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INTERPOLATION THEOREMS FOR REARRANGEMENT INVARIANT p-SPACES OF FUNCTIONS, for $0<p<1$

by Nicolae Popa

Abstract In this paper we extend two interpolation theorems in the setting of rearrangemant invariant p-spaces, for $0<p<1$. Some applications of these theorems are given, particularly we extend Theorem 2.c.6 - [3] proving that the Haar system is an unconditional basis in a rearrangement invariant p-space X iff the Boyd indidices p_X and q_X verify the relations $1<p_X$ and $q_X<\infty$.

In the sequel we assume all the vector spaces to be real. Moreover p is a real positive number between 0 and 1. We use often notions introduced in [4] .without additional explanation.

Let X be a rearrangement invariant (r.i.) p-space on I, where I is either $[0,1]$ or $[0,\infty)$.

Interesting examples of r.i. p-spaces are p-Orlicz and p-Lorentz spaces.

Let M be a p-Orlicz function on $[0,\infty)$, i.e. a continuous non decreasing and p-convexe (that is $M[(\alpha x^p+\beta y^p)^{1/p}]\leq \leq\alpha M(x)+\beta M(y)$, where $x,y\in[0,\infty)$, $\alpha,\beta\geq 0$ such that $\alpha+\beta=1$) function such that $M(0)=0$, $M(1)=1$ and $\lim\limits_{t\to\infty} M(t)=\infty$.

Let a be equal to 1 or to $+\infty$. The p-Orlicz space $L_M(0,a)$

is the space of all equivalence classes of Lebesgue measurable functions f on $[0,a)$ such that $\int_0^a M(|f(t)|/\rho)\,dt<\infty$ for some $\rho>0$. We consider also the subspace $H_M(0,a)=\{f:[0,a)\to\mathbb{R}\ ;\ \int_0^a M(|f(t)|/\rho)\,dt<\infty \text{ for all } \rho>0\}$. It is not difficult to prove that $L_M(0,a)$ and $H_M(0,a)$ endowed with the norm $\|f\|= =\inf\{\rho>0;\ \int_0^a M(|f(t)|/\rho)\,dt\le 1\}$ are r.i. p-spaces on $[0,a)$.

Let now $0<q<\infty$ and let W be a positive continuous non-increasing function on $(0,\infty)$ such that $\lim_{t\to 0} W(t)=\infty$, $\lim_{t\to\infty} W(t)=0$, $\int_0^1 W(t)\,dt=1$ and $\int_0^\infty W(t)\,dt=\infty$.

We denote by $L_{W,q}(0,\infty)$ the space of all Lebesgue measurable functions f on $[0,\infty)$ such that $\|f\|=\left(\int_0^\infty [f^*(t)]^q W(t)\,dt\right)^{1/q}<\infty$, where $0<p\le q<\infty$ and $f^*(t)=\inf_{\mu(E)=t}\ \sup_{s\in[0,\infty)\setminus E}|f(s)|$ (μ is the Lebesgue measure).

We can define similarly $L_{W,q}(0,1)$. $L_{W,q}(0,\infty)$ and $L_{W,q}(0,1)$ are called p-Lorentz spaces. These spaces are maximal rearrangement invariant p-spaces for $p\le q<\infty$.

The r.i. p-spaces are used to prove some interpolation theorems. For example we extend a theorem of Calderon. (See Theorem 2.a.10 - [3]).

First we introduce some auxiliary notions. Let $f,g\in L_p(0,1)$, $0<p\le 1$. We write $f \underset{p}{\prec} g$, whenever $\int_0^s [f^*(t)]^p dt\le \int_0^s [g^*(t)]^p dt$, for all $0<s\le 1$. Consequently it is extended the relation $f\prec g$ introduced on page 125 - [3], denoted in the sequel by $f \underset{1}{\prec} g$. Obviously $f \underset{p}{\prec} g \iff |f|^p \underset{1}{\prec} |g|^p$. Thus $f \underset{p}{\prec} g$ is equivalent to $|f| \underset{p}{\prec} |g|$, $f^* \underset{p}{\prec} g^*$ or $\lambda f \underset{p}{\prec} \lambda g$ for a real $\lambda\ne 0$. It is clear that $f \underset{p}{\prec} g$ and $g \underset{p}{\prec} h$ imply $f \underset{p}{\prec} h$.

$f \underset{p}{\prec} g$ and $g \underset{p}{\prec} h \iff f^*=g^*$.

Moreover $(f_1 \oplus f_2)^* \underset{p}{\prec} f_1^* \oplus f_2^*$, where $f_1 \oplus f_2 = (f_1^p + f_2^p)^{1/p}$.

Indeed, the relation above is equivalent, for $f_1, f_2 \geqslant 0$, to $(f_1^p + f_2^p)^* \underset{1}{\prec} (f_1^p)^* + (f_2^p)^*$, which is proved on page 125 -[3].

Moreover, we have the following property. <u>Assume that</u> $g \underset{p}{\prec} f_1 \oplus f_2$, <u>for positive functions</u> g, f_1, f_2. <u>Then there exist the positive functions</u> g_1, g_2 <u>such that</u> $g = g_1 \oplus g_2$ <u>and</u> $g_i \underset{p}{\prec} f_i$, $i = 1,2$.

Indeed, $g^p \underset{1}{\prec} f_1^p + f_2^p$ and, by Proposition 2.a.7 -[3], there exist $g_1', g_2' \geqslant 0$ in $L_1(0,1)$ such that $g_1' + g_2' = g^p$ and $g_i' \underset{1}{\prec} f_i^p$, $i = 1,2$. We conclude denoting $(g_i')^{1/p}$ by g_i, $i = 1,2$.

We extend now Proposition 2.a.8 -[3] for $0 < p \leqslant 1$.

<u>Proposition 1</u> <u>Let X be a r.i. p-space on</u> $[0,1]$. <u>Assume that</u> $g \underset{p}{\prec} f$ <u>and</u> $f \in X$. <u>Then</u> $g \in X$ <u>and</u> $\|g\| \leqslant \|f\|$.

<u>Proof</u> The case $p = 1$ constitute Proposition 2.a.8 -[3]. Let $0 < p < 1$. Then $g^p \underset{1}{\prec} f^p$ and, by Proposition 2.a.8 -[3] it follows that $g^p \in X_{(p)} = \{f : I \longmapsto \mathbb{R};\ f^{1/p} \in X\}$ and $\|g\|^p = |||g^p||| \leqslant$ $\leqslant |||f^p||| = \|f\|^p$. ∎

An operator T from a p-Banach space X taking values into a p-Banach lattice Y is said to be <u>quasilinear</u> if

1) $|T(\alpha x)| = |\alpha|\,|Tx|$, for all scalars α and $x \in X$.

2) There exists a constant $C < \infty$ such that
$$|T(x_1 + x_2)| \leqslant C\,(|Tx_1| + |Tx_2|),\ x_1, x_2 \in X.$$

A quasilinear operator T is <u>bounded</u> if $\|T\| = \sup\{\|Tx\|;\ \|x\| \leqslant 1\} < \infty$.

The following theorem extends Theorem 2.a.10 -[3].

<u>Theorem 2</u> <u>Let X be a r.i. p-space on</u> $[0,1]$. <u>Let T be a quasilinear operator defined on</u> $L_p(0,1)$, <u>which is simultaneously bounded on</u> $L_\infty(0,1)$ <u>and</u> $L_p(0,1)$.

<u>Then T applies X into X and moreover</u>

$$\|T\|_X \leq 2^{1/p-1} C \max\left(\|T\|_p, \|T\|_\infty\right),$$

<u>where C is the constant aforementionned.</u>

<u>Proof</u> Let $f \in X$ and $0 < s < 1$.

Put

$$g_s(t) = \begin{cases} f(t) - f^*(s) & \text{if } f(t) > f^*(s) \\ f(t) + f^*(s) & \text{if } f(t) < -f^*(s) \\ 0 & \text{if } |f(t)| \leq f^*(s) \end{cases}$$

and $h_s(t) = f(t) - g_s(t)$.

It is clear that $\|h_s\|_\infty = f^*(s)$ and denoting by

$A = \{t \in [0,1];\ f(t) > f^*(s)\}$, $B = \{t \in [0,1];\ f(t) < -f^*(s)\}$

we have $\mu(A \cup B) = \mu\{t \in [0,1];\ |f(t)| > f^*(s)\} = d_f(f^*(s)) \leq s$.

Hence

$$\|g_s\|_p^p + s[f^*(s)]^p = \int_0^1 [g_s(t)]^p dt + s[f^*(s)]^p =$$

$$\int_A \{[f(t) - f^*(s)]^p + [f^*(s)]^p\} dt + \int_B \{[f^*(s)]^p + |f(t) + f^*(s)|^p\} dt +$$

$$(*) \quad + [s - \mu(A \cup B)] \cdot [f^*(s)]^p \leq 2^{1-p}\left\{\int_{A \cup B} |f(t)|^p dt + (s - \mu(A \cup B))[f^*(s)]^p\right\} \leq$$

$$\leq \left(\text{since } \int_0^{\mu(A \cup B)} [f^*(t)]^p dt = \sup_{\mu(\sigma) = s} \int_\sigma |f(t)|^p dt\right) \leq$$

$$\leq 2^{1-p}\left[\int_0^{\mu(A \cup B)} [f^*(t)]^p dt + \int_{\mu(A \cup B)}^{s} [f^*(s)]^p dt\right] \leq 2^{1-p} \int_0^s [f^*(t)]^p dt.$$

Since $|Tf| \leq C(|Tg_s| + |Th_s|)$ we have

$$\int_0^s [(Tf)^*(t)]^p dt = \int_0^s \{[(Tf)(t)]^p\}^* dt \leq (\text{since } f \leq g \text{ implies } f^* \leq g^*) \leq$$

$$\leq C^p \int_0^s [(|Tg_s| + |Th_s|)^p]^* dt \leq C^p \int_0^s (|Tg_s|^p + |Th_s|^p)^* dt \leq$$

$$\leq \left(\text{since } (f_1 \oplus f_2)^* \underset{p}{\prec} f_1^* \oplus f_2^*\right) \leq C^p\left[\int_0^s (|Tg_s|^p)^* dt + \int_0^s (|Th_s|^p)^* dt\right] \leq$$

$$\leq C^p(\|Tg_s\|_p^p + s\|Th_s\|_\infty^p) \leq C^p \max(\|T\|_p^p, \|T\|_\infty^p)(\|g_s\|_p^p + s[f^*(s)]^p) \leq$$

$$\leq (\text{by } (*)) \leq 2^{1-p} C^p \max(\|T\|_p^p, \|T\|_\infty^p) \int_0^s [f^*(t)]^p dt.$$

Consequently $Tf \underset{p}{\prec} 2^{1/p-1} C \max(\|T\|_p, \|T\|_\infty) \cdot f$.

Hence, by Proposition 1, it follows that $Tf \in X$ and $\|Tf\| \leq$

$$\leq 2^{1/p-1} C \max(\|T\|_p, \|T\|_\infty) \|f\|_X.$$

Simple examples of simultaneously continuous operators on L_p and L_∞ are natural projections $P_A(f) = f\chi_A$, for $f \in L_\infty(0,1)$, where $A \subset [0,1]$ is a Lebesgue measurable subset. Another, more intricate example is given by $Tf(x) = \sum_{n=1}^{\infty} 1/(n^{3/p}) \cdot f(x^{1/n})$, where $f \in L_p(0,1)$ and $x \in [0,1]$.

Indeed Theorem 3.2. - [2] shows us that, for every sequence of Borel functions on $[0,1]$, $(a_n)_{n=1}^{\infty}$ and for every sequence of measurable functions $\sigma_n : [0,1] \to [0,1]$ such that

$$(**) \qquad \sup_{\mu(B)>0} \frac{1}{\mu(B)} \sum_{n=1}^{\infty} \int_{\sigma_n^{-1}(B)} |a_n(x)|^p d\mu(x) = M < \infty,$$

$Tf(x) = \sum_{n=1}^{\infty} a_n(x) f(\sigma_n(x))$, where $f \in L_p(0,1)$ and $x \in [0,1]$, defines a bounded operator $T : L_p(0,1) \longrightarrow L_p(0,1)$ such that $\|T\| = M^{1/p} < \infty$. It is not difficult to prove the condition (**) for every Borel set B. Consequently T is a continuous operator on $L_p(0,1)$. Since $\|Tf\|_\infty \leq \left(\sum_{n=1}^{\infty} \frac{1}{n^{3/p}}\right)\|f\|_\infty$ for $f \in L_\infty(0,1)$, it follows that T is a bounded operator on $L_\infty(0,1)$ too. By Theorem 2 it follows that T applies X into X and is bounded on it, where X is a r.i. p-space.

As an interesting application of Theorem 2 we give the following example of a complemented subspace of a r.i. p-space on $[0,1]$.

<u>Corrolary 3</u> <u>Let X be a r.i. p-space, $0 < p < 1$, and let Σ_0 be a σ-subalgebra of the σ-algebra $\mathcal{B}$ of all Borel subsets of $[0,1]$ containing the sets of Lebesgue measure equal to zero. If there exist $A \in \mathcal{B}$ and $\varepsilon > 0$ such that</u>

(1) $\qquad \mu(A \cap B) \geq \varepsilon \mu(B)$ <u>for</u> $B \in \Sigma_0$

and

(2) <u>for all $C \subseteq A$, $C \in \mathcal{B}$, there exists $B \in \Sigma_0$ such that</u> $B \cap A = C$,

then $X(\Sigma_0) = \{f \in X;$ <u>f being Σ_0-measurable function</u>$\}$ <u>is a</u>

complemented subspace of X.

Proof Let P_A be the natural projection of $L_p(0,1)$ onto $L_p(A)$. By (1) it follows that the restriction of P_A on $L_p(\Sigma_0) = L_p((0,1), \Sigma_0, \mu)$ has a continuous inverse and (2) shows that P_A maps $L_p(\Sigma_0)$ onto $L_p(A)$. Hence $P_A|_{L_p(\Sigma_0)}: L_p(\Sigma_0) \longrightarrow L_p(A)$ is a linear homeomorphism. Consequently $T = QP_A$, where $Q = [P_A|_{L_p(\Sigma_0)}]^{-1}$, is a continuous projection from $L_p(0,1)$ onto $L_p(\Sigma_0)$. Using (1) it follows that $\|P_A f\|_\infty = \|f\|_\infty$ for all $f \in L_\infty(\Sigma_0) = L_\infty((0,1), \Sigma_0, \mu)$ and by (2) we get that $P_A(L_\infty(\Sigma_0)) = L_\infty(A)$. Thus $T = QP_A$ is a continuous projection from $L_\infty(0,1)$ onto $L_\infty(\Sigma_0)$. Applying Theorem 2 we get that T is a continuous projection from X into X. If $f \in X \subset L_p(0,1)$, then $Tf \in L_p(\Sigma_0) \cap X \subset X(\Sigma_0)$. Conversely, if $g \in X(\Sigma_0) \subset L_p(\Sigma_0)$, then $g = Tg$ and we are done. ∎

An example of a σ-algebra Σ_0 verifying the conditions (1) and (2) is $\Sigma_0 = \{B \cup C \cup D;\ B \subset [0, 1/2]$ a Borel set, $C = \tau(B)$, where $\tau(x) = x + 1/2$, for $x \in [0,1/2]$ and $\mu(D) = 0\}$.

Theorem 2 allows us to conclude that the linear operators simultaneously continuous on $L_\infty(0,1)$ and $L_p(0,1)$ act continuously on every r.i. p-space X. Since there exist interesting operators which are bounded only on some $L_q(0,1)$ with $p < q < \infty$, we shall study further the r.i. p-spaces X which are "between" $L_{p_1}(0,1)$ and $L_{p_2}(0,1)$, in the sense that every operator defined and bounded on these two spaces is defined and bounded also on X.

In this purpose we recall the definition of Boyd indices. For $0 < s < \infty$ we define the operator D_s as follows.

For every measurable function f on $[0,1]$, put:

$$(D_s f)(t) = \begin{cases} f(t/s) & t \le \min(1,s) \\ 0 & s < t \le 1. \end{cases}$$

Obviously $\|D_s\|_\infty \le 1$ and

$$\|D_s f\|_p^p = \sup_{\|f\|_p \le 1} \|D_s f\|_p^p = \begin{cases} \sup\limits_{\|f\|_p \le 1} \int_0^\wedge |f(t/s)|^p dt = s & \text{for } s < 1 \\ \sup\limits_{\|f\|_p \le 1} \int_0^\wedge |f(t/s)|^p dt \le s & \text{for } s \ge 1. \end{cases}$$

Consequently $\|D_s\|_p = s^{1/p}$ and, by Theorem 2, it follows that D_s acts continuously on X and $\|D_s\|_X \le 2^{1/p-1}\max(1, s^{1/p})$.

Moreover $(D_s f)^* \le D_s f^*$ for every f and $0 < s < \infty$. Consequently we can compute $\|D_s\|_X$ using only nonincreasing functions f. Since, for such a function f, we get $D_r f \le D_s f$, where $0 < r < s < \infty$, it is clear that $\|D_s\|$ is a nondecreasing function of s. Moreover $\|D_{rs}\| \le \|D_r\|\|D_s\|$ for all $0 < r, s < \infty$.

Now we can define the so-called Boyd indices p_X, q_X.

$$p_X = \lim_{s\to\infty} \frac{\log s}{\log\|D_s\|} = \sup_{s>1} \frac{\log s}{\log\|D_s\|},$$

$$q_X = \lim_{s\to 0^+} \frac{\log s}{\log\|D_s\|} = \inf_{0<s<1} \frac{\log s}{\log\|D_s\|}.$$

If $\|D_s\| = 1$ for some $s > 1$ we put $p_X = \infty$. Similarly, if $\|D_s\| = 1$ for all $s < 1$, we put $q_X = \infty$. Obviously $p_X = q_X = p$ for $X = L_p(0,1)$ where $0 < p \le \infty$.

<u>Proposition 4</u> <u>Let X be a r.i. p-space. Then</u>

1) $p \le p_X \le q_X \le \infty$

2) $p_{X_{(p)}} = p_X/p$ and $q_{X_{(p)}} = q_X/p$.

<u>Proof</u> 1) Since $\|D_s\| \le 2^{1/p-1} s^{1/p}$ for $s \ge 1$, we get

$$p_X = \lim_{s\to\infty} \frac{\log s}{\log\|D_s\|} \ge \lim_{s\to\infty} \frac{\log s}{\log 2^{1/p-1}s^{1/p}} = p.$$

But $\|D_s\| \cdot \|D_{s^{-1}}\| \ge \|D_{ss^{-1}}\| = 1$; consequently

$$p_X = \lim_{s\to\infty} \frac{\log s}{\log\|D_s\|} \le \lim_{s\to 0} \frac{\log s^{-1}}{\log\|D_{s^{-1}}\|} = q_X.$$

2) Obviously $\|D_s\|_{X_{(p)}} = \|D_s\|_X^p$. ∎

Proposition 5 Let X be a r.i. p-space on [0,1]. For every $p \le p_1 < p_X$ and $q_X < q_1 \le \infty$ we have $L_{q_1}(0,1) \subset X \subset L_{p_1}(0,1)$ the inclusions maps being continuous.

Proof Proposition 2.b.3 - [3] settles the case $p=1$. For $0<p<1$, by Proposition 4, we get $1 \le p_1/p < p_X/p = p_{X_{(p)}}$ and $q_{X_{(p)}} = q_X/p < q_1/p \le \infty$. Applying again Proposition 2.b.3 -[3] it follows that the inclusion maps $L_{q_1/p} \longrightarrow X_{(p)}$ and $X_{(p)} \longrightarrow L_{p_1/p}$ are continuous. Hence the inclusion maps $L_{q_1} \longrightarrow X$ and $X \longrightarrow L_{p_1}$ are also continuous. ∎

We extend now Proposition 2.b.5 -[3] for p-Orlicz spaces.

Proposition 6 Let $X = L_M(0,1)$ (resp. $X = H_M(0,1)$) a p-Orlicz space. Then

$$p_X = \sup\left\{p_1;\ \inf_{\lambda,t\ge 1} M(\lambda t)/M(\lambda).t^{p_1} > 0\right\},$$

$$q_X = \inf\left\{q_1;\ \sup_{\lambda,t\ge 1} M(\lambda t)/M(\lambda).t^{q_1} < \infty\right\}.$$

Proof The conclusion follows easily from Proposition 4, Proposition 2.b.5 -[3] and the remark that $X_{(p)} = L_{M_{(p)}}(0,1)$ (resp. $X_{(p)} = H_{M_{(p)}}(0,1)$) where $M_p(x) = M(x^{1/p})$. ∎

It is clear that Proposition 2.b.7 -[3] implies the following

Proposition 7 Let X be a r.i. p-space. Then

(i) $q_X < \infty$ if and only if X does not contain, for all integers n, almoust isometric copies of $l_\infty(n)$ spanned by disjoint functions having the same distribution function.

(ii) $p < p_X$ <u>if and only if X does not contain, for all integers n, almoust isometric copies of</u> $l_p(n)$ <u>spanned by disjoint functions having the same distribution function.</u>

We give further another interpolation theorem which extends the Boyd' interpolation theorem.

First of all we introduce the spaces $L_{r,q}$, where $p \leq r, q \leq \infty$.

Let (Ω, Σ, μ) be a measure space. For $p \leq r < \infty$ and $p \leq q < \infty$, denote by $L_{r,q}(\Omega, \Sigma, \mu)$ the space of all locally p-integrable functions with real values f defined on Ω such that

$$\|f\|_{r,q} = \left[(q/r)\int_0^\infty \left(t^{1/r} f^*(t)\right)^q \frac{dt}{t}\right]^{1/q} < \infty .$$

For $p \leq r \leq \infty$ we denote by $L_{r,\infty}(\Omega, \Sigma, \mu)$ the space of all measurable functions f such that $\|f\|_{r,\infty} = \sup_{t>0} t^{1/r} f^*(t) < \infty$. (See [1]).

Obviously $L_{q,q} = L_q$ and $\|f\|_{q,q} = \|f\|_q$. Moreover we have $\|f\|_{r,q_2} \leq \|f\|_{r,q_1}$ for $0 < q_1 \leq q_2 \leq \infty$ (see [1]), thus $L_{r,q_1} \subseteq L_{r,q_2}$. If (Ω, Σ, μ) is a probability space, then, applying Hölder's inequality, we get

$$L_{r_3,\infty} \subset L_{r_2,q_1} \subset L_{r_1,q_2}, \text{ where } 0 < r_1 < r_2 < r_3 \leq \infty \text{ and}$$

$q_1, q_2 > 0$.

The spaces $L_{r,q}$ are topologically complete metric linear spaces. (See [1]).

Let now $(\Omega_i, \Sigma_i, \mu_i)$ $i = 1,2$ be two measure spaces, let $p \leq r_1 \leq \infty$ and let T be a map defined on a subset of $L_{r_1}(\Omega_1)$ with values in the space of all measurable functions on Ω_2.

1) The map T is said to be of <u>strong type</u> (r_1, r_2) for a suitable $r_1 \in [p, \infty]$, if there exists a constant $M > 0$ such that $\|Tf\|_{r_2} \leq M \|f\|_{r_1}$ for every f from the domain of definition of T.

2) T is said to be of <u>weak type</u> (r_1,r_2) for some $r_2 \in [p,\infty]$ if there exists a constant $M>0$ such that

$$\|Tf\|_{r_2,\infty} \le M\|f\|_{r_1,p}$$

for every f from the domain of definition of T. We make the convention that, for $r_1=\infty$, instead of $\|f\|_{\infty,p}$ we put $\|f\|_{\infty,\infty}=\|f\|_\infty$.

It is clear that an operator of strong type (r_1,r_2) is also of weak type (r_1,r_2). Finally we remark that T is of weak type (r_1,r_2) if and only if there exists a constant $M>0$ such that

$$\sup_{t>0} t\left(\mu_2\{\omega\in\Omega_2;\ |Tf(\omega)|\ge t\}\right)^{1/r} \le M\left(p/r_1\int_0^\infty t^{p/r_1-1}[f^*(t)]^p dt\right)^{1/p}.$$

We can now extend Theorem 2.b.11 - [3].

<u>Theorem 8</u> <u>Let $0<p\le 1$ and $p\le p_1<q_1\le\infty$ and let T be a linear operator acting from $L_{p_1,p}(0,1)$ into the space of all measurable functions on $(0,1)$.</u>

<u>Assume that T is of weak types (p_1,p_1) and (q_1,q_1). Then for every r.i. p-space X of functions on $[0,1]$ such that $p_1<p_X$ and $q_X<q_1$, T maps X into itself and it is bounded on X.</u>

The following lemma is an extension of Lemma 2.b.12 -[3].

<u>Lemma 9</u> <u>Wit the same assumptions on T as in Theorem 8 there is a constant $M<\infty$ such that</u>

$$[(Tf)^*(2t)]^p \le M\left[\int_0^1 [f^*(tu)]^p u^{p/p_1-1}du+\int_1^\infty [f^*(tu)]^p u^{p/q_1-1}du\right]$$

<u>for every</u> $0<t\le 1/2$ <u>and</u> $f\in L_{p_1,p}(0,1)$.

<u>Proof</u> Suppose that T is of weak types (p_1,p_1) and (q_1,q_1) with the constants M_{p_1} and M_{q_1}. Let $f\in L_{p_1,p}(0,1)$ and for $u,t\in[0,1]$ set

$$g_t(u)=\begin{cases} f(u)-f^*(t) & \text{if } f(u)>f^*(t)\\ f(u)+f^*(t) & \text{if } f(u)<-f^*(t)\\ 0 & \text{if } |f(u)|\le f^*(t)\end{cases}$$

and $h_t(u)=f(u)-g_t(u)$.

It is clear that $g_t, h_t \in L_{p_1,p}(0,1)$ and we apply the fact that T is of weak type (p_1,p_1) to g_t and of weak type (q_1,q_1) to h_t. Note that $g_t^*(u)=0$ for $u\in[t,\infty)$ and $g_t^*(u)\le f^*(u)$ for $0<u<t$. Hence, for $t\in I$, we have

$$t^{p/p_1}\left[(Tg_t)^*(t)\right]^p \le M_{p_1}^p(p/p_1)\int_0^\infty \left[g_t^*(s)\right]^p s^{p/p_1-1}\,ds \le$$

$$\le M_{p_1}^p\cdot\frac{p}{p_1}\int_0^t \left[f^*(s)\right]^p s^{p/p_1-1}\,ds = M_{p_1}^p\cdot\frac{p}{p_1}\cdot t^{p/p_1}\int_0^1\left[f^*(tu)\right]^p u^{p/p_1-1}\,du.$$

Since $|h_t(u)| = \min\left(|f(u)|,\, f^*(t)\right)$, for $t\in[0,1]$, we have

$$t^{p/q_1}\left[(Tm_t)^*(t)\right]^p \le M_{q_1}^p\cdot\frac{p}{q_1}\cdot\int_0^\infty\left[h_t^*(s)\right]^p s^{p/q_1-1}\,ds \le$$

$$\le M_{q_1}^p\cdot\frac{p}{q_1}\cdot\left(\int_0^t\left[f^*(t)\right]^p s^{p/q_1-1}\,ds + \int_t^\infty\left[h_t^*(s)\right]^p s^{p/q_1-1}\,ds\right) =$$

$$= M_{q_1}^p\cdot\frac{p}{q_1}\cdot\left(\frac{q_1}{p}\left[f^*(t)\right]^p t^{p/q_1} + t^{p/q_1}\int_1^\infty\left[h_t^*(tu)\right]^p u^{p/q_1-1}\,du\right) \le$$

$$\le M_{q_1}^p\cdot\frac{p}{q_1}\, t^{p/q_1}\left(\frac{q_1}{p}\int_0^1\left[f^*(tu)\right]^p u^{p/p_1-1}\,du + \int_1^\infty\left[f^*(tu)\right]^p u^{p/q_1-1}\,du\right).$$

Since $|Tf|\le|Tg_t|+|Th_t|$ it follows that

$$\left[(Tf)^*(2t)\right]^p \le \left[(Tg_t)^*(t)+(Th_t)^*(t)\right]^p \le \left[(Tg_t)^*(t)\right]^p + \left[(Th_t)^*(t)\right]^p \le$$

$$\le\left(M_{p_1}^p\cdot\frac{p}{p_1}+M_{q_1}^p\right)\int_0^1\left[f^*(tu)\right]^p u^{p/p_1-1}\,du + M_{q_1}^p\cdot\frac{p}{q_1}\int_1^\infty\left[f^*(tu)\right]^p u^{p/q_1-1}\,du.$$

This proves our lemma with $M=\frac{p}{p_1}\cdot M_{p_1}^p + M_{q_1}^p$. ∎

<u>Proof of Theorem 8</u> Let p_o and q_o be such that $p_1<p_o<p_X$ and $q_X<q_o<q_1$. Then there is $s_o>1$ such that, for $s\ge s_o$, we have $p<\frac{\log s}{\log\|D_s\|}$. Consequently $\|D_s\|\le s^{1/p_o}$ for $s\ge s_o$.

Since $s\longrightarrow\frac{\log s}{\log\|D_s\|}$ is a continuous function on $(1,\infty)$, it follows that there is $K<\infty$ such that $\|D_s\|\le Ks^{1/p_o}$ for $2\le s\le\infty$.

Similarly, we can assume that $\|D_s\| \leq Ks^{1/q_o}$ for $0 < s \leq 2$. Let now $g \in X' = [X_{(p)}]'$ such that $\|g\|_{X'} = 1$. (We recall that $X_{(p)}$ endowed with the norm $|||f||| = \||f|^{1/p}\|^p$ is a Banach lattice of functions on $[0,1]$. See for example [4]).

Putting $\tilde{g}(t) = \begin{cases} g(t) & \text{if } t \leq 1 \\ 0 & \text{if } t > 1 \end{cases}$, we have

$$\int_0^1 \Big(\int_0^1 [f^*(tu/2)]^p g(t) u^{p/p_1 - 1} du \Big) dt = \int_0^1 u^{p/p_1 - 1} \Big(\int_0^\infty (D_{2/u} f^*)^p (t) \tilde{g}(t) dt \Big) du$$

$$\leq \int_0^1 \|(D_{2/u} f^*)^p\|_{X_{(p)}} \cdot u^{p/p_1 - 1} du \leq K^p 2^{p/p_o} \Big(\int_0^1 u^{p/p_1 - p/p_o - 1} du \Big) \|f\|_X^p =$$

$$= 2^{p/p_o} K^p \Big(\frac{p}{p_1} - \frac{p}{p_o} \Big)^{-1} \|f\|_X^p \text{ and}$$

$$\int_0^1 \Big(\int_1^\infty [f^*(tu/2)]^p g(t) u^{p/q_1 - 1} du \Big) dt = \int_1^\infty u^{p/q_1 - 1} \Big(\int_0^1 (D_{2/u} f^*)^p (t) g(t) dt \Big) du$$

$$\leq \int_1^\infty u^{p/q_1 - 1} \|D_{2/u}\|^p \|f\|_X^p du \leq \|f\|_X^p K^p 2^{p/p_o} \int_1^\infty u^{p/q_1 - p/q_o - 1} du =$$

$$= \|f\|_X^p 2^{p/q_o} K^p \Big(\frac{p}{q_o} - \frac{p}{q_1} \Big)^{-1} \quad \text{for } 0 < t \leq 1/2.$$

By Lemma 9 it follows that $\int_0^1 [(Tf)^*(t)]^p g(t) dt \leq M_o \|f\|_X^p$ for $g \in X'$ such that $\|g\|_{X'} = 1$. Here $M_o = MK^p \Big(\frac{p}{p_1} - \frac{p}{p_o} \Big)^{-1} \cdot 2^{p/p_o} + 2^{p/q_o} \Big(\frac{p}{q_o} - \frac{p}{q_1} \Big)^{-1}$, M being the constant appearing in Lemma 9.

Hence $(Tf)^p \in [X_{(p)}]''$. In other words $Tf \in \{[X_{(p)}]''\}^{(p)} = X''$.

(We recall that for a Banach lattice of functions Y, $Y^{(p)} = \{f : [0,1] \to \mathbb{R} \,;\, |f|^p \in Y\}$ endowed with the p-norm $|||f||| = \||f|^p\|_Y^{1/p}$ is a p-Banach lattice of functions).

Moreover $\|Tf\|_{X''}^p = \|(Tf)^p\|_{X_{(p)}} \leq M_o \|f\|_X^p$.

If X is maximal, then $Tf \in X$ and $\|Tf\|_X \leq M_o \|f\|_X$. Since $L_{q_o}(0,1)$ is a maximal r.i. p-space, then it follows as above that $T(L_{q_o}(0,1)) \subset L_{q_o}(0,1)$. X being the closure of $L_{q_o}(0,1)$ for the topology of X'', it follows that T maps X into X and it is bounded there. ∎

The p-Lorentz space $L_{3/2,3/4}(0,1)$ is an example of r.i. $(3/4)$-space non locally convex X such that $p_X > 1$.

Let $\mathcal{A}$ be a σ-subalgebra of $\mathcal{B}$ (the σ-algebra of all Borel sets of $[0,1]$) such that the Lebesgue measure restricted on $\mathcal{A}$ is σ-finite. For $f \in L_1(0,1)$, the Lebesgue-Nikodym theorem shows the existence of a unique $\mathcal{A}$-measurable and Lebesgue integrable function, denoted by $E^{\mathcal{A}} f$, which verifies the relation

$$\int_0^1 (E^{\mathcal{A}} f) g \, dt = \int_0^1 gf \, dt$$

for every bounded $\mathcal{A}$-measurable function g on $[0,1]$.

It is clear that $f \longrightarrow E^{\mathcal{A}} f$ is an idempotent operator. This operator is called the <u>conditional expectation</u> and has the norm one on $L_1(0,1)$ and $L_\infty(0,1)$, consequently on every $L_q(0,1)$ with $1 \leq q \leq \infty$.

<u>Corrolary 10</u> <u>With the notations of above, for $0 < p \leq 1 \leq p_1 < q_1 \leq \infty$ and for a r.i. p-space X on $[0,1]$, such that $p_1 < p_X$ and $q_X < q_1$, $E^{\mathcal{A}}$ maps X into itself and it is bounded on it.</u>

<u>Proof</u> Since $p_1 \geq 1$, then $E^{\mathcal{A}}$ is an operator of strong types (p_1, p_1) and (q_1, q_1). Thus by Theorem 8, $E^{\mathcal{A}}$ maps X into itself and its norm does not depend on $\mathcal{A}$. ∎

We give an interesting application of Corrolary 10.

Recall that the Haar system $(\chi_n)_{n=1}^{\infty}$ is given by $\chi_1(t) \equiv 1$ and, for $\ell = 1,2,\ldots,2^k$ and $k = 0,1,\ldots$, by

$$\chi_{2^k+\ell}(t) = \begin{cases} 1 & \text{for } t \in [(2\ell-2)2^{-k-1}, (2\ell-1)2^{-k-1}) \\ -1 & \text{for } t \in [(2\ell-1)2^{-k-1}, 2\ell \cdot 2^{-k-1}) \\ 0 & \text{otherwise}. \end{cases}$$

It is known that the Haar system is not a Schauder basis in $L_p(0,1)$, for $0 < p < 1$. (See [5]).

Thus it is natural to ask whenever the Haar system is a Schauder basis in a r.i. p-space, for $0<p<1$. In order to answer to this question we associate to the Haar system an increasing sequence of σ-algebras $\{\mathcal{A}_n\}_{n=1}^{\infty}$ of Lebesgue measurable subsets of $[0,1]$. σ-algebra $\mathcal{A}_1$ consists only of the vanishing set ϕ and $[0,1]$. For $n=2^k+l$, $1\leq l\leq 2^k$, $k\geq 0$, $\mathcal{A}_n$ is the σ-algebra spanned by $\mathcal{A}_{n-1}$ and the intervals $[(2l-2)2^{-k-1}, (2l-1)2^{-k-1})$, $[(2l-1)2^{-k-1}, 2l.2^{-k-1})$. It is clear that $\mathcal{A}_n$ is the smallest σ-algebra $\mathcal{A}$ such that the functions $\{\chi_1,\dots,\chi_n\}$ are $\mathcal{A}$-measurables.

We can now prove the following assertion.

Corrolary 11 If X is a separable r.i. p-space on $[0,1]$, such that $0<p<1\leq p_1<p_X\leq q_X<q_1\leq\infty$, then the Haar system $(\chi_n)_{n=1}^{\infty}$ is a Schauder basis of X.

Proof We assume that X is not a Banach space, otherwise the assertion is well-known. Thus X is not isomorphic to $L_\infty(0,1)$, that is $\lim_{t\to 0}\|\chi_{(0,t)}\|_X=0$. Consequently every simple function on $[0,1]$ can be approximated in the norm of X by the characteristic functions of dyadic intervals $[l2^{-k}, (l+1)2^{-k})$, $0\leq l\leq 2^k-1$, $k=0,1,\dots$.

It follows that the Haar system spans a dense subspace in X. Schauder basis in the closed subspace spanned by them in X.

Observe also that for $n<m$ and for every choice of scalars $\{a_i\}_{i=1}^{m}$ we have

$$E^{\mathcal{A}_n}\Big(\sum_{i=1}^{m} a_i\chi_i\Big)=\sum_{i=1}^{n} a_i\chi_i$$

and, by Corrolary 10, it follows that $\|E^{\mathcal{A}_n}\|\leq M$ for $n\in\mathbb{N}$. This implies that $(\chi_i)_{i=1}^{\infty}$ is a basic sequence in X. (See Theorem III 2.12 - [5]).

We are interested further to know whenever the Haar system is an unconditional basis in a r.i. p-space on $[0,1]$. We recall that a Schauder basis in X is said to be unconditional if the expansion of every element of X, with respect to this basis, converges unconditionally.

It is of some interest to remark that the relation $1 < p_X \leq q_X < \infty$ is a necessary and sufficient condition for the unconditionality of the basis $(\chi_n)_{n=1}^{\infty}$ in every r.i. p-space X. We extend in this way Theorem 2.c.6 - [3].

Theorem 12 Let X be a separable r.i. p-space on $[0,1]$. The Haar system $(\chi_n)_{n=1}^{\infty}$ is an unconditional basis in X if and only if $1 < p_X \leq q_X < \infty$.

Proof If $1 < p_X$ and $q_X < \infty$ then by Theorem 8 and by Theorem 2.c.5 -[3] it follows that the projections P_σ from X onto $[\chi_i]_{i\in\sigma}$, where $\sigma \subseteq \mathbb{N}$ is a closed subset, are uniformly bounded, that is the basis $(\chi_i)_{i=1}^{\infty}$ is unconditional.

Conversely, assume that $(\chi_i)_{i=1}^{\infty}$ is an unconditional basis in X. By Proposition 4, $p_{X_{(p)}} = p_X/p$; consequently Theorem 2.b.6 -[3] shows us that $l_{p_{X_{(p)}}}(n)$ spanned by positive disjoint elements having the same distribution function are uniformly contained in $X_{(p)}$. It follows that X contains uniformly the spaces $l_{p_X}(n)$ spanned by positive disjoints functions having the same distribution function.

In other words there is $M>0$ such that for all $n \in \mathbb{N}$ there are 2^n disjoint functions $(u_i)_{i=1}^{2^n}$ having the same distribution function such that $\|u_i\| = 1$ and verifying the inequality

$$M\Big(\sum_{i=1}^{2^n}\|u_i\|^{p_X}\Big)^{1/p_X} \geq \Big\|\sum_{i=1}^{2^n} u_i\Big\| \geq 1/M\Big(\sum_{i=1}^{2^n}\|u_i\|^{p_X}\Big)^{1/p_X}.$$

Let $(h_i)_{i=1}^{2^n}$ " the Haar system over $(u_i)_{i=1}^{2^n}$ " defined by

$$h_1 = 2^{-n/p_X}(u_1 + \cdots + u_{2^n})$$

$$h_2 = 2^{-n/p_X}(u_1 + \cdots + u_{2^{n-1}} - u_{2^{n-1}+1} - \cdots - u_{2^n})$$

$$\vdots$$

$$h_{2^{n-1}+1} = 2^{-n/p_X}(u_1 - u_2)$$

$$\vdots$$

$$h_{2^n} = 2^{-n/p_X}(u_{2^n-1} - u_{2^n}).$$

Since $L_\infty(0,1)$ is dense in X we can assume that u_i is a finite linear combination of characteristic functions of intervals $((l_j-1)2^{-k}, l_j \cdot 2^{-k})$ for some k independent of i. Applying a suitable automorphism of $[0,1]$ we can suppose that on the first 2^n dyadic intervals of length 2^{-k} every u_i is nonzero exactly on some of these intervals and takes there an value independent of i, say β_1. The same fact is also true for the following 2^n dyadic intervals of length 2^{-k}, where β_1 is replaced by β_2 and so on.

Thus, for some $m \in \mathbb{N}$ and some scalars $(\beta_j)_{j=1}^m$ we have

$$u_i = \sum_{j=1}^m \beta_j \chi_{[(i-1+(j-1)2^n)2^{-k}, (i+(j-1)2^n)2^{-k})}, \quad 1 \le i \le 2^n$$

Remark that

$$2^{n/p_X} h_2 = u_1 + \cdots + u_{2^{n-1}} - u_{2^{n-1}+1} - \cdots - u_{2^n} = \sum_{j=1}^m \beta_j \chi_{2^{k-n+1}+j}$$

$$2^{n/p_X} h_3 = u_1 + \cdots + u_{2^{n-2}} - u_{2^{n-2}+1} - \cdots - u_{2^{n-1}} = \sum_{j=1}^m \beta_j \chi_{2^{k-n+2}+2j-1}$$

$$2^{n/p_X} h_4 = u_{2^{n-1}+1} + \cdots + u_{2^{n-1}+2^{n-2}} - u_{2^{n-1}+2^{n-2}+1} - \cdots - u_{2^n} =$$

$$= \sum_{j=1}^m \beta_j \chi_{2^{k-n+2}+2j} \quad \text{and so on.}$$

In other words $\{h_j\}_{j=1}^{2^n}$ constitutes a block basis for a permutation π of the Haar basis $(\chi_n)_{n=1}^\infty$ of X. Thus the unconditionality constant of $\{h_j\}_{j=2}^{2^n}$, K_n, is less than K_X, the unconditionality constant of the basis $(\chi_n)_{n=1}^\infty$ of X.

On the other hand, let $T_n : [u_i]_{i=1}^{2^n} \longrightarrow l_{p_X}(2^n)$ given by $T_n(u_i) = e_i$, $1 \le i \le 2^n$, be an isomorphism which satisfies $\|T_n\| \cdot \|T_n^{-1}\| \le M$ for all $n \in \mathbb{N}$.

If $S_n : \ell_{p_X}(2^n) \longrightarrow L_{p_X}(0,1)$ is the isometry given by $S_n(e_i) = \chi_{[\frac{i-1}{2^n}, \frac{i}{2^n})} 2^{n/p_X}$, then $U_n = S_n \circ T_n : [u_i]_{i=1}^{2^n} \longrightarrow L_{p_X}(0,1)$ verifies the condition $\|U_n\| \cdot \|U_n^{-1}\| \le M$ and, moreover, $U_n(h_i) = \chi_i$, $1 \le i \le 2^n$. Thus the unconditionality constant of the system $(h_i)_{i=1}^{2^n}$ is the same, up to a factor M, as that of first 2^n elements of Haar system in $L_{p_X}(0,1)$. Consequently, if $p_X \le 1$, then $K_n \xrightarrow[n]{} \infty$ and therefore $K_X = \infty$, contradictorily. Hence $1 < p_X$ and similarly we can prove that $q_X < \infty$. ∎

For example the Haar system is an unconditional basis in $L_{3/2,3/4}(0,1)$, in spite of the fact that this space is not locally-convex.

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