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I. Introduction

Let Ω be a bounded domain of an Euclidian space R^N with sufficiently smooth boundary Γ. We denote $V = H_0^1(\Omega)$ the usual Sobolev space, $V^* = H^{-1}(\Omega)$ its dual via the inner product $(.,.)$ of space $H = L^2(\Omega)$.

If X is a Banach space, with norm $|\cdot|_X$, then $L^p(o,T;X)$ is the Banach space of all continuous, X-valued functions on $[o,T]$ and $W^{1,p}(o,T;X)$ is the Sobolev space of all functions $f: [o,T] \rightarrow X$ such that $f, f_t \in L^p(o,T;X)$.

We shall be concerned in this paper with the following distributed control problem:

(P) Minimize $$\int_o^T \left(\frac{1}{2} |y - y_d|_H^2 + \Psi(u) \right) dt$$

over the set of all continuous functions $y \in W^{1,2}(o,T;H)$ and $u \in L^2(o,T;U)$ subject to the following equations:

(1.1) $v_t(t,x) - \Delta y(t,x) = Bu(t)(x) + f(t,x)$ a.e. Q,

$v(t,x) \in \beta(y(t,x))$ a.e. Q,

(1.2) $v(o,x) = v_o(x)$ a.e. Ω,

(1.3) $y(t,x) = 0$ a.e. Σ

where β is a given maximal monotone graph in R^2, $\Psi : H \rightarrow]-\infty, +\infty]$, f, y_d and v_o are given, Q and Σ denote the cylinder $\Omega \times]o,T[$ and its lateral face $\Gamma \times]o,T[$, respectively.

We have found necessary conditions for optimality in problem (P). They are obtained by using an abstract approximating scheme of the control process. This method is employed by V.Barbu [1], [2] who has studied the distributed control problem for semi-linear parabolic equations and the boundary control problem for linear equations with nonlinear boundary conditions. See also D.Tiba [7] for a similar argument in the case of nonlinear hyperbolic control problems.

For the investigation of the boundary control problem governed by the equation (1.1) we quote Ch.Saguez [6].

We shall assume that the following conditions are satisfied:

1°. The maximal monotone graph β is everywhere defined and verifying :

(1.4) $(\beta(y) - \beta(z))(y-z) \geq \alpha(y-z)^2$, $\alpha > 0$

for all $y,z \in R$.

2°. $B : U \rightarrow H$ is linear, continuous. U is a Hilbert space of controls.

3°. f, $y_d \in L^2(o,T;H)$.

4°. $v_o \in H$, $y_o = \beta^{-1}(v_o) \in V$.

5°. $\psi : H \to]-\infty, +\infty]$ is a proper, convex, lower semicontinuous function.

It must be emphasized that (1.1)-(1.3) represent the general description of free boundary problems. In particular, when the graph β is given by:

$$(1.6) \quad \beta(r) = \begin{cases} r - r_o & \text{if } r > r_o \\ [-\rho, o] & \text{if } r = r_o \\ K(r-r_o) - \rho & \text{if } r < r_o \end{cases}$$

where K, ρ are positive constants, the problem (1.1)-(1.3) reduces to the Stefan boundary problem (see e.g. [6] or [5] p.196).

In expressing necessary conditions for problem (P) we shall use the generalized gradient of Clarke [4] , defined by :

$$(1.7) \quad D\beta(y) = \text{Conv}\left\{ w \in R^N ; w = \lim_{y_n \to y} \nabla\beta(y_n) \right\}$$

when β is assumed locally Lipschitz.

The main results of the paper are stated in section 4.

2. Approximating Control Problem

Let θ , Π be the operators from $L^2(o,T; H)$ into itself, defined by $\theta g = y$, $\Pi g = v$, where y and v satisfy:

(2.1) $v_t - \Delta y = g \quad , \quad v \in \beta(y)$ a.e. Q ,

(2.2) $v(o) = v_o$ a.e. Ω ,

(2.3) $y(t,x) = 0$ a.e. Σ .

Let ρ be a fixed C_o^∞ function on the real axis such that $\int_{-\infty}^{\infty} \rho = 1$, $\rho \geqslant 0$, $\rho(\tau) = 0$ for $|\tau| \geqslant 1$, $\rho(\tau) = \rho(-\tau)$.

Denote $\gamma = \beta - \alpha I$. According to assumption 1°, γ is a maximal monotone graph. We define the mollifier of β by:

$$\beta^\varepsilon(y) = \alpha y + \int_{-1}^{1} \gamma_\varepsilon(y - \varepsilon^2\tau)\rho(\tau)d\tau =$$

(2.4) $$= \alpha y + \varepsilon^{-2}\int_{-\infty}^{\infty} \gamma_\varepsilon(\tau)\rho\left(\frac{y-\tau}{\varepsilon^2}\right) d\tau =$$

$$= \alpha y + \gamma^\varepsilon(y) \quad ,$$

where $\gamma_\varepsilon = \gamma((I + \varepsilon\gamma)^{-1})$ is the Yosida regularization of γ .

Taking β^ε instead of β in (2.1), we define the operators Θ_ε , Π_ε in the same way.

Lemma 2.1. Let assumption 1° be satisfied. Then $dom(\Theta) = dom(\Pi) = L^2(o,T; H)$ and Θ, Π are weakly-strongly continuous from $L^2(o,T; H)$ in $C(o,T; H)$ and $C(o,T; V^*)$ respectively. Moreover the following estimates hold:

(2.5) $$|(\Theta g)_t|_{L^2(o,T;H)} + |\Theta g|_{L^\infty(o,T;V)} \leqslant C(1+|g|_{L^2(o,T;H)})$$

(2.6) $$|(\Pi g)_t|_{L^2(o,T;V^*)} + |\Pi g|_{L^\infty(o,T;H)} \leq C(1+ |g|_{L^2(o,T;H)})$$

<u>for all</u> $g \in L^2(o,T;H)$, <u>where</u> $C = C(\alpha, |v_o|_H, |y_o|_V)$ <u>is a constant.</u>

<u>Proof.</u>

Let $A : H \longrightarrow H$, $\mathrm{dom}(A) = H^1_o(\Omega) \cap H^2(\Omega)$, be given by :

$$Ay = -\Delta y .$$

Consider the following problem:

(2.7) $$\beta(y_\lambda(t))_t + A_\lambda y_\lambda(t) \ni g(t) \qquad \text{a.e. } [o,T] ,$$

(2.8) $$\beta(y_\lambda(o)) \ni v_o$$

which is equivalent to:

(2.9) $$y_\lambda(t) = \beta^{-1}\left\{v_o + \int_o^t [g(s) - A_\lambda y_\lambda(s)]\, ds\right\} .$$

Here A_λ is the Yosida approximation of operator A. Since β^{-1}, A_λ are Lipschitz, Eq. (2.9) has a unique solution $y_\lambda \in W^{1,2}(o,T;\ H)$ and there exists $v_\lambda \in W^{1,2}(o,T;\ H)$ such that :

(2.1o) $$v_\lambda(t) = v_o + \int_o^t [g(s) - A_\lambda y_\lambda(s)]\, ds \in \beta(y_\lambda) .$$

Multiplying (2.7) by $(y_\lambda)_t$, respectively $\beta(y_\lambda(t))$, a standard argument involving (1.4) implies the estimates:

(2.11) $$|(y_\lambda)_t|_{L^2(o,T;H)} + |(I+\lambda A)^{-1}y_\lambda|_{L^\infty(o,T;V)} \leq C(1+ + |g|_{L^2(o,T;H)}) ,$$

(2.12) $$|(v_\lambda)_t|_{L^2(o,T;V^*)} + |v_\lambda|_{L^\infty(o,T;H)} \leq C(1+|g|_{L^2(o,T;H)})$$

where $C = C(\alpha , |v_o|_H , |y_o|_V)$ is a constant.

Since in (2.11) we also get $\{\lambda^{\frac{1}{2}} A_\lambda y_\lambda\}$ bounded in $L^\infty(o,T; H)$, the Arzela-Ascoli theorem yields:

(2.13) $(I + \lambda A)^{-1}y_\lambda \to y$ strongly in $C(o,T;H)$, weakly* in $L^\infty(o,T; V)$

(2.14) $y_\lambda \to y$ strongly in $C(o,T; H)$,

(2.15) $(y_\lambda)_t \to y_t$ weakly in $L^2(o,T;H)$,

(2.16) $v_\lambda \to v \in \beta(y)$ strongly in $C(o,T; V^*)$ weakly* in $L^\infty(o,T; H)$,

(2.17) $(v_\lambda)_t \to v_t$ weakly in $L^2(o,T; V^*)$.

Then, letting $\lambda \to 0$ in (2.7) and (2.8) we see that y is a solution to problem (2.1)-(2.3) and therefore $\mathrm{dom}(\theta) = \mathrm{dom}(\Pi) = L^2(o,T; H)$.

If y and z are two solution to problem (2.1)-(2.3) we have:

(2.18) $$\beta(y)_t - \beta(z)_t - \Delta(y-z) \ni 0.$$

Taking the inner product in V^* by $\beta(y) - \beta(z)$ in the above equation, we infer that

(2.19) $$\frac{d}{dt}\,|\beta(y(t)) - \beta(z(t))|^2_{V^*} + \alpha|y(t) - z(t)|^2_H \leq$$

$$\leq 0 \qquad\qquad \text{a.e. } [o,T]$$

and therefore $y = z$, i.e. θ and Π are single valued. Moreover, from this it follows that the limits y and v are independent of the subsequence λ.

Letting $\lambda \to 0$ in (2.11) and (2.12) we obtain (2.5) an (2.6).

The weakly-strong continuity of θ and Π is an easy consequence now.

<u>Lemma 2.2. Let</u> $g, g_\varepsilon \in L^2(o,T;H)$, $g_\varepsilon \longrightarrow g$ <u>weakly in</u> $L^2(o,T;H)$ <u>as</u> $\varepsilon \to 0$. <u>Then</u>:

(2.2o) $$\theta_\varepsilon(g_\varepsilon) \longrightarrow \theta g \quad \text{strongly in } C(o,T;H)$$
$$\text{weakly}^* \text{ in } L^\infty(o,T;V),$$

(2.21) $$(\theta_\varepsilon(g_\varepsilon))_t \longrightarrow (\theta g)_t \quad \text{weakly in } L^2(o,T;H),$$

(2.22) $$\Pi_\varepsilon(g_\varepsilon) \longrightarrow v \in \beta(\theta(g)) \quad \text{strongly in } C(o,T;V^*)$$
$$\text{weakly}^* \text{ in } L^\infty(o,T;H)$$

(2.23) $\quad (\Pi_\varepsilon(g_\varepsilon))_t \rightarrow v_t$ $\quad$ weakly in $L^2(o,T; V^*)$

This is a variant of Lemma 2.1.

Approximating control problem. For each $\varepsilon > 0$ we denote by $\Psi_\varepsilon : H \rightarrow]-\infty, +\infty]$ the regularized function of Ψ :

(2.24) $$\Psi_\varepsilon(h) = \inf \left\{ \frac{1}{2\varepsilon} | h - v|_H^2 + \Psi(v); \quad v \in H \right\}$$

We consider the control problem

(P_ε) Minimize $$\int_0^T \left(\frac{1}{2} | y(t) - y_d(t)|_H^2 + \frac{1}{2} | u(t) - u^*(t)|_U^2 + \Psi_\varepsilon (u(t)) \right) dt$$

in $y \in W^{1,2}(o,T; H)$ and $u \in L^2(o,T;U)$ subject to the following equations:

(2.25) $$\beta^\varepsilon(y(t,x))_t - \Delta y(t,x) = Bu(t)(x) + f(t,x) \quad \text{a.e. } Q ,$$

(2.26) $$y(o,x) = y_o(x) \quad \text{a.e. } \Omega ,$$

(2.27) $$y(t,x) = 0 \quad \text{a.e. } \Sigma ,$$

where u^* is an optimal control in problem (P) and $y^* \in W^{1,2}(o,T; H)$ is the corresponding optimal state. We have:

Lemma 2.3. For each $\varepsilon > 0$ problem (P_ε) has at least one solution $(y_\varepsilon, u_\varepsilon) \in W^{1,2}(o,T;H) \times L^2(o,T; U)$.

Proof.

It is a standard argument since the functional

$\phi_\varepsilon : L^2(o,T; U) \longrightarrow]-\infty, +\infty]$ defined by :

$$(2.28) \qquad \phi_\varepsilon(u) = \int_0^T \left(\frac{1}{2} |\theta_\varepsilon(Bu + f) - y_d|_H^2 + \frac{1}{2} |u-u^*|_U^2 + \right.$$

$$\left. + \psi_\varepsilon(u) \right) dt$$

is coercive (uniformly in ε !) and weakly lower semicontinuous on $L^2(o,T; U)$.

<u>Lemma 2.4</u>. <u>Let</u> $(y_\varepsilon, u_\varepsilon)$ <u>be a solution to</u> (P_ε). <u>Then</u>

$$(2.29) \qquad y_\varepsilon \longrightarrow y^* \qquad \underline{\text{strongly in}} \ C(o,T; H)$$
$$\underline{\text{weakly}^* \text{ in}} \ L^\infty(o,T; V) ,$$

$$(2.30) \qquad u_\varepsilon \longrightarrow u^* \qquad \underline{\text{strongly in}} \ L^2(o,T; U) .$$

<u>Proof</u>.

Since $(y_\varepsilon, u_\varepsilon)$ is a solution to (P_ε), we get

$$\int_0^T \left(\frac{1}{2} | y_\varepsilon(t) - y_d(t)|_H^2 + \frac{1}{2} |u_\varepsilon(t) - u^*(t)|_U^2 + \psi_\varepsilon(u_\varepsilon(t))\right) dt \leqslant$$

$$\leqslant \int_0^T \left(\frac{1}{2} | \theta_\varepsilon(Bu^* + f) - y_d|_H^2 + \psi_\varepsilon(u^*)\right) dt \leqslant C_1$$

with C_1 a constant independent of ε .

Since ϕ_ε are uniformly coercive we obtain $\{ u_\varepsilon \}$ bounded in $L^2(o,T; U)$. Extracting a convenient subsequence, again denoted ε , we infer

$$(2.31) \qquad u_\varepsilon \longrightarrow u^o \quad \text{weakly in } L^2(o,T; U)$$

and by <u>Lemma 2.2</u>:

(2.32) $\quad y_\varepsilon \longrightarrow y^o = \theta(u^o)$ strongly in $C(o,T;H)$

and weakly* in $L^\infty(o,T;V)$.

Passing to the limit in the above inequality, it yields :

$$\int_0^T \left(\frac{1}{2}|y^o - y_d|_H^2 + \frac{1}{2}|u^o - u^*|_U^2 + \Psi(u^o)\right) dt \leq$$

$$\leq \int_0^T \left(\frac{1}{2}|y^* - y_d|_H^2 + \Psi(u^*)\right) dt.$$

Since (y^*, u^*) is an optimal pair for (P), we deduce $u^o = u^*$ and (2.29), (2.3o).

3. Necessary Conditions

__Lemma 3.1. Let assumption__ 1^o __be satisfied. Then, for all__ $g \in L^2(o,T;H)$, __there exists a linear operator__ $\nabla\theta_\varepsilon(g) : L^2(o,T;H) \rightarrow L^2(o,T;H)$ __defined by :__

(3.1) $$\nabla\theta_\varepsilon(g) = \text{weak} - \lim_{\lambda\to o} \frac{\theta_\varepsilon(g+\lambda w) - \theta_\varepsilon(g)}{\lambda}$$

__Moreover__

(3.2) $$\nabla\beta^\varepsilon(\theta_\varepsilon(g))\,\nabla\theta_\varepsilon(g)\,w(t) - \Delta\int_0^t \nabla\theta_\varepsilon(g)\,w(s)\,ds = \int_0^t w(s)\,ds,$$

(3.3) $$|\nabla\theta_\varepsilon(g)\,w|_{L^2(o,t;H)} \leq C\int_0^t |w(s)|_{V^*}\,ds,\quad t\in[o,T].$$

__Proof.__

Denote $y^\lambda = \theta_\varepsilon(g+\lambda w)$, $v^\lambda = \beta^\varepsilon(y^\lambda)$, $y = \theta_\varepsilon(g)$ and $v = \beta^\varepsilon(y)$. We have :

$$(3.4) \qquad v^\lambda - v - \Delta\int_0^t (y^\lambda - y) = \lambda\int_0^t w \,.$$

Multiply by $(y^\lambda - y)$ and use 1^o to get :

$$(3.5) \quad \alpha\,|y^\lambda(t) - y(t)|_H^2 - \frac{1}{2}\frac{d}{dt}\left(\Delta\int_0^t (y^\lambda - y), \int_0^t (y^\lambda - y)\right) \leq$$

$$\leq \lambda\left(\int_0^t w, y^\lambda - y\right).$$

Put $z^\lambda = \dfrac{y^\lambda - y}{\lambda}$. Then by (3.5) we infer $\left\{\int_0^t z^\lambda\right\}$ bounded in $L^\infty(o,T;\ V)$ and $\{z^\lambda\}$ bounded in $L^2(o,T;\ H)$.

Therefore

$$\int_0^t z^\lambda \longrightarrow S \qquad \text{strongly in } C(o,T;\ H)$$
$$\qquad\qquad \text{weakly}^* \text{ in } L^\infty(o,T;\ V)\,,$$

$$z^\lambda \longrightarrow S_t \qquad \text{weakly in } L^2(o,T;\ H)\,.$$

Since β^ε is Lipschitz of constant $\frac{1}{\varepsilon}$ (ε is fixed) we see that $\dfrac{v^\lambda - v}{\lambda}$ bounded in $L^2(o,T;\ H)$. Moreover, using the Lebesgue theorem, we get

$$\frac{v^\lambda - v}{\lambda} \longrightarrow \nabla\beta^\varepsilon(y)\,.\,S_t$$

weakly in $L^2(o,T;\ H)$.

Dividing by λ and passing to the limit in (3.4) we obtain (3.2). The solution in (3.2) is unique, so $\{z^\lambda\}$ is convergent on the initial sequence. As concerns (3.3), we start from

$$v_t^\lambda - v_t - \Delta(y^\lambda - y) = \lambda w$$

multiply by $v^\lambda - v$ in the scalar product of V^* and deduce :

$$\frac{1}{2}\,\frac{d}{dt}\,|v^\lambda - v|^2_{V^*} + (v^\lambda - v, y^\lambda - y) \leqslant \lambda\,|w(t)|_{V^*} \,.$$

$$.\,|v^\lambda(t) - v(t)|_{V^*} \qquad \text{a.e. } [o,T]$$

By hypothesis 1^o, it yields:

$$|z^\lambda|_{L^2(o,t;\,H)} \leqslant C \int_o^t |w(t)|_{V^*}$$

and (3.3) follows.

Remark 3.2. By virtue of (3.3), the operator $\nabla\theta_\varepsilon(g)$ may be extended continuously from $L^1(o,T;\,V^*)$ into $L^2(o,T;H)$, still satisfying inequality (3.3). So, we can define the operator

$$\nabla\theta_\varepsilon(g)^* \,:\, L^2(o,T;\,H) \longrightarrow L^\infty(o,T;\,V)$$

the adjoint of $\nabla\theta_\varepsilon(g)$.

Lemma 3.3. *For all* $q,\ g \in L^2(o,T;\,H)$ *it holds*

(3.6) $\quad |\nabla\theta_\varepsilon(g)^*\,q|_{L^\infty(t,T;V)} \leqslant C\,|q|_{L^2(t,T;H)}$

and the equality

(3.7) $\quad p = -\ \nabla\theta_\varepsilon(g)^*\,q$

is equivalent to the following one :

$$(3.8) \qquad \nabla\beta^{\varepsilon}(\theta_{\varepsilon}(g))p_t + \Delta p = q \qquad \text{a.e. } Q,$$

$$p(T,x) = 0 \qquad \text{a.e. } \Omega.$$

<u>Proof</u>

The proof is based on the definition of the adjoint and we omit it.

<u>Lemma 3.4</u>. <u>For each</u> $\varepsilon > 0$ <u>there exists</u> $p_{\varepsilon} \in L^{\infty}(o,T;V)$ <u>such that</u> :

$$(3.9) \qquad p_{\varepsilon} = -\nabla\theta_{\varepsilon}(Bu_{\varepsilon} + f)^{*}(y_{\varepsilon} - y_d)$$

$$(3.1o) \qquad B^{*} p_{\varepsilon}(t) = \partial\Psi_{\varepsilon}(u_{\varepsilon}(t)) + u_{\varepsilon}(t) - u^{*}(t) \qquad \text{a.e. } [o,T].$$

<u>Proof</u>

Since $(y_{\varepsilon}, u_{\varepsilon})$ is an optimal pair for (P_{ε}), by a standard argument we infer :

$$(3.11) \qquad \int_o^T (y_{\varepsilon}(t) - y_d(t), \nabla\theta_{\varepsilon}(Bu_{\varepsilon} + f)Bw(t)) + $$
$$+ (\partial\Psi_{\varepsilon}(u_{\varepsilon}(t)) + u_{\varepsilon}(t) - u^{*}(t), w(t))\, dt = 0$$

for all $w \in L^2(o,T;H)$.

Summarising the above lemmas we can write :

<u>Proposition 3.5</u>. <u>Under hypotheses</u> 1^o–5^o, <u>for each</u> $\varepsilon > o$, <u>problem</u> (P_{ε}) <u>has at least one optimal pair</u> $(y_{\varepsilon}, u_{\varepsilon}) \in W^{1,2}(o,T;H) \times L^2(o,T;U)$. <u>There exists</u> $p_{\varepsilon} \in L^{\infty}(o,T;V)$ <u>such that</u> :

$$\beta^{\varepsilon}(y_{\varepsilon})_t - \Delta y_{\varepsilon} = Bu_{\varepsilon} + f \qquad \text{a.e. } Q,$$

$$(3.12) \qquad \nabla\beta^{\varepsilon}(y_{\varepsilon}).(p_{\varepsilon})_t + \Delta p_{\varepsilon} = y_{\varepsilon} - y_d \qquad \text{a.e. } Q,$$

$$y_{\varepsilon}(o,x) = y_o(x), \; p_{\varepsilon}(T,x) = 0 \qquad \text{a.e. } \Omega,$$

$$y_{\varepsilon}(t,x) = 0, \; p_{\varepsilon}(t,x) = 0 \qquad \text{a.e. } \Sigma,$$

$$B^* p_\varepsilon(t) = \partial\psi_\varepsilon(u_\varepsilon(t)) + u_\varepsilon(t) - u^*(t) \qquad \text{a.e. } [o,T] .$$

Moreover $y_\varepsilon \to y^*$ strongly in $C(o,T; H)$, $u_\varepsilon \to u^*$ strongly in $L^2(o,T; U)$ and

(3.13) $p_\varepsilon \longrightarrow p^*$ strongly in $C(o,T; H)$,
weakly* in $L^\infty(o,T; V)$,

(3.14) $(p_\varepsilon)_t \longrightarrow p^*_t$ weakly in $L^2(o,T; H)$,

where:

(3.15) $B^* p^*(t) \in \partial\psi(u^*(t))$ a.e. $[o,T]$.

Proof

We have to prove (3.13)-(3.15). Multiplying (3.12) by $(p_\varepsilon)_t$ and noticing that $\nabla\beta^\varepsilon(y) \geqslant \alpha > 0$, we get :

$\{p_\varepsilon\}$ bounded in $L^\infty(o,T; V)$,

$\{(p_\varepsilon)_t\}$ bounded in $L^2(o,T; H)$.

From this it follows (3.13), (3.14). Relation (3.15) is an easy consequence of the demiclosedness of maximal monotone operators.

Remark 3.6. We can compare Proposition 3.5. with some results of Saguez [6] , Ch.4. However in some outstanding cases more can be said about the adjoint state p^* .

4. Examples

Example 1. The Stefan problem

We shall consider β given by (1.6).

We make the only assumption:

(4.1) $\quad \text{mes}\left\{(t,x) \in Q \ ; \ y^*(t,x) = r_o\right\} = 0$

which seems quite reasonable from the physical point of view.

<u>Theorem 4.1. Under assumptions</u> 1^o-5^o <u>and</u> (4.1) <u>there exists</u> $p^* \in W^{1,2}(o,T;\, H) \cap L^\infty(o,T;\, V)$ <u>which satisfies together with</u> y^*, u^* <u>equations</u> (1.1)-(1.3), (3.15) <u>and</u>

(4.2) $\quad \nabla\beta(y^*(t,x)).p^*_t(t,x) + \Delta p^*(t,x) = y^*(t,x) - y_d(t,x) \qquad \text{a.e. } Q,$

(4.3) $\quad p^*(T,x) = 0 \qquad \text{a.e. } \Omega,$

(4.4) $\quad p^*(t,x) = 0 \qquad \text{a.e. } \Sigma.$

<u>Proof</u>

We have to pass to the limit in (3.12). This is a difficult point because both the sequences $\left\{\nabla\beta^\varepsilon(y_\varepsilon)\right\}$, $\left\{(p_\varepsilon)_t\right\}$ are at most weakly convergent.

For convenience, we suppose $k \geqslant 1$, so we can choose $\alpha = 1$ and $\beta(r) = r + \gamma(r)$.

A detailed calculation gives :

(4.5) $$\gamma_\varepsilon(r) = \begin{cases} -r_o & r > r_o - \varepsilon r_o \\ \dfrac{r - r_o}{\varepsilon} & r \in [r_o - \varepsilon\delta,\ r_o - \varepsilon r_o] \\ \dfrac{mr - mr_o - \delta}{1 + \varepsilon m} & r < r_o - \varepsilon\delta \end{cases}$$

(4.6) $$\nabla\gamma_\varepsilon(r) = \begin{cases} 0 & r > r_o - \varepsilon r_o \\ \dfrac{1}{\varepsilon} & r \in (r_o - \varepsilon\delta,\ r_o - \varepsilon r_o) \\ \dfrac{m}{1+\varepsilon m} & r < r_o - \varepsilon\delta \end{cases}$$

where $m = k - 1 \geqslant 0$, $\delta = r_o + \rho > r_o$.

We deduce :

(4.7) $$(r - r_o)\nabla \gamma_\varepsilon(r) = \gamma_\varepsilon(r) - g_\varepsilon(r)$$

where:

(4.8) $$g_\varepsilon(r) = \begin{cases} - r_o & r > r_o - \varepsilon r_o \\ 0 & r \in (r_o - \varepsilon\delta, r_o - \varepsilon r_o) \\ - \dfrac{\delta}{1 + \varepsilon m} & r < r_o - \varepsilon\delta \end{cases}$$

It follows :

$$(r - r_o)\nabla \beta^\varepsilon(r) = (r-r_o)\ (1 + \nabla \gamma^\varepsilon(r)) =$$

$$= r - r_o + (r-r_o)\int_{-1}^{1} \nabla \gamma_\varepsilon(r- \varepsilon^2\theta)\,\rho(\theta)\,d\theta =$$

$$= r - r_o + \varepsilon^2\int_{-1}^{1} \nabla \gamma_\varepsilon(r - \varepsilon^2\theta)\,\theta\,\rho(\theta)\,d\theta +$$

$$+ \gamma^\varepsilon(r) - g^\varepsilon(r)$$

where :

(4.9) $$g^\varepsilon(r) = \int_{-1}^{1} g_\varepsilon(r - \varepsilon^2\theta)\,\rho(\theta)\,d\theta .$$

Since γ_ε is Lipschitz, we have

$|\varepsilon \nabla \gamma_\varepsilon(r)| \leqslant 1$, hence

$$h^\varepsilon(r) = \varepsilon^2\int_{-1}^{1} \nabla \gamma_\varepsilon(r - \varepsilon^2\theta)\,\theta\,\rho(\theta)\,d\theta \longrightarrow 0$$

uniformly in r.

Next we can write :

$$(4.10)\quad (y_\varepsilon(t,x) - r_o)\nabla\beta^\varepsilon(y_\varepsilon(t,x)) = \beta^\varepsilon(y_\varepsilon(t,x)) - r_o - g^\varepsilon(y_\varepsilon(t,x)) + h^\varepsilon(y_\varepsilon(t,x)) .$$

The term $\{g^\varepsilon(y_\varepsilon)\}$ is bounded in $L^\infty(Q)$ by (4.8), (4.9).

As concerns $\beta^\varepsilon(y_\varepsilon)$ we know from Lemma 2.2 that

$$(4.11)\quad \beta^\varepsilon(y_\varepsilon) \longrightarrow \beta(y^*) \quad \text{weakly}^* \text{ in } L^\infty(o,T;\, H) .$$

However by (4.5) we get

$$|\gamma_\varepsilon(r)| \leq C(1 + |r|), \quad \forall r$$

that is $|\gamma^\varepsilon(r)| \leq C(1 + |r|)$.

Since y_ε is bounded in $L^\infty(o,T;\, H_o^1(\Omega))$, the Sobolev embedding theorem gives $\{\beta^\varepsilon(y_\varepsilon)\}$ bounded in $L^s(Q)$ with some $s > 2$.

Because $y_\varepsilon(t,x) \to y^*(t,x)$ a.e. Q, we can deduce easily that $\beta^\varepsilon(y_\varepsilon(t,x)) \to \beta(y^*(t,x))$ a.e. Q . Here we use (4.5) and (4.1) essentially.

Now, it is wellknown that it yields (4.12) $\beta^\varepsilon(y_\varepsilon) \to \beta(y^*)$ strongly in $L^2(Q)$.

From (4.10) we see that

$$(4.13)\quad (y_\varepsilon - r_o)\nabla\beta^\varepsilon(y_\varepsilon) \cdot (p_\varepsilon)_t \longrightarrow$$

$$\longrightarrow (\beta(y^*) - r_o)\, p_t^* + w$$

weakly in $L^1(Q)$, where w is the weak limit in $L^2(Q)$ (on a subsequence) of $g^\varepsilon(y_\varepsilon) \cdot (p_\varepsilon)_t$.

On the other hand, by (2.25) we obtain:

$$\int_{\Omega}\int_0^T (\beta^{\varepsilon}(y_{\varepsilon})_t - \beta^{\lambda}(y_{\lambda})_t)(y_{\varepsilon} - y_{\lambda})dx\,dt -$$

$$- \int_{\Omega}\int_0^T \Delta(y_{\varepsilon} - y_{\lambda})(y_{\varepsilon} - y_{\lambda})\,dx\,dt = \int_{\Omega}\int_0^T B(u_{\varepsilon} - u_{\lambda}).(y_{\varepsilon} - y_{\lambda})dx\,dt\,.$$

Integrating by parts, we obtain the equality :

$$(4.14)\quad \int_{\Omega}(\beta^{\varepsilon}(y_{\varepsilon}(T,x)) - \beta^{\lambda}(y_{\lambda}(T,x)))(y_{\varepsilon}(T,x) - y_{\lambda}(T,x))\,dx -$$

$$- \int_0^T\int_{\Omega}(\beta^{\varepsilon}(y_{\varepsilon}) - \beta^{\lambda}(y_{\lambda}))((y_{\varepsilon})_t - (y_{\lambda})_t)\,dx\,dt \;+$$

$$+ \int_0^T\int_{\Omega}|\nabla(y_{\varepsilon} - y_{\lambda})|^2\,dxdt = \int_0^T\int_{\Omega}B(u_{\varepsilon} - u_{\lambda})(y_{\varepsilon} - y_{\lambda})dxdt$$

Taking into account (4.12), (4.11) and (4.14) we infer $y_{\varepsilon} \to y^*$ strongly in $L^2(0,T;\,V)$.

From (3.12), (3.13) we have

$$\nabla\beta^{\varepsilon}(y_{\varepsilon}).(p_{\varepsilon})_t \longrightarrow \ell \qquad \text{weakly}^* \text{ in } L^{\infty}(0,T;\,V^*).$$

It follows

$$(4.15)\quad (y_{\varepsilon} - r_0)\,.\nabla\beta^{\varepsilon}(y_{\varepsilon}).(p_{\varepsilon})_t \longrightarrow (y^* - r_0)\,.\,\ell$$

at least in distributions. In fact the convergence is true in the weak topology of $L^1(Q)$. Combining (4.13) and (4.15) we establish :

$$(4.16)\quad (\beta(y^*) - r_0)\,p_t^* + w = (y^* - r_0)\,.\,\ell, \quad \text{a.e. } Q\,,$$

Using again (4.1) and the fact that $g^{\varepsilon}(y_{\varepsilon})$ is bounded in $L^{\infty}(Q)$, we deduce that $g^{\varepsilon}(y_{\varepsilon}) \to g(y^*)$ strongly in $L^2(Q)$, where

$$g(y^*(t,x)) = \begin{cases} -r_o & y^*(t,x) > r_o \\ -\delta & y^*(t,x) < r_o \end{cases}$$

is defined a.e. Q .

Then $w(t,x) = g(y^*(t,x)) \cdot p_t^*(t,x)$ a.e. Q .

After a short calculation, we arrive at :

$$(\beta(y^*) - r_o)\, p_t^* + w = s \cdot p_t^* \qquad \text{a.e. } Q$$

with :

$$s(t,x) = \begin{cases} y^*(t,x) - r_o & y^*(t,x) > r_o \\ k(y^*(t,x) - r_o) & y^*(t,x) < r_o \end{cases}$$

that is $s(t,x) = \nabla\beta(y^*(t,x)) \cdot (y^*(t,x) - r_o)$ a.e. Q .

By (4.16), we obtain :

$$(y^* - r_o)\, .\nabla\beta(y^*) \cdot p_t^* = (y^* - r_o) \cdot \ell \qquad \text{a.e. } Q .$$

As $y^* \neq r_o$ a.e. Q from (4.1), we have

$$\ell(t,x) = \nabla\beta(y^*(t,x)) \cdot p_t^*(t,x) \qquad \text{a.e. } Q .$$

Therefore we can pass to the limit in (3.12) and prove (4.2)-(4.4).

Example 2. The convex case

If β is more regular, for instance locally Lipschitz, hypothesis (4.1) is not necessary.

The main assumptions are now :

(4.17) β is locally Lipschitz on R and supplies :

$$|\nabla\beta(y) \cdot y| \leq C\,(\beta(y) + y^2 + 1) \qquad \text{a.e. } R .$$

(4.18) $\beta = \xi - \mu$

where $\xi, \mu : R \to]-\infty, +\infty]$ are convex functions defined on the whole real axis .

To begin with, we state :

Proposition 4.2. Under hypothesis (4.17) if (y^*, u^*) is an optimal pair of (P), then there exists $p^* \in W^{1,2}(o,T;H) \cap L^\infty(o,T;V)$, $h \in L^1(Q)$ such that

(4.19) $h + \Delta p^* = y^* - y_d$ a.e. Q ,

$p^*(T,x) = 0$ a.e. Ω ,

$p^*(t,x) = 0$ a.e. Σ ,

(4.2o) $\nabla\beta^\varepsilon(y_\varepsilon) \cdot (p_\varepsilon)_t \longrightarrow h$ weakly in $L^1(Q)$.

Proof

The argument we use is similar to the one given in Barbu [1], Theorem 2.

By (4.17) it follows after some computation involving (2.4) that

$$|\nabla\beta^\varepsilon(y) \cdot y| \leq C\ (|\beta^\varepsilon(y)| + y^2 + 1)$$

with C a constant independent of ε .

We denote for every $\varepsilon > 0$ and n a positive integer

$$E_n^\varepsilon = \{(x,t) \in Q;\ |y_\varepsilon(t,x)| \leq n\}$$

We have $|\nabla\beta^\varepsilon(y_\varepsilon(x,t))| \leq C_n$ for $(x,t) \in E_n^\varepsilon$.

Let E be an arbitrary measurable set of Q.

We have :

$$\left|\int_E (p_\varepsilon)_t \cdot \nabla\beta^\varepsilon(y_\varepsilon)dxdt\right| \leq \int_{E\cap E_n^\varepsilon} |(p_\varepsilon)_t| \cdot |\nabla\beta^\varepsilon(y_\varepsilon)| +$$

$$+ \int_{E \setminus E_n^\varepsilon} |(p_\varepsilon)_t| \cdot |\nabla \beta^\varepsilon(y_\varepsilon)| \leq C_n \int_E |(p_\varepsilon)_t| \, dxdt +$$

$$+ C n^{-1} \int_{E \setminus E_n^\varepsilon} |\beta^\varepsilon(y_\varepsilon)| \cdot |(p_\varepsilon)_t| + C \int_{E \setminus E_n^\varepsilon} |y_\varepsilon| \cdot |(p_\varepsilon)_t| + C n^{-1} .$$

Since $\{\beta^\varepsilon(y_\varepsilon)\}$, $\{(p_\varepsilon)_t\}$ are bounded in $L^2(Q)$ and $\{y_\varepsilon\}$ is bounded in $L^\infty(o,T; H_o^1(\Omega))$ we see that the family $\{\int_E (p_\varepsilon)_t \nabla \beta^\varepsilon(y_\varepsilon) \, dx \, dt\}$ is equicontinuous and in virtue of the Dunford-Pettis criterion, weakly compact in $L^1(Q)$.

This finishes the proof.

Theorem 4.3. Under hypotheses (4.17), (4.18) if (y^*, u^*) is an optimal pair for problem (P) then there exists $p^* \in W^{1,2}(o,T; H) \cap L^\infty(o,T; V)$ such that:

$$D\beta(y^*(t,x)) \cdot p_t^*(t,x) + \Delta p^*(t,x) = y^*(t,x) - y_d(t,x), \text{ a.e. } Q,$$

$$p^*(T,x) = 0 \qquad \text{a.e. } \Omega,$$

$$p^*(t,x) = 0 \qquad \text{a.e. } \Sigma,$$

where $D\beta$ is the Clarke generalized gradient of the locally Lipschitz mapping β.

Proof

We shall use a trick due to Tiba [7].

For the sake of simplicity we first take β convex. We have to identify function h in (4.2o). We are interested to pass to the limit in the product $\nabla \gamma^\varepsilon(y_\varepsilon) \cdot (p_\varepsilon)_t$. Function γ is convex too. We write :

$$(p_\varepsilon)_t = p_\varepsilon^+ - p_\varepsilon^-$$

where p_ε^+, p_ε^- are the positive and the negative part of $(p_\varepsilon)_t$ up to an additive constant, such that they are strictly positive.

On a certain subsequence we have:

$p_\varepsilon^+ \longrightarrow v^+$; $p_\varepsilon^- \longrightarrow v^-$ weakly in $L^2(Q)$,

$$p_t^* = v^+ - v^- .$$

A more precise calculation of $\nabla \gamma^\varepsilon(y)$, available for locally Lipschitz functions γ, is :

$$(4.21) \quad \nabla \gamma^\varepsilon(y) = \int_{-\infty}^{\infty} \frac{\nabla \gamma((I+\varepsilon\gamma)^{-1}(y-\varepsilon^2\theta))}{1+\varepsilon \nabla\gamma((I+\varepsilon\gamma)^{-1}(y-\varepsilon^2\theta))} \rho(\theta)d\theta$$

Since γ is convex, instead of $\nabla\gamma$ we can set $\partial\gamma$, the subdifferential of γ.

From Lemma 2.2. , it is Known that $y_\varepsilon \longrightarrow y^*$ strongly in $L^2(Q)$. The Egorov theorem shows that for every $\eta > 0$ there is $Q_\eta \subset Q$ with mes $(Q - Q_\eta) < \eta$ and $y_\varepsilon \longrightarrow y^*$ uniformly on Q_η.

We study the weak convergence of $\nabla \gamma^\varepsilon(y_\varepsilon) \cdot p_\varepsilon^+$ in $L^2(Q_\eta)$. Consider any $f \in L^2(Q_\eta)$.

Then :

$$\int_{Q_\eta} \nabla \gamma^\varepsilon(y_\varepsilon) \cdot p_\varepsilon^+ \cdot f \, dx\, dt = \int_{-1}^{1} \rho(\theta)\, d\theta \int_{Q_\eta} p_\varepsilon^+ \cdot \partial\gamma(.) \cdot \frac{1}{1+\varepsilon\partial\gamma(.)} \cdot$$

$$\cdot \; f \; dx\, dt.$$

As on Q_η we have $(I+\varepsilon\gamma)^{-1}(y_\varepsilon - \varepsilon^2\theta)$ uniformly bounded, we need to consider only the integral :

$$(4.22)\quad \int_{Q_\eta} p_\varepsilon^+ \cdot \partial\gamma((I+\varepsilon\gamma)^{-1}(y_\varepsilon - \varepsilon^2\theta)).f\,dxdt, \quad \theta\in[-1,1]$$

Define the saddle function :

$$K(p,y) = \begin{cases} p\,\gamma(y) & p\geq 0 \\ -\infty & p<0 \end{cases}$$

which is closed, proper. The subdifferential of K is given by :

$$(4.23)\quad \partial K(p,y) = [-\gamma(y),\ p\,\partial\gamma(y)] \quad , \quad (p,y)\in \mathrm{dom}\,(\partial K)$$

We denote by $\widetilde{\partial} K$ the realization of the maximal monotone operator ∂K in $L^2(Q_\eta)\times L^2(Q_\eta)$. By (4.23) we infer :

$$(4.24)\quad [-\gamma((I+\varepsilon\gamma)^{-1}(y_\varepsilon - \varepsilon^2\theta)),\ p_\varepsilon^+\partial\gamma(.)]\in \widetilde{\partial} K\ (p_\varepsilon^+ ,\ (I+\varepsilon\gamma)^{-1}(y_\varepsilon - \varepsilon^2\theta)) \quad \text{a.e. } Q_\eta .$$

We remark that :

$$(4.25)\quad [-\gamma(.),\ p_\varepsilon^+ .\partial\gamma(.)] \rightarrow [-\gamma(y^*),\ \tilde{h}]$$

$$(4.26)\quad [p_\varepsilon^+ ,\ (I+\varepsilon\gamma)^{-1}(y_\varepsilon - \varepsilon^2\theta)] \rightarrow [v^+ ,\ y^*]$$

weakly in $L^2(Q_\eta)\times L^2(Q_\eta)$.

We can also verify the following condition:

$$(4.27)\quad \lim_{\lambda,\varepsilon\to 0} \Big([p_\varepsilon^+ ,\ (I+\varepsilon\gamma)^{-1}(y_\varepsilon - \varepsilon^2\theta)] - [p_\lambda^+ ,\ (I+\lambda\gamma)^{-1}(y_\lambda - \lambda^2\theta)] ,\ [-\gamma((I+\varepsilon\gamma)^{-1}(y_\varepsilon - \varepsilon^2\theta)),\ p_\varepsilon^+ .\partial\gamma(.)] - [-\gamma((I+\lambda\gamma)^{-1}(y_\lambda - \lambda^2\theta),\ p_\lambda^+ .\partial\gamma(.)] \Big)_{L^2(Q_\eta)\times L^2(Q_\eta)} = 0$$

because $(I+\varepsilon\gamma)^{-1}(y_\varepsilon - \varepsilon^2\theta) \longrightarrow y^*$ and

$\gamma((I+\varepsilon\gamma)^{-1}(y_\varepsilon - \varepsilon^2\theta)) \longrightarrow \gamma(y^*)$ uniformly on Q_η.

Applying a wellknown property of monotone operators (Barbu [3] , p.42) we get from (4.24)-(4.27) that :

(4.28) $\left[-\beta(y^*), \tilde{h}\right] \in \partial\tilde{K}(v^+, y^*)$

therefore $\tilde{h}(t,x) \in v^+(t,x).\partial\gamma(y^*(t,x))$ a.e. Q .

As the maximal monotone operators are demiclosed, from $(I+\varepsilon\gamma)^{-1}(y_\varepsilon - \varepsilon^2\theta) \longrightarrow y^*$ uniformly on Q_η and $\{\partial\gamma^\varepsilon(y_\varepsilon)\}$ bounded in $L^2(Q_\eta)$ we infer (4.29) $\partial\gamma^\varepsilon(y_\varepsilon) \rightarrow b \in \partial\gamma(y^*)$ weakly in $L^2(Q_\eta)$ for every $\eta > 0$.

We show that $\tilde{h}(t,x) = v^+(t,x).b(t,x)$ a.e.Q .

Assume that $\tilde{h}(t,x) = v^+(t,x).\ b_1(t,x) \in v(t,x)$.

$.\partial\gamma(y^*(t,x))$ a.e. Q .

Let $C \geq 0$ be some constant. In a similar manner we prove that

$$(p_\varepsilon^+ + C)\ .\partial\gamma^\varepsilon(y_\varepsilon) \longrightarrow (v^+ + C).\ b_C$$

weakly in $L^2(Q_\eta)$, with $b_C(t,x) \in \partial\gamma(y^*(t,x))$ a.e. Q .

On the other hand, by (4.28) and (4.29) we get :

$$(p_\varepsilon^+ + C)\ .\partial\gamma^\varepsilon(y_\varepsilon) \longrightarrow v^+.\ b_1 + C.b$$

weakly in $L^2(Q_\eta)$.

We obtain the equality :

$$(4.30) \quad v^{+}.\, b_1 + C.\, b = (v^{+} + C)\cdot b_c, \text{ a.e. } Q_\eta$$

for every $C \geq 0$.

Family $\{b_c\}$ is bounded in $L^\infty(Q_\eta)$ since $\partial\gamma$ is locally bounded on the real axis. By (4.30) we derive :

$$v^{+}(b_1 - b_c) = C\,(b_c - b)$$

and making $C \to 0$ it yields $b_c \to b_1$ uniformly on Q_η. (We can suppose $v\,(t,x) \geq \tilde{m} > 0$ on Q_η from the Luzin theorem, modifying with a positive constant p_ε^{+}, p_ε^{-} if necessary).

Now let $C_1, C_2 \geq 0$ be two arbitrary constants. We have:

$$(p_\varepsilon^{+} + C_2)\partial\gamma^\varepsilon(y_\varepsilon) \longrightarrow (v^{+} + C_2)\,.\, b_{c_2}$$

$$(p_\varepsilon^{+} + C_1 + C_2 - C_1)\partial\gamma^\varepsilon(y_\varepsilon) \longrightarrow (v^{+} + C_1).\, b_{c_1} + (C_2 - C_1)\,.b$$

weakly in $L^2(Q_\eta)$. That is :

$$(v^{+} + C_2).\, b_{c_2} = (v^{+} + C_1).\, b_{c_1} + (C_2 - C_1)b$$

and equivalently :

$$v^{+}(b_{c_2} - b_{c_1}) + C_2.b_{c_2} = C_1.b_{c_1} + (C_2 - C_1).b.$$

Take $C_2 = 3C_1 = 3C$ and divide by C_2 :

$$v^{+}.\,\frac{b_{3C} - b_c}{3C} + b_{3C} = \frac{1}{3}\, b_c + \frac{2}{3}\, b\,.$$

If $C \to 0$, it yields that $\frac{b_{3C} - b_C}{3C}$ has a limit a.e.,

which we denote h_3 :

(4.31) $\quad v^+ . h_3 = \frac{2}{3} (b - b_1) \quad$ a.e. Q_η .

Take again $C_2 = \sqrt{3}\ C_1$ and denote $h_{\sqrt{3}}$ the limit

we obtain in this case. We have:

(4.32) $\quad v^+ . h_{\sqrt{3}} = (1 - \frac{1}{\sqrt{3}}) (b - b_1)$.

On the other hand, from :

$$\frac{b_{3C} - b_C}{3C} = \frac{b_{3C} - b_{\sqrt{3}C}}{3C} \qquad \frac{b_{\sqrt{3}C} - b_C}{3C}$$

when $C \to 0$, we deduce :

(4.33) $\quad h_3 = \frac{2h_{\sqrt{3}}}{\sqrt{3}}$

Relations (4.31)-(4.33) show $b = b_1$ a.e. Q .

Now, return to the sequence $p_\varepsilon^- . \partial\gamma^\varepsilon(y_\varepsilon)$.

Reasoning in the same way, we prove :

(4.34) $\quad p_\varepsilon^- . \partial\gamma^\varepsilon(y_\varepsilon) \rightharpoonup \underset{\sim}{h} = v^- . b$

weakly in $L^2(Q_\eta)$.

Subtracting (4.34) and (4.28) we obtain

$(p_\varepsilon)_t . \partial\gamma^\varepsilon(y_\varepsilon) \to \underline{h} = p_t^* . b \in p_t^* . \partial\gamma(y^*)$

weakly in $L^2(Q_\eta)$.

When $\eta \to 0$, we infer in (4.19)

$$h(t,x) \in p_t^* \cdot \partial\beta(y^*) \quad \text{a.e. } Q .$$

In the case (4.18) is valid, the argument follows the same lines and instead of $\partial\beta$, the subdifferential of a convex function, it appears $D\beta$, the generalized gradient of the locally Lipschitz function .

This ends the proof.

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