2 + 1 \right). \tag{5.13}$$

Now, we can choose the arbitrary constants γ_1, γ_2 and γ_3 so that $c_1 > o$. The proof of the theorem is complete.

On the basis of the above result we shall establish an estimate similar to (4.6). Thus, we have

Theorem 5.2. Let $(\bar{x}, \bar{\theta}, \bar{\xi})(X, t)$ be a smooth admissible thermodynamic process defined on $R \times [o, t_o]$, residing in the region of state space where the elastic material is a definite conductor of heat. We suppose that corresponding supply terms $(\bar{b}, \bar{r})(X, t) \in L^\infty(R \times [o, t_o])$. Then there are positive constants δ_3 and d_1, d_2, d_3, d_4 with the following property:

If $(x, \theta, \xi)(X, t)$ is any smooth admissible process defined on $R \times [o, t_o]$, with supply terms $(b, r)(X, t) \in L^\infty(R \times [o, t_o])$, and such that

$$|F(X,t) - \bar{F}(X,t)| + |\theta(X,t) - \bar{\theta}(X,t)| + |\xi(X,t) - \bar{\xi}(X,t)| +$$
$$+ |g(X,t) - \bar{g}(X,t)| < \delta_3 , \quad (X,t) \in R \times [o, t_o] , \tag{5.14}$$

$$(v_i - \bar{v}_i)(T_{Ai} - \bar{T}_{Ai})N_A \leq o , \quad -(\theta - \bar{\theta})(Q_A - \bar{Q}_A)N_A \leq o , \quad \text{on} \quad \partial R \times [o, t_o] , \tag{5.15}$$

then we have, for any $\tau \in [o, s]$, $s \in [o, t_o]$,

$$\int_R \rho(X)\left\{(v_i - \bar{v}_i)(v_i - \bar{v}_i) + \frac{\partial^2 \psi^*(\bar{F}, \bar{\theta}, \bar{\xi}; X)}{\partial F_{iA} \partial F_{jB}}(F_{iA} - \bar{F}_{iA})(F_{jB} - \bar{F}_{jB}) - \right.$$
$$\left. - \frac{\partial^2 \psi^*(\bar{F}, \bar{\theta}, \bar{\xi}; X)}{\partial \theta^2}(\theta - \bar{\theta})^2\right\}(X, \tau)\, dX + d_1 \|(\xi - \bar{\xi})(\cdot, \tau)\|^2_{L^2(R)} -$$

$$- \pi_1'^2 \|(\underset{\sim}{F}-\overline{\underset{\sim}{F}})(\cdot,\tau)\|^2_{L^2(R)} \leq d_2 \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}}, \underset{\sim}{F}-\overline{\underset{\sim}{F}}, \theta-\overline{\theta}, \underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,o)\|^2_{L^2(R)} +$$

$$+ d_3 \int_0^\tau \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}}, \underset{\sim}{F}-\overline{\underset{\sim}{F}}, \theta-\overline{\theta}, \underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,t)\|^2_{L^2(R)} dt +$$

$$+ d_4 \int_0^\tau \|(\underset{\sim}{b}-\overline{\underset{\sim}{b}}, r-\overline{r})(\cdot,t)\|_{L^2(R)} \cdot \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}}, \underset{\sim}{F}-\overline{\underset{\sim}{F}}, \theta-\overline{\theta}, \underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,t)\|_{L^2(R)} dt \quad , \qquad (5.16)$$

for arbitrary constant π_1'.

Proof. The relations (2.8) and (2.1o) and Theorem 5.1, as well as the boundary conditions (5.15) imply that there are positive constants δ_2 and d_5, d_6 and d_7 such that, whenever (5.4) holds, we have

$$\dot{D}(t) \leq d_5 \|(\underset{\sim}{F}-\overline{\underset{\sim}{F}}, \theta-\overline{\theta}, \underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,t)\|^2_{L^2(R)} + d_6 \|(\underset{\sim}{b}-\overline{\underset{\sim}{b}}, r-\overline{r})(\cdot,t)\|_{L^2(R)} \cdot$$

$$\cdot \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}}, \theta-\overline{\theta})(\cdot,t)\|_{L^2(R)} - d_7 \|(\underset{\sim}{g}-\overline{\underset{\sim}{g}})(\cdot,t)\|^2_{L^2(R)} \quad , \quad t \in [o,t_o] . \qquad (5.17)$$

We now proceed as in the proof of Theorem 4.1. We fix $s \in [o,t_o]$, integrate (5.17) over $[o,\tau]$, $\tau \in [o,s]$, such that we obtain

$$D(\tau) \leq D(o) + d_5 \int_0^\tau \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}}, \underset{\sim}{F}-\overline{\underset{\sim}{F}}, \theta-\overline{\theta}, \underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,t)\|^2_{L^2(R)} dt +$$

$$+ d_6 \int_0^\tau \|(\underset{\sim}{b}-\overline{\underset{\sim}{b}}, r-\overline{r})(\cdot,t)\|_{L^2(R)} \cdot \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}}, \underset{\sim}{F}-\overline{\underset{\sim}{F}}, \theta-\overline{\theta}, \underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot,t)\|_{L^2(R)} dt -$$

$$- d_7 \int_0^\tau \|(\underset{\sim}{g}-\overline{\underset{\sim}{g}})(\cdot,t)\|^2_{L^2(R)} dt. \qquad (5.18)$$

Combining (4.9) with (3.1), we conclude that there is a positive constant δ_1 with the property that, when (4.4) is satisfied, the inequality (4.1o) holds. We choose $\delta_3 = \min(\delta_1, \delta_2)$ in (5.14). By using the estimate (5.18) into (4.1o), the application of the Schwarz inequality and the inequality (4.11) to the last terms in (4.1o) yields

$$\int_R \rho(\underset{\sim}{X})\Big\{(v_i-\bar v_i)(v_i-\bar v_i) + \frac{\partial^2\psi^*(\underset{\sim}{\bar F},\bar\theta,\underset{\sim}{\bar\xi};\underset{\sim}{X})}{\partial F_{iA}\,\partial F_{jB}}(F_{iA}-\bar F_{iA})(F_{jB}-\bar F_{jB}) -$$

$$- \frac{\partial^2\psi^*(\underset{\sim}{\bar F},\bar\theta,\underset{\sim}{\bar\xi};X)}{\partial\theta^2}(\theta-\theta)^2\Big\}(\underset{\sim}{X},\tau)\,dX \le 2D(o) + 2d_5\int_0^\tau \|(\underset{\sim}{v}-\underset{\sim}{\bar v},\underset{\sim}{F}-\underset{\sim}{\bar F},\theta-$$

$$-\theta,\underset{\sim}{\xi}-\underset{\sim}{\bar\xi})(\cdot,t)\|^2_{L^2(R)}dt + 2d_6\int_0^\tau \|(\underset{\sim}{b}-\underset{\sim}{\bar b},r-\bar r)(\cdot,t)\|_{L^2(R)}\cdot$$

$$\cdot\|(\underset{\sim}{v}-\underset{\sim}{\bar v},\underset{\sim}{F}-\underset{\sim}{\bar F},\theta-\bar\theta,\underset{\sim}{\xi}-\underset{\sim}{\bar\xi})(\cdot,t)\|_{L^2(R)}dt - 2d_7\int_0^\tau \|(\underset{\sim}{g}-\underset{\sim}{\bar g})(\cdot,t)\|^2_{L^2(R)}dt +$$

$$+ \Big(a_7^2 + \frac{a_8^2}{\pi_1'^2}\Big)\|(\underset{\sim}{\xi}-\underset{\sim}{\bar\xi})(\cdot,\tau)\|^2_{L^2(R)} + \pi_1'^2\|(\underset{\sim}{F}-\underset{\sim}{\bar F})(\cdot,\tau)\|^2_{L^2(R)}. \qquad (5.19)$$

On the basis of the relation (4.14) we deduce

$$\|(\underset{\sim}{\xi}-\underset{\sim}{\bar\xi})(\cdot,\tau)\|^2_{L^2(R)} = \|(\underset{\sim}{\xi}-\underset{\sim}{\bar\xi})(\cdot,o)\|^2_{L^2(R)} +$$

$$+ 2\int_0^\tau\int_R \Big[(\xi_\alpha-\bar\xi_\alpha)(f_\alpha-\bar f_\alpha)\Big](\underset{\sim}{X},t)\,dX\,dt. \qquad (5.2o)$$

We use the relation (5.6) into (5.2o). Then, the application of the Schwarz inequality and the inequality (4.11) implies that for arbitrary constants $\pi_2' > o$ and π_3', we have

$$\pi_2'\|(\underset{\sim}{\xi}-\underset{\sim}{\bar\xi})(\cdot,\tau)\|^2_{L^2(R)} \le \pi_2'\|(\underset{\sim}{\xi}-\underset{\sim}{\bar\xi})(\cdot,o)\|^2_{L^2(R)} + \pi_2'\pi_3'^2\int_0^\tau \|(\underset{\sim}{g}-\underset{\sim}{\bar g})(\cdot,t)\|^2_{L^2(R)}dt +$$

$$+ \pi_2' d_8\int_0^\tau \|(\underset{\sim}{F}-\underset{\sim}{\bar F},\theta-\bar\theta,\underset{\sim}{\xi}-\underset{\sim}{\bar\xi})(\cdot,t)\|^2_{L^2(R)}dt, \qquad (5.21)$$

where

$$d_8 = \max\Big(\frac{d_9^2}{\pi_3'^2} + d_{1o}^2 + d_{11}^2 + 1,\ d_{12}^2\Big),$$

$$d_9 = \max|\underset{\sim}{\bar F}(\underset{\sim}{X},t)|, \quad d_{1o} = \max|\underset{\sim}{\bar q}(\underset{\sim}{X},t)|; \quad d_{11}^2 = 2\max|\underset{\sim}{\bar s}(\underset{\sim}{X},t)|,$$

$$d_{12} = \max|\bar{p}(\underset{\sim}{X},t)| \quad , \quad \text{on} \quad \bar{R}\times[o,t_o] \;. \tag{5.22}$$

By taking into account the estimate (4.16), from (5.19) and (5.21), we get

$$\int_R \rho(\underset{\sim}{X})\Big\{(v_i-\bar{v}_i)(v_i-\bar{v}_i) + \frac{\partial^2\psi^*(\bar{\underset{\sim}{F}},\bar{\theta},\bar{\underset{\sim}{\xi}};\underset{\sim}{X})}{\partial F_{iA}\partial F_{jB}}(F_{iA}-\bar{F}_{iA})(F_{jB}-\bar{F}_{jB}) -$$

$$- \frac{\partial^2\psi^*(\bar{\underset{\sim}{F}},\bar{\theta},\bar{\underset{\sim}{\xi}};\underset{\sim}{X})}{\partial\theta^2}(\theta-\bar{\theta})^2\Big\}(\underset{\sim}{X},\tau)\,dX + d_1\|(\underset{\sim}{\xi}-\bar{\underset{\sim}{\xi}})(\cdot,\tau)\|^2_{L^2(R)} -$$

$$- \pi_1'^2\|(\underset{\sim}{F}-\bar{\underset{\sim}{F}})(\cdot,\tau)\|^2_{L^2(R)} \leq d_2\|(\underset{\sim}{v}-\bar{\underset{\sim}{v}},\underset{\sim}{F}-\bar{\underset{\sim}{F}},\theta-\bar{\theta},\underset{\sim}{\xi}-\bar{\underset{\sim}{\xi}})(\cdot,o)\|^2_{L^2(R)} + \tag{5.23}$$

$$+ d_3\int_0^\tau\|(\underset{\sim}{v}-\bar{\underset{\sim}{v}},\underset{\sim}{F}-\bar{\underset{\sim}{F}},\theta-\bar{\theta},\underset{\sim}{\xi}-\bar{\underset{\sim}{\xi}})(\cdot,t)\|^2_{L^2(R)}dt + d_4\int_0^\tau\|(\underset{\sim}{b}-\bar{\underset{\sim}{b}},r-\bar{r})(\cdot,t)\|_{L^2(R)}\cdot$$

$$\cdot\|(\underset{\sim}{v}-\bar{\underset{\sim}{v}},\underset{\sim}{F}-\bar{\underset{\sim}{F}},\theta-\bar{\theta},\underset{\sim}{\xi}-\bar{\underset{\sim}{\xi}})(\cdot,t)\|_{L^2(R)}dt - (2d_7-\pi_2'\pi_3'^2)\int_0^\tau\|(\underset{\sim}{\xi}-\bar{\underset{\sim}{\xi}})(\cdot,t)\|^2_{L^2(R)}dt \;,$$

where

$$d_1 = \pi_2' - \Big(a_7^2 + \frac{a_8^2}{\pi_1'^2}\Big) \;,\; d_2 = 2a_{10} + \pi_2' \;,\; d_3 = 2d_5 + \pi_2' d_8 \;,\; d_4 = 2d_6 . \tag{5.24}$$

We now choose the arbitrary constants $\pi_2' > o$ and π_3' such that $d_1 > o$ and $2d_7 - \pi_2'\pi_3'^2 > o$. This implies the estimate (5.16) and the proof is complete.

As in Section 4, we now establish the uniqueness and continuous dependence results.

a) <u>Continuous dependence in a neighborhood of a positive smooth admissible process.</u>

<u>Theorem 5.3. Let</u> $(\bar{\underset{\sim}{x}},\bar{\theta},\bar{\underset{\sim}{\xi}})(\underset{\sim}{X},t)$ <u>and</u> $(\underset{\sim}{x},\theta,\underset{\sim}{\xi})(\underset{\sim}{X},t)$ <u>be as in Theorem 5.2. We assume that the smooth admissible thermodynamic process</u> $(\bar{\underset{\sim}{x}},\bar{\theta},\bar{\underset{\sim}{\xi}})(\underset{\sim}{X},t)$ <u>is positive. Then there are positive constants</u> δ_3, α_1', M_1' <u>and</u> N_1' <u>with the property that,</u>

whenever (5.14) holds, for any $s \in [0, t_0]$,

$$\|(\underset{\sim}{v}-\overline{\underset{\sim}{v}}, \underset{\sim}{F}-\overline{\underset{\sim}{F}}, \theta-\overline{\theta}, \underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot, s)\|_{L^2(R)} \leq M_1' e^{\alpha_1' s} \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}}, \underset{\sim}{F}-\overline{\underset{\sim}{F}}, \theta-\overline{\theta}, \underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot, 0)\|_{L^2(R)} +$$

$$+ N_1' e^{\alpha_1' s} \int_0^s \|(\underset{\sim}{b}-\overline{\underset{\sim}{b}}, r-\overline{r})(\cdot, t)\|_{L^2(R)} dt. \qquad (5.25)$$

Theorem 5.4. Let $(\overline{\underset{\sim}{x}}, \overline{\theta}, \overline{\underset{\sim}{\xi}})(\underset{\sim}{X}, t)$ and $(\underset{\sim}{x}, \theta, \underset{\sim}{\xi})(\underset{\sim}{X}, t)$ be as in Theorem 5.3. We assume that the corresponding supply terms $(\overline{\underset{\sim}{b}}, \overline{r})(\underset{\sim}{X}, t)$ and $(\underset{\sim}{b}, r)(\underset{\sim}{X}, t)$ coincide on $R \times [0, t_0]$ and that both processes originate from the same state, that is,

$$\underset{\sim}{x}(\underset{\sim}{X}, 0) = \overline{\underset{\sim}{x}}(\underset{\sim}{X}, 0), \quad \underset{\sim}{v}(\underset{\sim}{X}, 0) = \overline{\underset{\sim}{v}}(\underset{\sim}{X}, 0),$$

$$\theta(\underset{\sim}{X}, 0) = \overline{\theta}(\underset{\sim}{X}, 0), \quad \underset{\sim}{\xi}(\underset{\sim}{X}, 0) = \overline{\underset{\sim}{\xi}}(\underset{\sim}{X}, 0), \quad \text{on } \overline{R}. \qquad (5.26)$$

Then $(\overline{\underset{\sim}{x}}, \overline{\theta}, \overline{\underset{\sim}{\xi}})(\underset{\sim}{X}, t)$ and $(\underset{\sim}{x}, \theta, \underset{\sim}{\xi})(\underset{\sim}{X}, t)$ coincide on $\overline{R} \times [0, t_0]$.

b) Continuous dependence of smooth admissible processes in the strong ellipticity region.

Theorem 5.5. Let $(\overline{\underset{\sim}{x}}, \overline{\theta}, \overline{\underset{\sim}{\xi}})(\underset{\sim}{X}, t)$ and $(\underset{\sim}{x}, \theta, \underset{\sim}{\xi})(\underset{\sim}{X}, t)$ be as in Theorem 5.2. Moreover, we assume that

$$\underset{\sim}{x}(\underset{\sim}{X}, t) = \overline{\underset{\sim}{x}}(\underset{\sim}{X}, t), \quad \text{on } \partial R \times [0, t_0], \qquad (5.27)$$

and that the smooth admissible thermodynamic process $(\overline{\underset{\sim}{x}}, \overline{\theta}, \overline{\underset{\sim}{\xi}})(\underset{\sim}{X}, t)$ resides in the strong ellipticity region. Then there are positive constants δ_3, α_2', β_2', M_2' and N_2' with the property that, whenever (5.14) holds, for any $s \in [0, t_0]$,

$$\|(\underset{\sim}{v}-\overline{\underset{\sim}{v}}, \underset{\sim}{F}-\overline{\underset{\sim}{F}}, \theta-\overline{\theta}, \underset{\sim}{\xi}-\overline{\underset{\sim}{\xi}})(\cdot, s)\|_{L^2(R)} \leq M_2' \exp(\alpha_2' s + \beta_2' s^2) \|(\underset{\sim}{v}-\overline{\underset{\sim}{v}}, \underset{\sim}{F}-\overline{\underset{\sim}{F}}, \theta-\overline{\theta}, \underset{\sim}{\xi}-$$

$$-\overline{\underset{\sim}{\xi}})(\cdot, 0)\|_{L^2(R)} + N_2' \exp(\alpha_2' s + \beta_2' s^2) \int_0^s \|(\underset{\sim}{b}-\overline{\underset{\sim}{b}}, r-\overline{r})(\cdot, t)\|_{L^2(R)} dt. \qquad (5.28)$$

Theorem 5.6. Let $(\bar{x},\bar{\theta},\bar{\xi})(X,t)$ and $(x,\theta,\xi)(X,t)$ be as in Theorem 5.5. We assume that the corresponding supply terms $(\bar{b},\bar{r})(X,t)$ and $(b,r)(X,t)$ coincide on $R\times[o,t_o]$ and that both processes originate from the same state, that is,

$$x(X,o) = \bar{x}(X,o) \quad , \quad v(X,o) = \bar{v}(X,o) \quad ,$$

$$\theta(X,o) = \bar{\theta}(X,o) \quad , \quad \xi(X,o) = \bar{\xi}(X,o) \quad , \quad \text{on} \quad \bar{R}.$$

Then $(\bar{x},\bar{\theta},\bar{\xi})(X,t)$ and $(x,\theta,\xi)(X,t)$ coincide on $\bar{R}\times[o,t_o]$.

We omit the proofs for Theorems 5.3-5.6 , which are identical with those in Theorems 4.2-4.5.

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REFERENCES

1. Coleman, B. D. and M. E. Gurtin, Thermodynamics with internal state variables. J. Chem. Phys. 47(1967) 597-613.
2. Bowen, R. M., On the thermochemistry of reacting materials. J. Chem. Phys. 49(1968) 1625-1637.
3. Cristescu, N. and I. Suliciu, Viscoplasticity (in Romanian), Editura tehnică, Bucharest, 1976.
4. Truesdell, C. A., Rational thermodynamics. A course of lectures on selected topics. McGraw-Hill, New York, 1969.
5. Valanis, K. C., Proper tensorial formulation of the internal variable theory. The endochronic time spectrum. Arch. of Mechanics 29(1977) 173-185.
6. Kratochvil, J., Meaning of internal variables in plasticity. Lett. Appl. Engng Sci. 16(1978) 4o3-411.
7. Nachlinger, R. R. and J. W. Nunziato, Wave propagation and uniqueness theorems for elastic materials with internal state variables. Int. J. Engng Sci. 14(1976) 31-38.
8. Bowen, R. M. and C. -C. Wang, Thermodynamic influences on acceleration waves in inhomogeneous isotropic elastic bodies with internal state variables. Arch. Rational Mech. Analysis 41(1971) 287-318.
9. Bowen, R. M. and P. J. Chen, Acceleration waves in anisotropic thermoelastic materials with internal state variables. Acta Mechanica 15(1972) 95-1o4.
1o. Suliciu, I., Symmetric waves in materials with internal state variables. Arch. of Mechanics 27(1975) 841-856.
11. Suliciu, I., On the existence and symmetry of acceleration waves in materials with internal state variables. Mech. Res. Comm. 2 (1975) 79-83.

12. Mihăilescu, M. and I. Suliciu, Finite and symmetric thermomechanical waves in materials with internal state variables. Int. J. Solids Structures 12(1976) 559-575.

13. Suliciu, I. and M. Mihăilescu-Suliciu, On the semilinear theory for materials with internal state variables. An. St. Univ. Iaşi, Matematică, 24(1978) 4o3-416.

14. Kosinski, W. and P. Perzyna, Analysis of acceleration waves in materials with internal parameters. Arch. of Mechanics 24(1972) 629-643.

15. Kosinski, W., Behaviour of acceleration and shock waves in materials with internal state variables. Int. J. Nonlinear Mech. 9 (1974) 481-499.

16. Kosinski, W., On the global behaviour of one-dimensional acceleration waves in a material with internal variables. Arch. of Mechanics 27(1975) 231-243.

17. Gurtin, M. E., Thermodynamics and the energy criterion for stability. Arch. Rational Mech. Analysis 52(1973) 93-1o3.

18. Gurtin, M. E., Thermodynamics and stability. Arch. Rational Mech. Analysis 59(1975) 63-96.

19. Kosinski, W., A uniqueness theorem for the dynamic initial-displacement boundary-value problem in the theory with internal state variables. Quart. Appl. Math. 38(198o) 129-134.

2o. Dafermos, C. M., The second law of thermodynamics and stability. Arch. Rational Mech. Analysis 7o(1979) 167-179.

21. Chiriţă, S., Uniqueness and continuous data dependence in dynamical problems of nonlinear thermoelasticity. J. Thermall Stresses (forthcoming).

22. Fichera, G., Existence theorems in elasticity. Handbuch der Physik VIa/2. Berlin, Heidelberg, New York, Springer, 1972.

23. Wheeler, L., A uniqueness theorem for the displacement problem in finite elastodynamics. Arch. Rational Mech. Analysis 63(1977) 183-189.

24. Coleman, B. D. and M. E. Gurtin, Waves in materials with memory. III. Thermodynamic influences in the growth and decay of acceleration waves. Arch. Rational Mech. Analysis 19(1965) 266-298.

25. Coleman, B. D. and M. E. Gurtin, Waves in materials with memory. IV. Thermodynamics and the velocity of three-dimensional acceleration waves. Arch. Rational Mech. Analysis 19(1965) 317-338.

26. Chen, P. J., Thermodynamic influences on the propagation and the growth of acceleration waves in elastic materials. Arch. Rational Mech. Analysis 31(1968) 228-254.

27. Chadwick, P. and P. K. Currie, The propagation and growth of acceleration waves in heat-conducting elastic materials. Arch. Rational Mech. Analysis 49(1972) 137-158.

28. Gurtin, M. E. and S. J. Spector, On stability and uniqueness in finite elasticity. Arch. Rational Mech. Analysis 7o(1979)153-165.