ON THE CONVERGENCE OF A CLASS OF NEWTON-LIKE METHODS

by

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1. Introduction

In a previous paper [13] we have studied a class of iterative procedures of the form

(1) $$Fx_n+\delta F(x_{p_n},x_{q_n})(x_{n+1}-x_n)=0\ , \qquad n=0,1,2,\ldots$$

where F was a nonlinear operator between two Banach spaces, δF a strongly consistent approximation of F', and $(p_n)_{n\geq 0}$, $(q_n)_{n\geq 0}$ two sequences of integers satisfying the condition

(2) $$-1=q_o\leq q_n\leq n\ , \quad 0=p_o\leq p_n\leq n\ , \qquad q_n\leq p_n\ .$$

This iterative procedure reduces to Newton's method for $p_n=q_n=n$ and to the secant method for $p_n=n$, $q_n=n-1$. The choice

(3) $$p_{km+j}=q_{km+j}=km\ , \qquad j=0,1,\ldots,m-1;\ k=0,1,2,\ldots$$

where m is a fixed positive integer leads to an iterative procedure investigated in [2], [3], [15], [20], [22], [24]. If

(4) $$p_{km+j}=km\ ,\ q_{km+j}=km-1\ , \qquad j=0,1,\ldots,m-1;\ k=0,1,2,\ldots$$

then (1) reduces to the so called multi-step secant method (see [5], [9], [16], [20]).

It can be proved that the sequence $(z_k)_{k\geq 0}$, with, $z_k=x_{km}$ has the order of convergence equal to m+1 in case of choice (3), and $\frac{m+\sqrt{m^2+4}}{2}$ in case of choice (4). The parameter m can be chosen according to the dimension of the space as to maximize the efficiency of the procedure (see [20]).

Usually $\delta F(x,y)\neq\delta F(y,x)$ so that together with (1) we may also consider the iterative procedure

(5) $$Fx_n+ F(x_{q_n},x_{p_n})(x_{n+1}-x_n)=0\ , \qquad n=0,1,2,\ldots\ .$$

More generally we can set

$$(6) \qquad Fx_n + A_n(x_{n+1} - x_n) = 0, \qquad n=0,1,2,\ldots$$

where for each n, A_n can be taken either $\delta F(x_{p_n}, x_{q_n})$ or $\delta F(x_{q_n}, x_{p_n})$ i.e:

$$(7) \qquad A_n \in \{\delta F(x_{p_n}, x_{q_n}), \ \delta F(x_{q_n}, x_{p_n})\}, \quad n=0,1,2,\ldots$$

In the first part of the present paper we shall make a semi-local analysis for this iterative procedure obtaining a slight generalization of a result contained in [13]. This will improve also a result of J.E.Dennis [5].

In the second part of the paper we shall investigate the monotonicity properties of an iterative procedure of type (6), (7) generalizing some results of [11], [17], [19], [23], [25].

2. Notation

If X and Y are two linear spaces (L-spaces) then we denote by L(X,Y) the set of all linear operators from X into Y. If X and Y are topological linear spaces (TL-spaces) then B(X,Y) denotes the set of bounded linear operators from X into Y. If X and Y are normed spaces (N-spaces) then the space B(X,Y) is endowed with the operator norm. All the norms will be denoted by the symbol $\|\cdot\|$.

A subset K of an L-space X is called cone if $K+K\subset K$ and $\alpha K\subset K$ for $\alpha>0$. The cone K is called proper if $K\cap\{-K\}=\{0\}$. If K is a proper cone of X then the relation "$\le$", defined by $x\le y \Longleftrightarrow y-x\in K$ is a partial ordering in X. An L-space X endowed with such a relation is called a partially ordered linear space (POL-space). Two elements x and y of X are called comparable if either $x\le y$ or $y\le x$ holds. If x and y are comparable then we denote by $x\wedge y$ (resp. $x\vee y$) the minimum (resp. maximum) of x and y. If $a\le b$ then we denote by $[a,b]$ the set $\{x\in X;\ a\le x\le b\}$. If u, v are two comparable points of X then we denote by $\langle u,v\rangle$ either the set $[u,v]$, if $u\le v$, or the set $[v,u]$, if $u\ge v$.

A TL-space partially ordered by a closed proper cone is called a partially ordered topological linear space (POTL-space). A POTL-space X is called normal if, given a local basis $\mathcal{U}$ for the topology, there exists a positive number η so that if $0\le z\in U\in\mathcal{U}$ then $[0,z]\subset\eta U$. A POTL-space is called regular if every order bounded increasing sequence has a limit. We note that any regular partially ordered Banach space is normal but the reverse is not true. For example the space C[0,1] with the natural partial ordering is normal but not regular. All finite dimensional POTL-spaces are both normal and regular.

Let X and Y be two POL-spaces and G an operator from X into Y.

G is called: nonnegative if $Gz\geq 0$ for all $z\geq 0$; inverse nonnegative if $Gz\geq 0$ implies $z\geq 0$; isotone if $z_1\leq z_2$ implies $Gz_1\leq Gz_2$, antitone if $z_1\leq z_2$ implies $Gz_1\geq Gz_2$. If G is nonnegative we write $G\geq 0$. If G and H are two operators such that $G-H\geq 0$ then we write $H\leq G$. The space $L(X,Y)$ with the relation "$\leq$" defined above becomes a POL-space. Let T be an operator belonging to $L(X,Y)$. An operator $S\varepsilon L(Y,X)$ is called a left (resp.right) subinverse of T if $ST\leq I$ (resp.$TS\leq I$) where I denotes the identity operator in X (resp. in Y). S is called a subinverse of T if it is a left as well as a right subinverse of T.

3. Semi-local convergence

3.1. Theorem. Let F be a nonlinear operator defined on a convex subset D of a Banach space X, with values in a Banach space Y, and let x_o, x_{-1} be two points from the interior $\mathring{D}$ of D satisfying the inequality

(8) $$|x_o-x_{-1}|\leq c\ .$$

Suppose F is Fréchet differentiable on $\mathring{D}$ and there exists a mapping $\delta F:\mathring{D}\times\mathring{D}\to B(X,Y)$ such that the linear operator A_o, where A_o is either $\delta F(x_o,x_{-1})$ or $\delta F(x_{-1},x_o)$, is invertible, its inverse T_o is continuous and:

(9) $$|T_oFx_o|\leq b,$$

(10) $$|T_o(\delta F(x,y)-F'(z))|\leq a(|x-z|+|y-z|),\qquad x,y,z\varepsilon\mathring{D}\ .$$

Let $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ be two sequences of integers satisfying condition (2).

If the constants a, b, c introduced above satisfy the inequality

(11) $$ac+2(ab)^{1/2}\leq 1$$

and if the set $D_c=\{x\varepsilon D;\ f \text{ is continuous at } x\}$ contains the closed ball U with center $x_1=x_o-T_oFx_o$ and radius $r_1=\frac{1}{2a}\{1-a(2b+c)-[(1-ac)^2-4ab]^{1/2}\}$, then the iterative procedure described by (6), (7) is well defined (i.e. for each n there exists $x_{n+1}\varepsilon\mathring{D}$ satisfying (6)), the sequence $(x_n)_{n\geq 1}$ produced by it converges to a root $x^*\varepsilon U$ of the equation $Fx=0$ and the following estimates hold:

(12) $$|x_n-x^*|\leq t_o-|x_n-x_o|-[(t_o-|x_n-x_o|)^2-(|x_n-x_{p_{n-1}}|+ +|x_{n-1}-x_{p_{n-1}}|+|x_{p_{n-1}}-x_{q_{n-1}}|)|x_n-x_{n-1}|]^{1/2}\leq t_n-d\ ,$$

(13) $$|x_n-x^*|\geq[(t_o-2^{-1}(|x_{p_n}-x_{q_n}|+|x_{p_n}-x_o|+|x_{q_n}-x_o|)- -|x_n-x_{p_n}|)^2+(2t_o-|x_{p_n}-x_o|-|x_{q_n}-x_o|)|x_n-x_{n+1}|]^{1/2}-$$

$$-t_o+2^{-1}(|x_{p_n}-x_{q_n}|+|x_{p_n}-x_c|+|x_{q_n}-x_o|)+|x_n-x_{p_n}|$$

where

(14) $$d=\frac{1}{2a}[(1-ac)^2-4ab]^{1/2},$$

(15) $$t_{-1}=\frac{1+ac}{2a},\quad t_o=\frac{1-ac}{2a},\quad t_{n+1}=t_n-\frac{t_n^2-d^2}{t_{p_n}+t_{q_n}},\qquad n=0,1,2,\ldots .$$

Proof. We shall follow closely the proof of Theorem 1 of [13]. First we observe that the linear operator $P=\delta F(u,v)$ is invertible for all $u,v\in\mathring{D}$ with

(16) $$|u-x_o|+|v-x_o|<2t_o .$$

Indeed from (10) it follows that

$$|I-T_oP|=|T_o(A_o-P)|\le|T_o(P-F'(x_o))|+|T_o(F'(x_o)-A_o)|\le$$

$$\le a(|u-x_o|+|v-x_o|+|x_o-x_{-1}|)<1$$

so that, according to Banach's lemma P is invertible and

(17) $$|(T_oP)^{-1}|\le[1-a(|u-x_o|+|v-x_o|+c)]^{-1} .$$

Let us note that condition (10) implies the following Lipschitz condition for F':

(18) $$|T_o(F'(u)-F'(v))|\le 2a|u-v|,\qquad u,v\in\mathring{D} .$$

Using the integral representation

(19) $$Fx-Fy=[\int_0^1 F'(y+t(x-y))dt](x-y)$$

we deduce that

(20) $$|T_o[Fx-Fy-F'(u)(x-y)]|\le a(|x-u|+|y-u|)|x-y|$$

for all $x,y,u\in\mathring{D}$.

Finally from (10) and (20) we have

(21) $$|T_o[Fx-Fy-\delta F(u,v)(x-y)]|\le a(|x-u|+|y-u|+|u-v|)|x-y|$$

for all $x,y,u,v\in\mathring{D}$. By a continuity argument (19), (20) and (21) remain valid if x and/or y belong to D_c.

Using the above inequalities we shall prove that

(22) $$|x_n-x_{n+1}|\le t_n-t_{n+1}$$

for $n=-1,0,1,2,\ldots$

It is easy to see that the sequence $(t_n)_{n\ge-1}$ given by (15) is decreasing and converges to d. If k is a nonnegative integer and if (22) holds for $n\le k-1$ then:

$$|x_o-x_n|\le t_o-t_n<t_o-d\ ,\quad |x_1-x_n|\le t_1-t_n<t_1-d=r_1$$

for $n\le k$. This shows that (16) is satisfied for $u=x_i$ and $v=x_j$ with $i,j\le k$. Thus (22) assures the fact that (6) is well defined.

For $n=-1$ and $n=0$ (22) reduces to $|x_{-1}-x_o|\le c$ and $|x_o-x_1|\le b$ (compare with (8) and (9)). Suppose (22) holds for $n=-1,0,\dots,k$, where $k\ge 0$. Denote $T_n=A_n^{-1}$.

Using (6), (17) and (21) we may write:

$$|x_{k+1}-x_{k+2}|=|T_{k+1}Fx_{k+1}|=|(T_oA_{k+1})^{-1}T_o[Fx_{k+1}-Fx_k-A_k(x_{k+1}-x_k)]|\le$$

$$\le\frac{a(|x_{k+1}-x_{p_k}|+|x_k-x_{p_k}|+|x_{p_k}-x_{q_k}|)}{1-a(|x_{p_{k+1}}-x_o|+|x_{q_{k+1}}-x_o|+c)}|x_k-x_{k+1}|\le$$

$$\le\frac{a(t_{p_k}-t_{k+1}+t_{p_k}-t_k+t_{q_k}-t_{p_k})}{1-a(t_o-t_{p_{k+1}}+t_o-t_{q_{k+1}}+t_{-1}-t_o)}(t_k-t_{k+1})=$$

$$=\frac{t_{p_k}+t_{q_k}-t_{k+1}-t_k}{t_{p_{k+1}}+t_{q_{k+1}}}(t_k-t_{k+1})=t_{k+1}-t_{k+2}\ .$$

We have thus proved that (22) holds for all n. From the completeness of X it follows that the sequence $(x_n)_{n\ge 0}$ converges to a point x^* and that

(23) $$|x_n-x^*|\le t_n-d\ .$$

From (6), (7) and (22) we obtain the inequality

(24) $$|T_oFx_{k+1}|\le a(|x_{k+1}-x_{p_k}|+|x_k-x_{p_k}|+|x_{p_k}-x_{q_k}|)|x_k-x_{k+1}|$$

wherefrom it follows that $Fx^*=0$.

Let us take now $x=x_n$ and $y=x^*$ in (19) and denote $S=\int_0^1 F'(x^*+t(x_n-x^*))dt$. According to (22) and (23) it follows that

$$|x_n-x_o|+|x^*-x_o|+|x_o-x_{-1}|\le 2|x_n-x_o|+|x_n-x^*|+c<$$

$$<c+2(|x_n-x_o|+|x_n-x^*|)\le 2(t_o-t_n+t_n-d)+c\le 2t_o+c=1/a\ .$$

Using (10) and Banach's lemma, one can prove that S is invertible and

(25) $$|(T_oS)^{-1}|\le[1-a(2|x_n-x_o|+|x_n-x^*|+c)]^{-1}$$

(see also the proof of inequality (17).) According to (24) and (25) we have

$$|x_n-x^*|=|S^{-1}Fx_n|\le|(T_oS)^{-1}||T_oFx_n|\le$$

$$\leq \frac{a(|x_n - x_{p_{n-1}}| + |x_{n-1} - x_{p_{n-1}}| + |x_{p_{n-1}} - x_{q_{n-1}}|)}{1 - a(2|x_n - x_o| + |x_n - x^*| + c)} |x_n - x_{n-1}|$$

and it is easy to see that the above inequality together with the fact $|x_n - x^*| < t_o$ implies the estimate (12).

Using the identity

$$x_{n+1} - x_n = x^* - x_n + (T_o A_n)^{-1} T_o (Fx^* - Fx_n) - A_n(x^* - x_n))$$

and the inequalities (17) and (21) we obtain

$$|x_{n+1} - x_n| \leq \frac{a(2|x_n - x_{p_n}| + |x^* - x_n| + |x_{p_n} - x_{q_n}|)}{1 - a(|x_{p_n} - x_o| + |x_{q_n} - x_o| + c)} |x_n - x^*| + |x_n - x^*| .$$

This inequality implies the lower bound (13) ∎

Let us observe that by taking $x=y$ in (10) we deduce that the Fréchet derivative of F satisfies a Lipschitz condition of the form

$$|T_o(F'(y) - F'(z))| \leq 2a|y - z| .$$

Conversely if the above condition is satisfied then (10) is also fulfilled, taking for example $\delta F(x,y) = \int_0^1 F'(tx + (1-t)y)dt$. It follows that for $x_o = x_{-1}$ and $c=0$ the hypothesis of Theorem 3.1 reduces to the hypothesis of the affine invariant version of the Kantorovich theorem [6]. We obtain thus the following:

3.2. <u>Corollary</u>. Let X, Y be two Banach spaces and D a convex subset of X. Let $F: D \subset X \to Y$ be a nonlinear operator, Fréchet differentiable on $\overset{o}{D}$. Suppose that, for an $x_o \in \overset{o}{D}$, $F'(x_o)$ is invertible and its inverse $T_o = [F'(x_o)]^{-1}$ is continuous. If there exist two constants a, b such that

$$4ab \leq 1, \quad |T_o F x_o| \leq b, \quad |T_o(F'(x) - F'(y))| \leq 2a|x - y|, \quad x, y \in \overset{o}{D},$$

and if $U := \{x \in X;\ |x - x_o + T_o F x_o| \leq \frac{1}{2a}[1 - 2ab - (1-4ab)^{1/2}]\} \subset D_c := \{x \in D;$ F is continuous at $x\}$, then Newton's method:

(26) $$Fx_n + F'(x_n)(x_{n+1} - x_n) = 0 \qquad n = 0, 1, 2, \ldots$$

is well defined, the sequence $(x_n)_{n \geq 0}$ produced by it converges to a root $x^* \in U$ of the equation $Fx=0$, and the following estimates hold:

(27) $$|x_n - x\ | \leq t_o - |x_n - x_o| - [(t_o - |x_n - x_o|)^2 - |x_n - x_o|^2]^{1/2},$$

(28) $$|x_n - x\ | \geq [(t_o - |x_n - x_o|)^2 + 2(t_o - |x_n - x_o|)|x_n - x_o|)|x_n - x_{n+1}|]^{1/2} - t_o + |x_n - x_o|,$$

where $t_o=(2a)^{-1}$ ∎

It is interesting to note that the error bounds (27) and (28), which follow from (12) and (13) for $c=0$ and $p_n=q_n=n$, are generally more accurate than the error estimates obtained especially for Newton's method by Gragg and Tapia [7], Miel [10], Potra and Pták [14] (see [12]). We also note that the error bounds obtained in Theorem 3.1 are sharp in the following sense:

3.3. Proposition. If $a>0$, $b\geq 0$, $c\geq 0$ are three constants satisfying inequality (11) then:

(i) There exist a function $F:R\to R$ and two points $x_o, x_{-1}\in R$ verifying the hypothesis of Theorem 3.1 and for which the estimates (12) are attained at each $n=1,2,3,\ldots$.

(ii) For any given $n\in Z_+$ there exist a function $f_n:R\to R$ and two points $x_o, x_{-1}\in R$ verifying the hypothesis of Theorem 3.1 and for which (13) holds with equality.

Proof. (i) $f(x)=x^2-d^2$, $x_o=t_o$, $x_{-1}=t_{-1}$;

(ii) $f_n(x)=x^2-d^2$ if $x\geq t_n$ and $f_n(x)=-x^2+4t_nx-2t_n^2-d^2$ if $x<t_n$;

$x_o=t_o$, $x_{-1}=t-1$ ∎

4. Monotonous convergence

In this section supposing that the operator F acts between two POL-spaces we shall use iterative procedures of type (6) in order to obtain monotonically convergent sequences enclosing the roots of the equation $Fx=0$.

First let us introduce some notation: Let x_o, x_{-1}, y_o be three comparable points belonging to the domain of definition D of the operator F such that

(29) $$x_o\in\langle x_{-1}, y_o\rangle\subset D .$$

We denote by D_1 the set

(30) $$D_1=\{(x,y)\in\langle x_{-1},y_o\rangle^2;\ x \text{ and } y \text{ are comparable}\}$$

and we consider a mapping A defined on D_1 and taking values linear operators. In the statement of the next theorems we shall use the following set of hypotheses and conclusions:

(31) $$x_o\leq y_o\ ,\ Fx_o\geq 0\geq Fy_o\ ,$$

(32) $$Fy-Fx\geq A(u,v)(y-x)\ ,\ u\vee v\leq x\leq v\leq v_o .$$

(H_1)

(33) $x_o \geq y_o$, $Fx_o \geq 0 \geq Fy_o$,

(34) $Fy-Fx \leq A(u,v)(y-x)$, $x_{-1} \leq x \leq y \leq u \wedge v$. (H_2)

(35) $x_o \geq y_o$, $Fx_o \leq 0 \leq Fy_o$,

(36) $Fy-Fx \geq A(u,v)(y-x)$, $x_{-1} \leq x \leq y \leq u \wedge v$. (H_3)

(37) $x_o \leq y_o$, $Fx_o \leq 0 \leq Fy_o$,

(38) $Fy-Fx \leq A(u,v)(y-x)$, $u \vee v \leq x \leq y \leq y_o$. (H_4)

$x_n \leq x_{n+1} \leq y_{n+1} \leq y_n$, $Fx_n \geq 0 \geq Fy_n$, $n=0,1,\ldots$ (C_1)

$x_n \geq x_{n+1} \geq y_{n+1} \geq y_n$, $Fx_n \geq 0 \geq Fy_n$, $n=0,1,\ldots$ (C_2)

$x_n \geq x_{n+1} \geq y_{n+1} \geq y_n$, $Fx_n \leq 0 \leq Fy_n$, $n=0,1,\ldots$ (C_3)

$x_n \leq x_{n+1} \leq y_{n+1} \leq y_n$, $Fx_n \leq 0 \leq Fy_n$, $n=0,1,\ldots$ (C_4)

4.1. Theorem. Consider a nonlinear operator $F:D \subset X \to Y$, where X is a regular POTL-space and Y a POTL-space, three comparable points x_o, x_{-1}, y_o of D satisfying condition (29), and a mapping $A:D_1 \to B(X,Y)$, where D_1 is the set defined by (30). Let $(p_n)_{n \geq 0}$, $(q_n)_{n \geq 0}$ be two sequences of integers satisfying condition (2). Assume that hypothesis (H_i) is satisfied for an $i \in \{1,2,3,4\}$. Assume moreover that the linear operator $(-1)^i A(u,v)$ has an injective nonnegative continuous left subinverse for all $(u,v) \in D_1$. Then:

1°. The iterative algorithm

(39) $Fx_n + A_n(x_{n+1} - x_n) = 0$

(40) $Fy_n + A_n(y_{n+1} - y_n) = 0$ $n=0,1,\ldots$

(41) $A_n \in \{A(x_{p_n}, x_{q_n}),\ A(x_{q_n}, x_{p_n})\}$

is well defined (i.e. for any $n \in Z_+$ there are x_{n+1} and y_{n+1} satisfying (39) and (40)).

2°. Conclusion (C_i) holds.

3°. There exist two comparable points x^*, y^* in $<x_o, y_o>$ such that $x^* = \lim_{n\to\infty} x_n$, $y^* = \lim_{n\to\infty} y_n$.

4°. If the operators $(-1)^i A_n$, $n=0,1,2,\ldots$ are inverse nonnegative then any solution of the equation $Fx=0$ in $<x_o, y_o>$ belongs to $<x^*, y^*>$ (i.e. $u \in <x_o, y_o>$ and $Fu=0$ imply $u \in <x^*, y^*>$).

Proof. We shall make the proof for the case i=1. Let B_o be a continuous nonsingular nonnegative left subinverse of $(-A_o)$ and let us consider the operator

$$H:[0,y_o-x_o]\to X\ ,\quad Hx=x-B_o(Fy_o-A_ox)\ .$$

It is easy to see that H is isotone and continuous. We also have:

$H(0)=-B_oFy_o\geq 0$,

$H(y_o-x_o)=y_o-x_o-B_oFx_o-B_o(Fy_o-Fx_o-A_o(y_o-x_o))\leq y_o-x_o-B_oFx_o\leq y_o-x_o$.

According to Kantorovich's theorem [8] the operator H has a fixed point $Hw=w\varepsilon[0,y_o-x_o]$. Taking $y_1=y_o-w$ we have

$$Fy_o+A_o(y_1-y_o)=0,\quad x_o\leq y_1\leq y_o\ .$$

Using (32) we deduce that

$$Fy_1=Fy_1-Fy_o-A_o(y_1-y_o)\leq 0\ .$$

Now let us define the operator

$$G:[0,y_1-x_o]\to X,\ Gx=x+B_o(Fx_o+A_ox)\ .$$

G is clearly continuous isotone and we have:

$G(0)=B_oFx_o\geq 0$,

$G(y_1-x_o)=y_1-x_o+B_oFy_1-B_o(Fy_1-Fx_o-A_o(y_1-x_o))\leq y_1-x_o+B_oFy_1\leq y_1-x_o$.

Applying again Kantorovich's theorem [8] we deduce the existence of a point $z\varepsilon[0,y_1-x_o]$ such that z=Gz. Taking $x_1=x_o+z$ it follows that

$$Fx_o+A_o(x_1-x_o)=0,\quad x_o\leq x_1\leq y_1\ .$$

Using the above relations and condition (32) we obtain

$$Fx_1=Fx_1-Fx_o-A_o(x_1-x_o)\geq 0\ .$$

Proceeding by induction we can show that there exist two sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ satisfying (C_1). The space X being regular it follows that there exist $x^*, y^*\varepsilon X$ such that $x^*=\lim_{n\to\infty} x_n$, $y^*=\lim_{n\to\infty} y_n$. We have obviously $x^*\leq y^*$.

If $x_o\leq u\leq y_o$ and Fu=0 then we can write

$$A_o(y_1-u)=A_oy_o-Fy_o-A_ou=A_o(y_o-u)-(Fy_o-Fu)\leq 0$$

and

$$A_o(x_1-u)=A_ox_o-Fx_o-A_ou=A_o(x_o-u)-(Fx_o-Fu)\geq 0.$$

If the operator $(-A_0)$ is inverse nonnegative then it follows that $x_1 \leq u \leq y_1$. Proceeding by induction we deduce that $x_n \leq u \leq y_n$ holds for all n. Hence $x^* \leq u \leq y^*$ □

To complete the statement of the above theorem we shall give some natural conditions under which the points x^* and y^* are solutions of the equation $Fx"0$.

4.2. Proposition. Under the hypothesis of Theorem 4.1 suppose that F is continuous at x^* and y^*. If one of the following conditions is satisfied:

(i) X is normal and there exists an operator $T \varepsilon L(X,Y)$, having a continuous nonnegative inverse, such that $A_n \leq T$ for sufficiently large n;

(ii) Y is normal and there exists an operator $S \varepsilon L(X,Y)$ such that $A_n \leq S$ for sufficiently large n;

(iii) the operators A_n , $n=0,1,2,\ldots$ are equicontinuous.

Then $Fx^* = Fy^* = 0$ □

The proof of this proposition is very simple and will be omitted (see [17]).

Let us note that if F has a linear continuous Gâteaux derivative $F'(x)$ at each point $x \varepsilon \langle x_{-1}, y_0 \rangle$ and if the mapping $F': \langle x_{-1}, y_0 \rangle \to B(X,Y)$ is isotone, then conditions (32) and (34) are satisfied taking respectively

(42) $A(u,v) = F'(u \vee v)$, $(u,v) \varepsilon D_1$

(43) $A(u,v) = F'(u \wedge v)$, $(u,v) \varepsilon D_1$.

If F' is antitone then (36) is fulfilled for A given by (43), and (38) is fulfilled for A given by (42). With the choice (3), (42) the results of Theorem 4.1 and Proposition 4.2 constitute a slight improvement of the result of M.H.Wolfe [25].

We also note that conditions (32) and (34) are satisfied if A is a divided difference of F on $\langle x_{-1}, y_0 \rangle$ (i.e. $A(u,v)(u-v) = Fu - Fv$) which is isotone in each argument, while conditions (36) and (38) are satisfied if A is a divided difference antitone in each argument. This shows that the results contained in Theorem 4.1 and Proposition 4.2 represent a generalization of the result obtained by J.W.Schmidt and H. Leonhardt [19] concerning the secant method (see also [17] and [21]).

Now let us make some remarks on the regularity assumption of the space X appearing in the statement of Theorem 4.1. This assumption was essentially used in proving that the iterative procedure (39)-(41) is well defined and that the sequences produced by it are convergent. In Proposition 4.2 we have given some sufficient conditions under which the limits of these sequences are roots of the equation $Fx=0$. We have already mentioned in Section 2 that the regularity condition is rather restrictive. In some cases the existence of the solution can be proved by other means without this assumption and iterative procedures are applied for enclosing the solution (see [1]). In the following theorem we shall show that an "explicit version" of the iterative procedure (39)-(41) can be used to this effect.

4.3. <u>Theorem</u>. Consider a nonlinear operator $F:D\subset X\to Y$, where X and Y are POL-spaces and let x_{-1},x_o,y_o be three comparable points of D satisfying condition (29). Consider also a mapping $A:D_1\to L(X,Y)$ where D_1 is the set defined by (30). Let $(p_n)_{n\geq 0}$ and $(q_n)_{n\geq 0}$ be two sequences of integers satisfying condition (2) and let i be a fixed integer between 1 and 4. Assume the operator $(-1)^i A(u,v)$ has a nonnegative subinverse for any $(u,v)\in D_1$ and hypothesis (H_i) is satisfied.

Then the iterative algorithm:

$$(44)\qquad x_{n+1}=x_n-B_nFx_n$$

$$n=0,1,2,\ldots$$

$$(45)\qquad y_{n+1}=y_n-B'_nFy_n$$

where $(-1)^iB_n$, $(-1)^iB'_n$ are nonnegative subinverses of $(-1)^iA_n$, $(-1)^iA'_n$ $\in\{(-1)^iA(x_{p_n},x_{q_n}),(-1)^iA(x_{q_n},x_{p_n})\}$, generates two sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ satisfying conclusion (C_i). Moreover for any solution $u\in$ $\in\langle x_o,y_o\rangle$ of the equation $Fx=0$ we have

$$(46)\qquad u\in\langle x_n,y_n\rangle\,,\qquad n=0,1,2,\ldots$$

<u>Proof</u>. We shall prove the theorem for $i=1$. In this case we have:

$$(47)\qquad B_o\leq 0,\ B'_o\leq 0,\ I\geq A_oB_o\,,\ I\geq B_oA_o\,,\ I\geq A'_oB'_o\,,\ I\geq B'_oA'_o\,.$$

From (31), (32), (44), (45) and (47) it follows that:

$$y_o-y_1=B'_oFy_o\geq 0\,,$$

$$y_1-x_o=y_o-x_o-B'_oFy_o\geq y_o-x_o-B'_o(fy_o-Fx_o)\geq B'_o(A'_o(y_o-x_o)-(Fy_o-Fx_o))\geq 0,$$

$$x_1-x_o=-B_oFx_o\geq 0\,,$$

$$y_o-x_1=y_o-x_o+B_oFx_o\geq y_o-x_o-B_o(Fy_o-Fx_o)\geq B_o(A_o(y_o-x_o)-(Fy_o-Fx_o))\geq 0.$$

Hence $x_1,y_1\in[x_o,y_o]$. Using again (31), (32), (44), (45) and (47) we obtain:

$$Fy_1=Fy_1+A'_o(y_o-y_1-B'_oFy_o)=Fy_1-A'_oB'_oFy_o+A'_o(y_o-y_1)\leq Fy_1-Fy_o+A'_o(y_o-y_1)\leq 0\,,$$

$$Fx_1=Fx_1-A_o(x_1-x_o+B_oFx_o)=Fx_1-A_oB_oFx_o-A_o(x_1-x_o)\geq Fx_1-Fx_o-A_o(x_1-x_o)\geq 0,$$

$$y_1-x_1 \geq y_1-x_1+B_0'Fx_1=y_0-x_1-B_0'(Fy_0-Fx_1) \geq B_0'(A_0'(y_0-x_1)-(Fy_0-Fx_1)) \geq 0 \ .$$

Thus we have proved that $x_0 \leq x_1 \leq y_1 \leq y_0$ and $Fx_1 \leq 0 \leq Fy_1$. Proceeding by induction we deduce that (C_1) is satisfied.

Finally, if $u \in [x_0, y_0]$ and $Fu=0$ then we may write:

$$y_1-u=y_0-u-B_0'Fy_0+B_0'Fu \quad B_0'(A_0'(y_0-u)-(Fy_0-Fu)) \geq 0 \ ,$$

$$u-x_1=u-x_0+B_0Fx_0-B_0Fu \quad B_0(A_0(u-x_0)-(Fu-Fx_0)) \geq 0 \ .$$

Hence $x_1 \leq u \leq y_1$ and, by induction, $x_n \leq u \leq y_n$ for $n=1,2,\ldots$ ∎

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