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AN APPROXIMATION THEOREM FOR MARKOV PROCESSES

by

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AN APPROXIMATION THEOREM FOR MARKOV PROCESSES

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An Approximation Theorem for Markov Processes

Vlad Bally

Watanabe /1/ approximates a Markov process by a sequence of regular step processes (R.S.P.) in the sense that the resolvents of this processes converge weakly to the resolvent of the Markov process.

Under some supplementary conditions we shall prove that the distributions of this R.S.P. converge to the distribution of the Markov process.

The initial assumptions in Watanabe's construction are as follows.

Let S be a locally compact space with demonstrable base (L.C.S.D.), U and open base and d, any metric of S. For each n, we can choose the systems U^n_i, $i \in N$ and V^n_i, $i \in N$ of sets in U satisfying the following conditions :

(1) Each $\overline{U^n_i}$ is compact and $d(U^n_i) \leq \frac{1}{n}$. (2) $V^n_i \subseteq U^n_i$

(3) $\bigcup_i V^n_i = S$ (4). For every compact set K only a finite number of V^n_i intersect with K.

We now define σ^n_k by :

$$\sigma^n_1(\omega) = D_{\overline{U^n_i}^c}(\omega) \quad \text{if} \quad X_0(\omega) \in V^n_i - \bigcup_{j<i} V^n_j$$

$$\sigma^n_k(\omega) = \sigma^n_{k-1}(\omega) + \sigma^n_1 \circ \theta_{\sigma^n_{k-1}}(\omega) \quad \text{for } k>1$$

(for a measurable set A, $D_A = \inf(t \mid X_t \in A)$)

The following result is Lemma 3.3. in Watanabe's work :

Let X be a standard process such that $G_\alpha(C(S)) \subseteq C(S)$ for $\alpha > 0$ and

(6) $\sup_{x \in S} G_0(x, S) < \infty$

(i) for each n, $q_n(x) = [E^x(\sigma_1^n)]^{-1}$ and

$\Pi_n(x, A) = P^x(X_{\sigma_1^n} \in A)$ represent the parameters of a R.S.P.

The corresponding R.S.P. are denoted by X^n and the Green operator of X^n, by $G_\alpha^{(n)}$.

(ii) X^n approximates X in the following sense :

(7) $\lim_n G_\alpha^{(n)} f^x = G_\alpha f^x$ for every $x \in S$ and $f \in C(S)$

We shall need instead of (7) a stronger result :

For any compact set K, $\alpha > 0$ and $f \in C(S)$

(8) $\lim_n G_\alpha^{(n)} f^x = G_\alpha f^x$ uniformly for $x \in K$

Considering the proof of Lemma 3.1. in /1/ it is obvious that to obtain (8) it is sufficient to prove the following condition :

(C) For any compact set $L \subseteq S$, $\alpha > 0$ and $\varepsilon > 0$, we may choose a compact set K such that $L \subset K$ and

(9) $\overline{\lim_n} \sup_{x \in L} G_\alpha^{(n)}(x, K^c) \leq \varepsilon$

Through this paper we shall consider on S a metric d such that $B_h(y \mid d(x,y) < h)$ has compact closure.

For $h > 0$, we define T_h^k, $k \in N$ by $T_h = \inf(t \mid d(X_0, X_t) > h)$ and $T_h^1 = T_h$, $T_h^{k+1} = T_h^k + T_h \circ \Theta_{T_h^k}$.

We shall also consider the function $q(h) = [\sup_{x \in S} E^x(T_h)]^{-1}$

We note that $q^{(h)}$ and $q_n^{(x)}$ are distinct notations.

The function $h \longrightarrow q^{(h)}$ is monotone, so we may choose for every n, h_n and d_n such that $\lim_n h_n = 0$, $d_n < h_n$ and

(10) $$\lim_n \frac{q(h_n - d_n)}{q(h_n + d_n)} = 1$$

(h_n will be continuity points for $h \longrightarrow q^{(h)}$).

Now we shall choose the above mentioned U_i^n and V_i^n in the following particular form :

$U_i^n = B_{h_n}(x_i)$, $V_i^n = B_{d_n}(x_i)$ with $x_i, \in \in N$ chosen such that condition (4) is fulfilled.

σ_i^n will be defined like above with respect to this system of sets.

The following two inequalities will be usfful in what it follows

(11) $$T_{h_n} - d_n \leq \sigma_1^n \leq T_{h_n + d_n}$$

If we assume that for X the following condition holds, ex. $d > 0$ such that for every $h > 0$, $\sup_{x \in S} E^x(T_h) \leq d \inf_{x \in S} E^x(T_h)$ then, by (11) we may conclude that

(12) $$d\, q(h_n - d_n) \geq q_n(x) \geq q(h_n + d_n)$$

Now we are able to formulate the results of the paper.

<u>Theorem.</u> Let X be a standard process with state space S, such that :

i) $G_\alpha(C(S)) \subseteq C(S)$ for every $\alpha > 0$

ii) $\lim_{t \to 0} P_t f^x = f(x)$ uniformly on S, for every $f \in C_c(S)$

iii) $\sup_{x \in S} G_0(x, S) < \infty$

iv) There is some $d > 0$ such that for every $h > 0$

$$\sup_{x \in S} E^x(T_h) \leq d \inf_{x \in S} E^x(T_h)$$

v) There is some $c > 0$ such that for every $x \in S$ and $h > 0$

$$E^x(D_{B_h^c(x)}) \geq c \, \| E^{(\cdot)}(D_{B_h^c(x)}) \|$$

$$(\; \| E^{(\cdot)}(\varphi) \| = \sup_{y \in S} | E^y(\varphi) | \;)$$

Then, $\lim_n P_n^\mu = P^\mu$ for every μ (probability measure on S).

(We denote by P_n^μ the distribution of the R.S.P. X^n which has initial measure μ).

We note that condition (iv) implies

(vi) $\lim_{h \to 0} \sup_{x \in S} E^x(T_h) = 0$

That is because for any $x \in S$ $\lim_{h \to 0} E^x(T_h) = 0$

The proof will go as follows: In the first part we establish the similitudes between X and X^n. We refere to Annexa 1, which presents the low of large numbers in two forms appropiate to our deal. The first two Lemmas assure that we may use the results in Annexa 1. We use it in Lemma 3 which is esential for all the proof. Roughly speaking this Lemma establish the similitude between the "time" of X^n and of X. Lemma 4 is a simple remark which assures that the "space" of X^n and X coincide.

These similitudes are used in all the following in order to evaluate quantities refering to X^n by their analougues with respect to X.

In the second part of the proof we establish the tightness of the sequence P_n^μ $n \in N$. The last part deals with the convergence of the finite dimensional distributions. We use here Watanabe's result in his stronger form (8).

To do it we prove first (C), and then we refere to Annexa 3 which permite us to check the finite dimensional distributions convergence by the convergence of the resolvents.

We first define for all $h > 0$

(13) $$\tilde{F}_h(t) = \inf_{x \in S} P^x(T_h < t) = 1 - \sup_{x \in S} P^x(T_h > t)$$

$\tilde{F}_h$ is infimum of a family of increasing functions which are right continuous and have left hand limits, then so is $\tilde{F}_h$.

Next, it is obvious that $\tilde{F}_h(o) = 0$, and so, in order to show that $\tilde{F}_h$ is a distribution function on R_+ it will be sufficient to see that

(14) $$\lim_{t \to \infty} F_h(t) = 1$$

By Chebishev's inequality $P^x(T_h > t) \leq \frac{1}{t} E^x(T_h)$ and so,

$$\tilde{F}_h(t) \geq 1 - \frac{1}{t} \sup_{x \in S} E^x(T_h)$$

Because $G_o(x,S) = E^x\left(\int_0^\infty 1_S \circ X_t \, dt\right) = E^x(J)$ with $J = D_{S^c}$

by (ii), $\sup_{x \in S} E^x(T_h) \leq \sup_{x \in S} E^x(J)$ and so (14) is proved.

We may now consider the distribution on R_+ corresponding to $\tilde{F}_h$. We denote it by F_h.

It is obvious that for any t and x

(15) $$F_h([0,t]) \leqslant P^x(T_h \leqslant t) \text{ and } F_h((t,\infty)) = \sup_{x\in S} P^x(T_h > t)$$

Let be $t_n = 2 \sup_{x\in S} \| E^{(\cdot)}(D_{B^c_{h_n}(x)})\| = 2 \sup_x \sup_y |E^y(D_{B^c_{h_n}(x)})|$

The following relations will be used to prove Lemma 1 :

(16)
a) $\lim_n t_n = 0$

b) $F_{h_n}((kt_n,\infty)) \leqslant \left(\frac{1}{2}\right)^k$

Because $T_h = D_{B^c_h(x)}$ P^x a.s., a) is consequence of (v) and (vi)

To prove b) we shall use exercise (10.25) in /2/ :

If $\alpha(t) = \sup_{x\in S} P^x(D_U > t)$ (U is a measurable set), then

(17) $$P^x(D_U > kt) \leqslant \alpha(t)^k$$

Next we consider $t = t_n$ and $U = D_{B^c_{h_n}}(x)$ and by Chebishev's inequality $P^y(D_{B^c_{h_n}(x)} > t_n) \leqslant \frac{1}{t_n} E^y(D_{B^c_{h_n}(x)})$

We take the supremum over all $y \in S$ and considering the definition of t_n and α_n we conclude that

$$\alpha_{D_{B_{h_n}}(x)}(t_n) \leqslant \frac{1}{2}$$

By (15) and (17) we get (16) :

$$F_{h_n}((kt_n,\infty)) = \sup_{x\in S} P^x(D_{B^c_{h_n}(x)} > kt_n) \leqslant \left(\frac{1}{2}\right)^k$$

<u>Lemma 1.</u> a) for every $a > 0$, $\lim_n \int_a^\infty z F_{h_n}(dz) \Big/ \int z F_{h_n}(dz) = 0$

b) $\int_0^\infty z F_{h_n}(dz) \leqslant \frac{4}{c} \sup_{x\in S} E^x(T_{h_n}) = \frac{4}{c}\,[q(h_n)]^{-1}$

where c is defined in (v).

Proof. If $k_n = \left[\frac{a}{t_n}\right]$ we have :

$$\int_a^\infty z\, F_{h_n}(dz) \leqslant \int_{k_n t_n}^\infty z\, F_{h_n}(dz) \leqslant (k_n+1)\, F_{h_n}[k_n t_n, \infty) + t_n \sum_{k>k_n} F_{h_n}([k t_n, \infty))$$

By (16) we obtain (18) $\quad \int_a^\infty z\, F_{h_n}(dz) \leqslant t_n (k_n+2)\, 2^{-k_n}$

It is obvious that $\quad \int_0^\infty z\, F_{h_n}(dz) \geqslant \int T_{h_n}\, P^x(d\omega)$

for every $x \in S$, and so, by (v)

$$\int_0^\infty z\, F_{h_n}(dz) \geqslant \sup_{x\in S} E^x(T_{h_n}) \geqslant c \sup_{x \in S} \| E^{(\cdot)}(D_{B^c_{h_n}(x)}) \| = \frac{c}{2} t_n$$

The right continuity of the trajectories assures that $D_{B^c_{h_n}(x)} > 0$

and then $t_n > 0$ and we may write :

$$\int_a^\infty z\, F_{h_n}(dz) \Big/ \int_0^\infty z\, F_{h_n}(dz) \leqslant \frac{t_n (k_n+2)\, 2^{-k_n}}{\frac{c}{2} t_n} = \frac{2}{c}(k_n+2)\, 2^{-k_n}$$

The last term vanishes when $n \to \infty$ and a) is proved .

$$\int_0^\infty z\, F_{h_n}(dz) \leqslant t_n F_{h_n}(0,\infty) + t_n \sum_{k=1}^\infty F_{h_n}(k t_n, \infty) \leqslant \frac{4}{c} \sup_{x\in S} E^x(T_{h_n})$$

and b) is also proved.

<u>Lemma 2.</u> For every $k \in N$ and $\ell > 0$

(a) $\quad P^\mu(\sigma^n_k > \ell) \leqslant F^{*k}_{h_n + d_n}(\ell, \infty)$

(b) $\quad P^\mu_n(\tau_k \leqslant \ell) \leqslant e^{*k}_{d\, q(h_n - d_n)}(0, \ell)$

τ_k is defined by : $\tau_1 = \inf(t \mid X_t \neq X_0)$, $\tau_k = \tau_{k-1} + \tau_1 \circ \theta_{\tau_{k-1}}$

and e_a is the exponential distribution with parameter a.

<u>Proof</u>. We note first that for any two distributions F and G on R_+

$$F * G(\ell, \infty) = \int_0^\infty F(\ell - t, \infty)\, G(dt)$$

It is obvious that if F and F' are such that $F(s,\infty) \leqslant F'(s,\infty)$

for every $s \in R_+$, then (*) $F * G(\ell,\infty) \leqslant F' * G(\ell, \infty)$

To prove (a) we procede by induction on k. For k = 1, (a) is (13).

Using the strong Markov property for two variables functions we obtain:

$$P^\mu(\sigma^n_k > \ell) = P^\mu(\sigma^n_{k-1} + \sigma^n_1 \circ \theta_{\sigma^n_{k-1}} > \ell) = \int P^{X_{\sigma^n_{k-1}(\omega)}}(\sigma^n_1 > \ell - \sigma^n_{k-1}(\omega))\, P^\mu(d\omega)$$

By (11) and (13), for every fixed ω ,

$$P^{X_{\sigma^n_{k-1}(\omega)}}(\sigma^n_1 > \ell - \sigma^n_{k-1}(\omega)) \leq P^{X_{\sigma^n_{k-1}(\omega)}}(T_{h_n+d_n} > \ell - \sigma^n_{k-1}(\omega)) \leq F_{h_n+d_n}(\ell - \sigma^n_{k-1}(\omega), \infty)$$

and so

$$P^{\mu}(\sigma^n_k > \ell) \leq P^{\mu}\circ(\sigma^n_{k-1})^{-1} * F_{h_n+d_n}(\ell, \infty)$$

Now, using the induction hypothesis and (*) we finish the proof.

In order to prove (b) we obtain in the same way as above

$$P_n^{\mu}(\tau_k \leq \ell) = \int P_n^{X^n_{\tau_{k-1}(\omega)}}(\tau_1 \leq \ell - \tau_{k-1}(\omega))\, P_n^{\mu}(d\omega)$$

With respect with $P_n^{X_{\tau_{k-1}(\omega)}}$, the stopping time τ_1 is exponentialy distributed with parameter $q_n(X^n_{\tau_{k-1}}(\omega)) \leq d\, q(h_n - d_n)$

(see (12)).

Then

$$P^{X^n_{\tau_{k-1}(\omega)}}(\tau_1 \leq \ell - \tau_{k-1}(\omega)) \leq e_{d q(h_n - d_n)}(0, \ell - \tau_{k-1}(\omega))$$ for every ω.

We conclude that $P_n^{\mu}(\tau_k \leq \ell) \leq P_n^{\mu}\circ\tau_k^{-1} * e_{d q(h_n - d_n)}(0,t)$

and the proof finishes like above.

Lemma 3. For fixed $k > 0$ and $\delta > 0$, we define :

$k' = \dfrac{16 k d}{c}$, $\delta' = \dfrac{32 \delta d}{c}$ with c defined in (v) and d in (iv).

$k_n = [2k\, q(h_n - d_n)\, d]$, $\ell_n = [2\delta\, q(h_n - d_n)\, d]$

(We shall use these notations through all the rest of the paper).

(a) $\lim_n P_n^{\mu}(\tau_{k_n} < k) = 0$

(b) $\lim_n P^{\mu}(\sigma^n_{k_n} > k') = 0$

(c) $\lim_n P_n^{\mu}(A_n^c) = 0$; $A_n = (\omega \mid \tau_i - \tau_j > \delta, \forall i,j \leq k_n, i-j \geq \ell_n)$

(d) $\lim_n P^{\mu}(B_n^c) = 0$; $B_n = (\omega \mid \sigma^n_i - \sigma^n_j < \delta', \forall i,j \leq k_n, 0 \leq i-j \leq \ell_n)$

We note that all these limits are uniform with respect to the family $(\mu \mid \mu$ probablity measure on S).

(e) $\overline{\lim_n} \sup_{x \in S} E^x(\sigma^n_{k_n} ; \sigma^n_{k_n} > k') = 0$

The idea of this Lemma is that both $\sigma^n_{k_n}$ and τ_{k_n} are sums of little quantities with the same meab value. If we take k_n (the number of terms in the sum) such that $k_n a_n \sim \ell$ (a_n is the mean value), then $\sigma^n_{k_n} \sim \ell$ and $\tau_{k_n} \sim \ell$ (assimtoticaly). This is the idea of the law of large numbers, and to prove the Lemma we shall refere to Annexa 1, which presents appropiate forms of this law.

Proof (a) By L.2.b. $P^\mu_n(\tau_{k_n} < k) \leq e^{*k_n}_{dq(h_n-d_n)}(0,k)$

$$\lim_n d\,q(h_n-d_n)^{-1} k_n = 2k$$ and so, by L.3. Annexa 1.

$$\lim_n e^{*k_n}_{dq(h_n-d_n)}(0,k) = 0$$ (independent of μ)

(b) By L.2.a. $P^\mu_n(\sigma^n_{k_n} > k') \leq F_{h_n+d_n}(k',\infty)$

By L.1.b., $M(F_{h_n+d_n}) \leq \frac{4}{c}\frac{1}{q(h_n+d_n)}$ and so

$$M(F_{h_n+d_n})k_n \leq \frac{4}{c}\frac{2kq(h_n-d_n)d}{q(h_n+d_n)}$$

and by (10) we obtain $\overline{\lim_n}\, M(F(h_n+d_n))k_n \leq \frac{4}{c}\cdot 2kd = \frac{8kd}{c} = \frac{k'}{2}$

L.1.a. assures that conditions in L.1. and C.2. of Annexa 1 are fullfiled and so $\lim_n F_{h_n+d_n}(k',\infty) = 0$ and (b) is proved.

To prove c), we note that for i, j with $i - j \geq \ell_n$,

$$\tau_i - \tau_j \geq \tau_{j+\ell_n} - \tau_j$$ and so,

$$A^c_n = (\omega \mid \exists\, i,j \leq k_n \text{ such that } i-j > \ell_n \text{ and } \tau_i - \tau_j < \delta) \subseteq \bigcup_{j \leq k_n} (\tau_{j+\ell_n} - \tau_j < \delta)$$

Then $$P^\mu_n(A^c_n) \leq \sum_{j \leq k_n} P^\mu_n(\tau_{j+\ell_n} - \tau_j < \delta)$$

By the Markov property $P^\mu_n(\tau_{j+\ell_n} - \tau_j < \delta) = E^\mu_n(P^{X^n_{\tau_j(\omega)}}(\tau_{\ell_n} < \delta))$

which is dominated by $e^{*\ell_n}_{dq(h_n-d_n)}(0,\delta)$ (see L.2.b.), and so

$$P^\mu_n(A^c_n) \leq k_n e^{*\ell_n}_{dq(h_n-d_n)}(0,\delta) = \frac{k_n}{\ell_n}\ell_n e^{*\ell_n}_{dq(h_n-d_n)}(0,\delta)$$

$\lim\limits_n \frac{k_n}{p_n} = \frac{k}{\delta} < \infty$ and $\lim\limits_n \frac{\ell_n}{d\,q(h_n - d_n)} = 2\delta$ then, by L.3. Annexa 1 the term in the right of the above inequality vanishes when $n \to \infty$

To prove d) we note that $\omega \in B_n^c$ implies that there is some $i, j \leqslant k_n$ such that $0 \leqslant i - j \leqslant \ell_n + 1$ and $\sigma_i^n - \sigma_j^n > \delta'$.

Then, there is some $p \leqslant \frac{k_n}{l_n + 1}$ such that $\sigma^n_{p(\ell_n+1)} - \sigma^n_{(p-1)(\ell_n+1)} > \frac{\delta'}{2}$

We conclude that
$$P^\mu(B_n^c) \leqslant \sum_{p < \frac{k_n}{\ell_n+1}} P^\mu\left(\sigma^n_{p(\ell_n+1)} - \sigma^n_{(p-1)(\ell_n+1)} > \frac{\delta'}{2}\right)$$

By the strong Markov property and L.2.a. we obtain :

$$P^\mu\left(\sigma^n_{p(\ell_n+1)} - \sigma^n_{(p-1)(\ell_n+1)} > \frac{\delta'}{2}\right) \leqslant F^{*(\ell_n+1)}_{h_n+d_n}\left(\frac{\delta'}{2}, \infty\right) \text{ for every } p,$$

and so $P^\mu(B_n^c) \leqslant \frac{k_n}{\ell_n+1} F^{*(\ell_n+1)}_{h_n+d_n}\left(\frac{\delta'}{2}, \infty\right)$

The proof ends like to the points b and c.

To prove e) we note that L.2.a. implies that

$$E^x(\sigma^n_{k_n};\ \sigma^n_{k_n} > k') = \int_{(z>k')} z\, P^x \circ \sigma^{n\,-1}_{k_n}(dz) \leqslant \int_{(z>k')} z\, F^{*k_n}_{h_n+d_n}(dz)$$

Because $\overline{\lim\limits_n}\, F^{*k_n}_{h_n+d_n} = \varepsilon_k$ with $k < k'$

the term in the right of the above inequality vanishes under $\overline{\lim\limits_n}\, \sup\limits_x$ and e) is proved.

<u>Lemma 4</u>. $(X^n_{\tau_k}, k \in N)$ has, with respect to P^μ_n the same distribution as $(X_{\sigma^n_k}, k \in N)$ with respect to P^μ.

<u>Proof</u>. That is because both of them are Markov chains with initial distribution μ and kernel $\Pi_n(x, dy) = P^x \circ X^{-1}_{\sigma^n_1}$

Now, in order to prove the relative compactness of the sequence P_n^μ, $n \in N$ we shall use T 2 page 429 in /3/, which we write down for processes with a L.C.D.B. state space and time $[0, \infty)$:

The tightness of P_n^μ, $n \in N$ is equivalent to the following conditions :

1) for every $\varepsilon > 0$ and $k > 0$ there is some compact set $K_\varepsilon \subset S$ such that $\overline{\lim_n}\ P_n^\mu(\cdot\ \omega \mid$ there is some $k > 0$ such that $X_t^n \notin K) \leq \varepsilon$

2) $\lim_{\delta \to 0} \overline{\lim_n}\ P_n^\mu(\omega \mid w''_{k,\delta}(X^n) > \varepsilon) = 0$ for every $k > 0$, $\varepsilon > 0$.

3) $\lim_{\delta \to 0} \overline{\lim_n}\ P_n^\mu(\omega \mid w_{[0,\delta)}(X^n(\omega)) > \varepsilon) = 0$ for every $\varepsilon > 0$

4) $\lim_{\delta \to 0} \overline{\lim_n}\ P_n^\mu(\omega \mid w_{[k-\delta, k)}(X^n(\omega)) > \varepsilon) = 0$ for every $\varepsilon > 0$, $k > 0$

$w_{[a,b)}(X) = \sup(d(X_t, X_s);\ a \leq t < s < b)$

$w''_{k,\delta}(X) = \sup(\min(d(X_{t'}, X_t), d(X_t, X_{t''}))$

over all t', t, t'' such that $0 \vee (t-\delta) \leq t' < t \leq t'' \leq k \wedge (t+\delta)$

To prove 1) it will be sufficient to show that for any $k > 0$, $\varepsilon > 0$ and any compact set $K \subset S$, there is some $k' > 0$ such that :

$$\overline{\lim_n}\ P_n^\mu(\exists t \leq k,\ X_t^n \notin K) \leq P^\mu(\exists t \leq k',\ X_t \notin K)$$

and then we refere to the tightness of P^μ itself.

By L.3.a. we know that $\overline{\lim_n}\ P_n^\mu(\exists t \leq k,\ X_t^n \notin K) =$

$$= \overline{\lim_n}\ P_n^\mu(\exists t \leq k,\ X_t^n \notin K,\ \tau_{k_n} > k) \leq \overline{\lim_n}\ P_n^\mu(\exists j \leq k_n,\ X^n_{\tau_j} \notin K)$$

(k_n is chosen with respect to k like in L.3.).

By L.4. we know that the last term in the above inequality is equal to $\overline{\lim_n}\ P^\mu(\omega \mid \exists j \leq k_n,\ X_{\sigma_j^n}(\omega) \notin K)$ and by L.3.b. that is $\overline{\lim_n}\ P^\mu(\omega \mid \exists j \leq k_n,\ X_{\sigma_j^n}(\omega) \notin K,\ \sigma_j^n < k')$ with k' chosen in L.3.

Because the terms under this limit are dominate by $P^\mu(\exists t < k',\ X_t \notin K)$, the proof of 1) is complete.

(In what will follow we shall frequently use the same way of passing from X^n to X).

To prove 4) we note first that $w_{[k-\delta,k)}(X) \leq 2 \sup_{k-\delta \leq t \leq k} d(X_t, X_{k-\delta})$

We have to show that for every $k > 0$ and $\varepsilon > 0$

(18) $\lim_{\delta \to 0} \overline{\lim_n}\ P_n^\mu(\sup_{k-\delta \leq t \leq k} d(X_t^n, X_{k-\delta}^n) > \varepsilon) = 0$

By the Markov property

$$P_n^\mu\left(\sup_{k-\delta\le t\le k} d(X_t^n, X_{k-\delta}^n)\ge\varepsilon\right)=\int P_n^x\left(\sup_{0\le t\le\delta} d(X_0^n,X_t^n)>\varepsilon\right)P_n^\mu\circ X_{k-\delta}^{n^{-1}}(dx)$$

Using L.4. we obtain in the same way like above

$$P_n^x\left(\sup_{0\le t\le\delta} d(X_0^n,X_t^n)>\varepsilon\right)\le P^x\left(\sup_{0\le t\le\delta'} d(X_0,X_t)>\varepsilon\right)+P_n^\mu(\tau_{\ell_n}>\delta)+P_n^\mu(\sigma_{\ell_n}^n<\delta')$$

with δ and ℓ_n like in L.3.

Because the convergence in L.3.a.b. is uniform with respect to $x\in S$ the last two terms venish under $\overline{\lim_n}\int$.

$\delta\to 0 \Rightarrow \delta'\to 0$ and so we have to prove that

$$\lim_{\delta'\to 0}\overline{\lim_n}\int P^x\left(\sup_{0\le t\le\delta'} d(X_t,X_0)>\varepsilon\right)P_n^\mu\circ X_{k-\delta}^{n^{-1}}(dx)=0$$

For a fixed $\eta>0$, 1) assures that we may choose a compact set K_η such that

$$\overline{\lim_n}\, P_n^\mu(\exists\, t<k,\ X_t^n\in K_\eta)\le\eta \tag{19}$$

We dominate the above integral by

$$\sup_{x\in K_\eta} E^x\left(\sup_{0\le t\le\delta'} d(X_t,X_0)>\varepsilon\right)+P_n^\mu\circ X_{k-\delta}^{n^{-1}}(K_\eta^c)$$

Annexa 2 assures that the first term vanishes under $\lim_{\delta'\to 0}$.

By (19), for every $\delta>0$

$$\lim_n P_n^\mu\circ X_{k-\delta}^{n^{-1}}(K_\eta^c)\le\eta$$

η is arbitrary small and the proof is complete.

An analogous proof goes for 3) and also for

$$\lim_{\delta\to 0}\overline{\lim_n}\, P_n^\mu\left(\sup_{t\le s\le t+\delta} d(X_t,X_s)>\varepsilon\right)=0 \text{ for every } t>0,\ \varepsilon>0 \tag{20}$$

This last relation will be used later.

To prove 2) we define for a fixed $k>0$ and $\delta>0$ a discrete correspondent of $w''_{k,\delta}$: $T_n: S^{k_n}\to R$.

$T_n(x_{1,2},\dots,x_{kn}) = \sup\min(d(x_i,x_j),\, d(x_i,x_p))$ with $i,j,p\in N,\ i-(\ell_n+1)\le j<i<p\le k\wedge(i+(\ell_n+1))$
(k_n and ℓ_n are those in L.3.).

By L.3.a.c.

$$\overline{\lim_n}\; P_n^\mu(W''_{k,\delta}(X^n)>\varepsilon) = \overline{\lim_n}\; P^\mu(W''_{k,\delta}(X^n)>\varepsilon, A_n, \tau_{k_n}>k)$$

We note that for $\omega \in A_n \cap (\tau_{k_n}>k)$

$$W''_{k,\delta}(X^n(\omega))>\varepsilon \Rightarrow T_n(X^n_{\tau_1},\dots,X^n_{\tau_{k_n}})>\varepsilon$$

Let be $0\leqslant t', t, t'' \leqslant k$ such that $t-\delta \leqslant t' < t < t'' \leqslant t+\delta$

and $d(X^n_{t'}, X^n_t)>\varepsilon$, $d(X^n_t, X^n_{t''})>\varepsilon$.

We define i by $\tau_i \leqslant t < \tau_{i+1}$ and , j and p the corresponding integers for t' and t''.

$t-t'<\infty \Rightarrow \tau_i(\omega)-\tau_{j+1}(\omega)<\delta$. Because $\omega\in A_n$ it follows that $i-(j+1)\leqslant l_n$, that is $i-j\leqslant l_n+1$.

In the same way we obtain $p-i \leqslant l_n+1$

Because $\omega \in (\tau_{k_n}>k)$, $t''\leqslant k \Rightarrow \tau_p \leqslant k \Rightarrow p\leqslant k_n$.

$X^n_t = X^n_{\tau_i}$, $X^n_{t'} = X^n_{\tau_j}$ and $X^n_{t''} = X^n_{\tau_p}$ implies that $\min(d(X^n_{\tau_i}, X^n_{\tau_j}), d(X^n_{\tau_i}, X^n_{\tau_p}))>\varepsilon$
and the above implication is proved.

We may now write

$$\overline{\lim_n}\; P_n^\mu(W''_{k,\delta}(X^n)>\varepsilon) \leqslant \overline{\lim_n}\; P_n^\mu(T_n(X^n_{\tau_1},\dots,X^n_{\tau_{k_n}})>\varepsilon)$$

By L.4. first and then by L.3.b.d. the last term is equal to

$$\overline{\lim_n}\; P^\mu(T_n(X_{\sigma_1^n},\dots,X_{\sigma^n_{k_n}})>\varepsilon, B_n, \sigma^n_{k_n}<k')$$

In the same way as above we may dominate this term by

$$P^\mu(W''_{k',\delta'}(X)>\varepsilon)$$

So we have proved that

$$\overline{\lim_n}\; P_n^\mu(W''_{k,\delta}(X)>\varepsilon) \leqslant P^\mu(W''_{k',\delta'}(X)>\varepsilon)$$

$k<\infty \Rightarrow k'<\infty$, $\delta\to 0 \Rightarrow \delta'\to 0$, and so, we may refer to the tightness of X, and the proof of 2 is complete.

To prove the convergence of the finite dimensional distributions we have to verify the hypothesis of L.3. Annexa 3.

The first one is an immediate consequence of i) in our theorem.

For 2) we have to verify condition (c) enounced in the beginning of the paper :

(c) for any compact set $L \subseteq S$, $\alpha > 0$ and $\varepsilon > 0$, there is some compact set $K \subseteq S$ such that

$$\lim_n \sup_{x \in L} G_\alpha^{(n)}(x, K^c) \leq \varepsilon$$

To do it we shall prove that for every compact set $\tilde{K} \subseteq S$ and $k > 0$ we may choose another compact set $K \subseteq S$ such that $\tilde{K} \subseteq K$ and

$$\text{(21)} \qquad \overline{\lim_n} \sup_{x \in L} G_\alpha^{(n)}(x, K^c) \leq e^{\alpha k'} \sup_{x \in L} G_\alpha(x, \tilde{K}^c) + e^{-\alpha k}$$

(k' in L.3.).

If (21) is true, we choose k such that $e^{-\alpha k} < \frac{\varepsilon}{2}$ and ε' such that $e^{\alpha k'} \varepsilon' < \frac{\varepsilon}{2}$

Then, L.2. Annexa 2 assures that there is some compact set $\tilde{K} \subseteq S$ such that $\sup_{x \in L} G_\alpha(x, \tilde{K}^c) < \varepsilon'$

Then, the compact K mentioned in (21) is the compact set needed in (c).

Now, to prove (21) we choose K in the following way :

If $A^n = \bigcup_{U_i^n \cap \tilde{K} \neq \emptyset} U_i^n$, then by the definition of U^n_i we may choose a compact set K such that $A^n \subseteq K$ for every $n \in \mathbb{N}$.

Next we prove the following inequality :

$$\text{(22)} \qquad E_n^x \int_{\tau_i}^{\tau_{i+1}} e^{-\alpha t} 1_{K^c} \circ X_t^n \, dt \leq E^x \int_{\sigma_i^n}^{\sigma_{i+1}^n} 1_{\tilde{K}^c} \circ X_t \, dt$$

Because $X_t^n = X_{\tau_i}^n$ for $\tau_i \leq t < \tau_{i+1}$ and $\tau_{i+1} - \tau_i = \tau_1 \circ \theta_{\tau_i}$, we have

$$\int_{\tau_i}^{\tau_{i+1}} e^{-\alpha t} 1_{K^c} \circ X_t^n \, dt = 1_{K^c} \circ X_{\tau_i}^n \int_{\tau_i}^{\tau_{i+1}} e^{-\alpha t} dt \leq 1_{K^c} \circ X_{\tau_i}^n \, \tau_1 \circ \theta_{\tau_i}$$

We dominate the term in the left of (22) by $E_n^x(1_{K^c} \circ X_{\tau_i}^n \, \tau_1 \circ \theta_{\tau_i})$, which by the strong Markov property is

$$E_n^x(1_{K^c} \circ X_{\tau_i}^n \, E^{X_{\tau_i}^n}(\tau_1)) = \Pi_n^i(1_{K^c} \circ q_n^{-1})$$

By the definition of σ_i^n and K_n we have

$$E^x \int_{\sigma_i^n}^{\sigma_{i+1}^n} 1_{\tilde{K}^c} \circ X_t \, dt \geq E^x \left(1_{K^c} \circ X_{\sigma_i^n} \int_{\sigma_i^n}^{\sigma_{i+1}^n} 1 \, dt \right)$$

In the same way like above, the last term is equal to $\Pi_n^i (1_{K^c} q_n^{-1})^x$ and so (22) is proved.

Next, to prove (21), we shall change X^n by X in the same way as above :

$$(23) \quad G_\alpha^{(n)}(x, K^c) = E_n^x \int_0^k e^{-\alpha t} 1_{K^c} \circ X_t^n \, dt + E_n^x \int_k^\infty e^{-\alpha t} 1_{K^c} \circ X_t^n \, dt$$

The second term in the sum is dominated by $e^{-\alpha k}$.

The first one is dominated by

$$E_n^x \left(\int_0^k e^{-\alpha t} 1_{K^c} \circ X_t^n \, dt ; \tau_{k_n} > k \right) + \frac{1}{\alpha} P_n^x (\tau_{k_n} \leq k)$$

By L.3. a we may ignore the second term in the above sum. The first one is dominate by

$$E_n^x \left(\int_0^{\tau_{k_n}} e^{-\alpha t} 1_{K^c} \circ X_t^n \, dt \right)$$ which by (22) is dominate

by $$E^x \left(\int_0^{\sigma_{k_n}^n} 1_{\tilde{K}^c} \circ X_t \, dt \right) \leq E^x \left(\int_0^{\sigma_{k_n}^n} 1_{\tilde{K}^c} \circ X_t \, dt ; \sigma_{k_n}^n < k' \right) + $$
$$+ E^x \left(\int_0^{\sigma_{k_n}^n} 1_{\tilde{K}^c} \circ X_t \, dt ; \sigma_{k_n}^n \geq k' \right)$$

$\int_0^{\sigma_{k_n}^n} 1_{\tilde{K}^c} \circ X_t \, dt \leq \sigma_{k_n}^n$ and therefore we may dominate the second term in the sum by $E^x (\sigma_{k_n}^n ; \sigma_{k_n}^n \geq k')$ which we may ignore (see L.3.e.).

To dominate the first term, we note that on $\sigma_{k_n}^n < k'$

$$\int_0^{\sigma_{k_n}^n} 1_{\tilde{K}^c} \circ X_t \, dt \leq \int_0^{k'} 1_{\tilde{K}^c} \circ X_t \, dt \leq e^{\alpha k'} \int_0^{k'} e^{-\alpha t} 1_{\tilde{K}^c} \circ X_t \, dt$$

and therefore

$$E^x \left(\int_0^{\sigma_{k_n}^n} 1_{\tilde{K}^c} \circ X_t \, dt ; \sigma_{k_n}^n < k' \right) \leq e^{\alpha k'} G_\alpha (x, \tilde{K}^c)$$

The proof of (21) is complete and also that of (c).

We verify now the last condition in L.3. Annexa 3. :

For every $t \geqslant 0$, $f \in U_b(S)$ and $\varepsilon > 0$, there is some $\delta_\varepsilon > 0$ such that (25) $\overline{\lim_n} \sup_{t \leq s \leq t+\delta_\varepsilon} E_n^\mu(|f(X_t^n) - f(X_s^n)|) \leq \varepsilon$

We choose $\eta_\varepsilon > 0$ such that $d(x,y) \leq \eta_\varepsilon \Rightarrow |f(x) - f(y)| \leq \frac{\varepsilon}{2}$

$$E_n^\mu(|f(X_t^n) - f(X_s^n)|) \leq E_n^\mu(|f(X_t^n) - f(X_s^n)|; d(X_t^n, X_s^n) < \eta_\varepsilon) +$$
$$+ 2\|f\| E_n^\mu(d(X_t^n, X_s^n) > \eta_\varepsilon)$$

The first term is less as $\varepsilon/2$ and therefore

$$\overline{\lim_n} \sup_{t \leq s \leq t+\delta_\varepsilon} E_n^\mu(|f(X_t^n) - f(X_s^n)|) \leq \frac{\varepsilon}{2} + 2\|f\| \overline{\lim_n} E_n^\mu(\sup_{t \leq s \leq t+\delta_\varepsilon} d(X_t^n, X_s^n) > \eta_\varepsilon)$$

(20) assures that we may choose δ_ε needed in (25) and the whole prove ends.

ANNEXA 1

Lemma 1. Let $(F_n)_{n \in N}$ be a sequence of distributions on R_+, $(k_n)_{n \in N}$ a sequence of positive integers such that $\lim_n k_n a_n = \ell$

with
$$a_n = \int_0^\infty z\, F_n(dz)$$

If: a) $\lim_n a_n = 0$

b) $\lim_n \frac{1}{a_n} \int_a^\infty z\, F_n(dz) = 0$ for every $a > 0$

Then $\lim_n F_n^{*k_n} = \varepsilon_\ell$

Proof. It will be sufficient to show that $\lim_n \varphi_n(t)^{k_n} = e^{-i\ell t}$

with $\varphi_n(t) = \int_0^\infty e^{-itz} F_n(dz)$ By a), $\lim_n F_n = \varepsilon_0$ and so $\lim_n \varphi_n(t) = 1$. We may organize the above limit in an exponential form, and then, we have to show that $\lim_n k_n(\varphi_n(t) - 1) = -i\ell t$.

By the choose of k_n, this is $\lim_n \frac{1}{a_n}(1 - \varphi_n(t)) = it$.

We write $1-\varphi_n(t)$ in the following form :

$$it\int_0^\infty z\,F_n(dz)+\int_0^\infty z\left(\frac{1-\cos tz}{z}\right)F_n(dz)+i\int_0^\infty z\left(\frac{\sin z}{z}-t\right)F_n(dz)$$

Both $z \longrightarrow \frac{1-\cos(tz)}{z}$ and $z \longrightarrow \frac{\sin t\,z}{z}-t$ are bounded continuous functions which vanishes when $z \to 0$, therefore it will be sufficient to show that for such a function α, $\lim\limits_n \int_0^\infty z\,\alpha(z)\,F_n(dz)=0$

Let M be such that $|\alpha(z)|<M$ for $z \geqslant 0$, and for an $\varepsilon>0$ $a_\varepsilon>0$ such that $z<a_\varepsilon \Rightarrow |\alpha(z)|\leqslant\varepsilon$.

$$\frac{1}{a_n}\left|\int_0^\infty z\,\alpha(z)\,F_n(dz)\right|\leqslant\frac{1}{a_n}\int_0^{a_\varepsilon} z\,|\alpha(z)|\,F_n(dz)+\frac{M}{a_n}\int_{a_\varepsilon}^\infty z\,F_n(dz)$$

By b) the second term of the sum vanishes when $n\to\infty$. The first one is dominated by ε, which is arbitrary small, and so the proof is complete.

<u>Corollary 2.</u> If $\overline{\lim\limits_n}\, k_n a_n<\ell$ then $\lim\limits_n F_n^{*k_n}(\ell+\varepsilon,\infty)=0$

<u>Lemma 3.</u> If $a_n\uparrow\infty$ and $\lim\limits_n \frac{k_n}{a_n}=\ell+\varepsilon$ with $k_n\in N$, $\varepsilon>0$. then $\lim\limits_n k_n\, e_{a_n}^{*k_n}(0,\ell)=0$

<u>Proof.</u> Let $(E,\mathcal{K},P)$ be a probability space a[illegible] $f_n: E\to R$ $n\in N$ a sequence of independent variables, e_1 distributed.

$\frac{1}{a_n} f_i$ $i\in N$ are independent and ε_{a_n} distributed, and so

$$e_{a_n}^{*k_n}(0,\ell)=P\left(\frac{1}{a_n}f_1+\ldots+\frac{1}{a_n}f_{k_n}<\ell\right)=P\left(\frac{f_1+\ldots+f_{k_n}}{k_n}\,\frac{k_n}{a_n}\leqslant\ell\right)$$

By the choose of k_n, for a sufficiently large n, we may dominate this term by $P\left(\frac{f_1+\ldots+f_{k_n}}{k_n}<\frac{1}{1+\frac{\varepsilon}{\ell}}\right)\leqslant$

$$\leqslant P\left(\left|\frac{f_1+\ldots+f_{k_n}}{k_n}-1\right|>\frac{\varepsilon/\ell}{1+\varepsilon/\ell}\right)=P\left(\left(\sum_{i=1}^{k}(f_i-1)\right)^4>k_n^4c^4\right)$$

with

$$c=\frac{\varepsilon/\ell}{1+\varepsilon/\ell}.$$

By Chebishev's inequality we dominate it by $\frac{1}{k_n^4 c^4}\int \left(\sum_{i=1}^{k_n}(f_i-1)\right)^4 dP$

Because f_i are independent with mean value 0, this is

$$\frac{1}{k_n^4 c^4}\left(\sum_{i=1}^{k_n}\int (f_i-1)^4 dP + \sum_{i\neq j}\int (f_i-1)^2 dP \int (f_j-1)^2 dP\right.$$

The other terms of the sum vanish. This sum is dominated by $\frac{Mk_n^2}{k_n^4 c}$, with M sufficiently large.

$$\text{So, } \overline{\lim_n}\; k_n e^{*k_n}_{a_n}(0,\ell) \le \overline{\lim_n}\; k_n \frac{Mk_n^2}{k_n^4} = 0$$

<u>ANNEXA 2.</u>

The first Proposition follows an idea exposed in /4/ Annexa 3.

<u>Proposition 1.</u> Let S be a L.C.D.S. with a metric d such that $B_h(x)$ is relatively compact, and X a standard process with semigroup $(P_t)_{t\geq 0}$.

If for every $f\in C_c(S)$, $\lim_{t\to 0} P_t f = f$ uniformly on S, then for every compact set $K\subseteq S$ and $\varepsilon>0$

$$(1)\quad \lim_{h\to 0}\ \sup_{x\in K} P^x\left(\sup_{0\le t\le h} d(X_0,X_h)>\varepsilon\right)=0$$

<u>Proof.</u> We note first that, to prove (1) it will be sufficient to show that for every $L\subseteq G\subseteq S$, L compact set and G a relatively compact open set

$$(2)\quad \lim_{h\to 0}\ \sup_{x\in L} P^x(D_{G^c}\le h)=0$$

If (2) is true, the proof of (1) goes like this : for every $x\in K$ we choose an open set V_x and a compact one K_x such that

$$x\in V_x\subseteq K_x\subseteq B_{\varepsilon/2}(x)$$

We choose V_{x_i} $i\in N$ a finite coverment of K. Then K_{x_i} $i\le n$ will also be a coverment of K. For any $x\in K$, there is some $i\le n$ such that

$x\in K_{x_i}\subseteq B_{\frac{\varepsilon}{2}}(x_i)$ therefore $B_{\frac{\varepsilon}{2}}(x_i)\subseteq B_{\varepsilon}(x)$ and so

$D_{B^c_{\varepsilon}(x)} \geq D_{B^c_{\varepsilon/2}(x)}$.

Because $\left(\sup_{t\le h} d(X_0,X_t)>\varepsilon\right) = \left(D_{B^c_{\varepsilon}(x)}\le h\right)$ P^x a.s.

we may conclude that

$$\sup_{x\in K} P^x(\sup_{t\le h} d(X_0, X_t)>\varepsilon) \le \max_{i\le n} \sup_{x\in K_i} P^x(D_{B^c_{\varepsilon/2}(x)} \le h)$$

Now we take $L_i = K_i$ and $G_i = B_{\varepsilon/2}(x_i)$ (which is relatively compact), and (2) $\Longrightarrow$ (1) is proved.

To prove (2), we choose a relatively compact open set U such that $L \subsetneq U \subsetneq \bar{U} \subsetneq G$ and note that

(3) $\quad P^x(D_{G^c} \le h) \le P^x(D_{G^c} \le h, X_h \in U) + P^x(X_h \in U^c)$

Let be $f \in C(S)$ such that $1_{U^c} \le f \le 1_{L^c}$

$$P^x(X_h \in U^c) \le P_h 1_{U^c}^x \le P_h f^x$$

For $x\in L$, $f(x) = 0$ and so

$$\sup_{x\in L} P^x(X_h \in U^c) \le \sup_{x\in L} P_h f^x = \sup_{x\in L} |P_h f^x - f(x)| = \sup_{x\in L} |P_h g^x - g(x)|$$

with $g = 1 - f$. Because $\bar{U}$ is compact, $g \in C_c(S)$ and by our hypothesis, this term vanishes when $h \longrightarrow 0$

To dominate the first term in (3) we note that $X_{D_{G^c}} \in G^c$ and so $P^x(D_{G^c} \le h, X_h \in U) = P^x(D_{G^c} \le h, X_{D_{G^c}} \in G^c, X_h \in U) =$

$$= \int_{(D_{G^c}\le h,\, X_{D_{G^c}} \in G^c)} P^{X_{D_{G^c}}(\omega)}(X_{h - D_{G^c}(\omega)} \in U)\, P^x(d\omega) \le \sup_{t\le h} \sup_{y\in G^c} P^y(X_t \in U)$$

For $f \in C(S)$ such that $1_{\bar{U}} \le f \le 1_G$,

$$P^y(X_t \in U) = P_t 1_U^y \le P_t f^y = |P_t f^y - f(y)|$$

for every $y \in G^c$. $\operatorname{sup} f \subseteq \bar{G}$ and we may use our hypothese. This ends the proof.

<u>Lemma 2.</u> Let Q be a kernel on S, L. C.D.S.

If $Q(C_b(S)) \subset C_b(S)$, then, for every compact set $L \subset S$ and $\varepsilon > 0$, there is some compact set $K_\varepsilon \subseteq S$ such that

$$\sup_{x\in L} Q(x, K_\varepsilon^c) \le \varepsilon$$

<u>Proof.</u> For every $x\in L$ we choose K_x, K'_x compact sets and $f_x \in C_b(S)$ such that $Q(x, K_x^c) < \varepsilon$, $K_x \subset \operatorname{Int} K'_x$ and $1_{K_x} \le f_x \le 1_{K'_x}$

Then, $Q(x, K_x^c) < \varepsilon \Rightarrow Q(x, 1 - f_x) < \varepsilon$

$y \longrightarrow Q(1-f_x)^y$ is a continuous function and we may choose V_x such that $Q(y, 1-f_x) < \varepsilon$ for every $y \in V_x$. Let V_{x_i} $i \leq n$ be a finite coverment of L. Then, the compact K_ε will be $\bigcup_{i \leq n} V_{x_i}$

Indeed, for an $y \in L$, let be i such that $y \in V_{x_i}$

$$Q(x, K_\varepsilon^c) \leq Q(y, K'^{c}_{x_i}) \leq Q(y, 1-f_{x_i}) < \varepsilon$$

ANNEXA 3.

We introduce first some notations :

$$R_+^k = ((t_1 \dots t_k) \mid t_i \geq 0), \quad s^k = (s_1, \dots, s_k), \quad ds^k = ds_1, \dots, ds_k$$

$$A(\alpha^k, s^k) = \exp\left(-\sum_{i=1}^{k} \alpha_i s_i\right) \text{ with } \alpha^k = (\alpha_1, \dots, \alpha_k)$$

For a permutation σ on $(1, 2, \dots, k)$ we denote

$$\Lambda_\sigma^k = ((s_1, \dots, s_k) \mid 0 \leq s_{\sigma(1)} \leq \dots \leq s_{\sigma(k)})$$

If σ is the idebtic permutation we ignore it and write Λ^k

We consider a standard process and for $0 \leq t_1 < \dots < t_k$ and $f_i \in C_b(S)$ $i \leq n$ we define :

$$G_{\alpha^k} f_1 \dots f_k = E^\mu \int_{R_+^k} A(\alpha^k, s^k) \prod_{i \leq k} f_i(X_{s_i})\, ds^k$$

We note that $G_{\alpha^k} f_1 \dots f_k$ is the Laplace transform for the distribution $F(dt^k) = h(t^k)dt^k$ with $h(t^k) = E^\mu\left(\prod_{i \leq k} f_i(X_{t_i})\right)$

We define also $H_{\sigma, \alpha^k} f_1 \dots f_k = E^\mu \int_{\Lambda_\sigma^k} A(\alpha^k, s^k) \prod_{i \leq k} f_i(X_{t_i})\, ds^k$

If σ is the identic permutation, we ignore it in the notation.

Because $m(\partial \Lambda_\sigma^k) = 0$ (m is the Lebesgue measure)

(1) $$G_{\alpha^k} f_1 \dots f_k = \sum_\sigma H_{\sigma, \alpha^k} f_1 \dots f_k$$

Lemma 1. Let X^n $n \in N$ and X be standard processes with state space S, L.C.S.D. such that :

i) $G_\alpha(C_b(S)) \subseteq C_b(S)$

ii) for every $f \in C_b(S)$ $\lim_n G_\alpha^{(n)} f = G_\alpha f$ uniformly on compacts.

Then, for every $0 \leq t_1 < \dots < t_k$ and $f_i \in C_b(S)$ $i \leq k$

$$\lim_n G_{\alpha^k}^{(n)} f_1 \dots f_k = G_{\alpha^k} f_1 \dots f_k$$

(G_α is the Green operator of X with parameter $\alpha \in R_+$)

Proof. By (1) it will be sufficient to show that

$$\lim_n H^{(n)}_{\sigma,\alpha^k} f_1 \dots f_k = H_{\sigma,\alpha^k} f_1 \dots f_k$$

It is no loss of generality to do it only when σ is the identic permutation. We shall do it by induction on k. For k = 1, that is ii).

By the Markov property we obtain :

$$E^\mu\Big(\prod_{i\le k} f_i(X_{s_i})\Big) = E^\mu\Big(\prod_{i\le k-1} f_i(X_{s_i}) E^{X_{s_{k-1}}}(f_k(X_{s_k - s_{k-1}}))\Big)$$

Then, applying twice Fubini's theorem we get $H_{\alpha^k} f_1 \dots f_k =$

$$= \int_{\Lambda^{k-1}} ds^{k-1} A(\alpha^{k-1}, s^{k-1}) E^\mu\Big(\prod_{i\le k-1} f_i(X_{s_i}) E^{X_{s_{k-1}}}\Big(\int_{s_{k-1}}^\infty e^{-\alpha_k s_k} f_k(X_{s_k - s_{k-1}}) ds_k\Big)\Big)$$

By the changement $s = s_k - s_{k-1}$, we get :

$$\int_{s_{k-1}}^\infty e^{-\alpha_k s_k} f_k(X_{s_k - s_{k-1}}) ds_k = e^{-\alpha_k s_k} \int_0^\infty e^{-\alpha_k s} f_k(X_s) ds$$

and therefore

$$E^{X_{s_{k-1}}}\Big(\int_{s_{k-1}}^\infty e^{-\alpha_k s_k} f(X_{s_k - s_{k-1}}) ds_k\Big) = e^{-\alpha_k s_k} G_\alpha f_k(X_{s_{k-1}})$$

Conditions i) assure that $G_{\alpha_k} f_k \in C_b(S)$ therefore

$$H_{\alpha^k} f_1 \dots f_k = H_{\beta^{k-1}} f_1 \dots f_{k-2}\, g_{k-1}$$

with $\beta^{k-1} = (\beta_1, \dots, \beta_{k-1})$, $\beta_i = \alpha_i$, $i \le k-2$, $\beta_{k-1} = \alpha_{k-1} + \alpha_k$

and $g_{k-1} = f_{k-1} G_{\alpha_k} f_k$

We may establish an analogous relation for every $n \in N$. In this case β^{k-1} will be same, but $g^n_{k-1} = f_{k-1} G^{(n)}_{\alpha_k} f_k$, which is no more continuous. Neverthless, the definition of $H^{(n)}_{\beta^{k-1}}$ makes sense, and we write :

$$H^{(n)}_{\beta^{k-1}} f_1 \dots f_{k-2}\, g^n_{k-1} - H_{\beta^{k-1}} f_1 \dots f_{k-2}\, g_{k-1} = d^n_1 \cdot d^n_2$$

with

$$d^n_1 = H^{(n)}_{\beta^{k-1}} f_1 \dots f_{k-2}\, g^n_{k-1} - H^{(n)}_{\beta^{k-1}} f_1 \dots f_{k-2}\, g_{k-1}$$ and

$$d^n_2 = H^{(n)}_{\beta^{k-1}} f_1 \dots f_{k-2}\, g_{k-1} - H_{\beta^{k-1}} f_1 \dots f_{k-2}\, g_{k-1}$$

$g_{k-1} \in C_b(S)$ and therefore, by the induction hypothesis, $\lim_n d^n_2 = 0$

For a fixed $\varepsilon > 0$, let $K_\varepsilon \subseteq S$ be a compact set such that

$$H_{\beta^{k-1}} f_1 \cdots f_{k-2} 1_{K^c_\varepsilon} \leq \varepsilon$$

We choose another compact set K' such that $K_\varepsilon \subseteq \operatorname{Int} K'$ and a function $\varphi \in C(S)$ such that $1_{K_\varepsilon} \leq \varphi \leq 1_{K'}$

$$d^n_1 = H^{(n)}_{\beta^{k-1}} f_1 \cdots f_{k-2}((g^n_{k-1} - g_{k-1})\varphi) + H^{(n)}_{\beta^{k-1}} f_1 \cdots f_{k-2}((g^n_{k-1} - g_{k-1})(1-\varphi))$$

We dominate the first term of the sum by

$$\prod_{i \leq k-2} \|f_i\| \sup_{x \in K'} |g^n_{k-1} - g_{k-1}| \leq \prod_{i \leq k-1} \|f_i\| \sup_{x \in K'} |G^{(n)}_{\alpha_k} f_k - G_{\alpha_k} f_k|$$

which, by ii), vanishes when $n \to \infty$

The second term is dominated by $2\|f_k\|\,\|f_{k-1}\|\, H^{(n)}_{\beta^{k-1}} f_1 \cdots f_{k-2}(1-\varphi)$ which by the induction hypothesis converges to

$$2\|f_k\|\,\|f_{k-1}\|\, H_{\beta^{k-1}} f_1 \cdots f_{k-2}(1-\varphi)$$

By the choose of K_ε and φ, this term is dominated by $2\|f_k\|\,\|f_{k-1}\|\,\varepsilon$.

ε is arbitrary small and so the proof is complete.

Lemma 2. Let F_n $n \in N$ and F be distributions on R^k_+, of the form

$$F_n(ds^k) = h_n(ds^k)\,ds^k$$

$$F(ds^k) = h(ds^k)\,ds^k$$

If i) $\lim_n F_n = F$

ii) h_n $n \in N$ are equal (with respect to $n \in N$) right continuous.

Then : $\lim_n h_n(t^k) = h(t^k)$ for every $t^k \in R^k_+$

Proof. Let us suppose that there is some $t^k \in R^k_+$ such that $\lim_n h_n(t^k) \neq h(t^k)$. Passing to a subsequence, we may considere that for some $\varepsilon > 0$, $h_n(t^k) > h(t^k) + \varepsilon$ $n \in N$.

Let δ_ε be such that $s^k = (s_1, \ldots, s_k)$ with $t_i \leq s_i < t_i + \delta_\varepsilon$ $i \leq k$ implies

$$|h_n(t^k) - h_n(s^k)| \leq \frac{\varepsilon}{3} \quad n \in N$$

$$|h(t^k) - h(s^k)| \leq \frac{\varepsilon}{3}$$

We define : $A = \prod_{i \leq k} [t_i, t_i + \delta_\varepsilon]$. $F(\partial A) = 0$ and therefore

$\lim_n F_n(A) = F(A)$, that is (2) $\lim_n \int_A h_n(s^k) - h(s^k)\, ds^k = 0$.

$$h_n(s^k)-h(s^k)=(h_n(s^k)-h_n(t^k))+(h_n(t^k)-h(t^k))+(h(t^k)-h(s^k)).$$

For $s^k \in A$, the first and the last term of the above sum are dominated by $\frac{\varepsilon}{3}$, and the middle term is greater than ε, so $h_n(s^k)-h(s^k) \geq \frac{\varepsilon}{3}$ for $s^k \in A$ which is in contradictory with (2).

Lemma 3. Let X_n $n \in N$ and X be standard processes.

If i) $G_\alpha(C(S)) \subseteq C(S)$

ii) $\lim_n G_\alpha^{(n)} f = G_\alpha f$ uniformly on compacts for every $\alpha > 0$ and $f \in C_b(S)$

iii) for every $f \in C_b(S)$ and $t \geq 0$

$$\lim_{\delta \to 0} \overline{\lim_n} \sup_{t \leq s \leq t+\delta} E_n^\mu(|f(X_t^n)-f(X_s^n)|)=0$$

Then, for every $t_1 < t_2 < \ldots < t_k$ and $f_i \in \mathcal{U}_b(S)$, $i \leq k$

$$\lim_n E_n^\mu\left(\prod_{i \leq k} f_i(X_{t_i}^n)\right)=E^\mu\left(\prod_{i \leq k} f_i(X_{t_i})\right) \tag{3}$$

Remark : The above relation is sufficient to assure the convergence of the finite dimensional distribution.

Proof. We consider

$$F_n(dt_1,\ldots,dt_k)=E_n^\mu(f_1(X_{t_1}^n)\ldots f_k(X_{t_k}^n))\,dt_1\ldots dt_k$$

$$F(dt_1\ldots dt_k)=E^\mu(f_1(X_{t_1})\ldots f_k(X_{t_k}))\,dt_1\ldots dt_k$$

By L 1 we know that $F_n \longrightarrow F$

By L 2 we know that to establish (3) it will be sufficient to prove that $(t_1\ldots t_k) \to E_n^\mu(f_1(X_{t_1})\ldots f_k(X_{t_k}))$ $n \in N$ are equal right continuous in every point $t^k \in R_+^k$, and this is a simple consequence of iii).

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