RENEWAL PROCESSES WITH COMPLETE CONNECTIONS

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Abstract. The aim of this paper is to extend results known for J-X processes and Markov renewal processes to a wider class of stochastic processes, namely those obtained replacing the Markovian component J_n by a chain with complete connections [5], [11]. The corresponding continuous time process is a generalization of semi-Markov processes. Such processes are suitable for modelling those real phenomena where the whole past is in a way involved in the conditional laws of the future.

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1. INTRODUCTION

Let $(W,\mathcal{W})$ be a measurable space and $Q(\cdot,\cdot)$ be a transition probability function from $(W,\mathcal{W})$ to $(W\times R_+,\mathcal{W}\times\mathcal{B}_+)$. It is well known that, for each $w\in W$ and $x_0\in X$, there exist a probability space $(\Omega,\mathcal{K},P_w)$ and a $W\times R_+$ valued sequence $(J_n,X_n)_{n\in N}$ of random variables, such that

$$P_w(J_0 = w , X_0 = 0) = 1 ,$$

$$P_w(J_{n+1}\in A ,X_{n+1}\in B \mid J_0,X_0,\dots,J_n,X_n) = Q(J_n, A\times B) ; P_w - \text{a.s.}$$

for every $n\in N$, $A\in\mathcal{W}, B\in\mathcal{B}_+$.

The sequence of random variables $(J_n, X_n)_{n\in N}$ is called a J-X process. It is easy to see that this process is a Markov chain of a special kind. The sequence $(J_n)_{n\in N}$ is also a Markov chain with state space $(W,\mathcal{W})$. The sequence $(X_n)_{n\in N}$ is a Markov chain-dependent process.

The process $(J_n,S_n)_{n\in N}$,where $S_n = X_0 + \dots + X_n$,which is also a Markov chain, is called a Renewal Markov process (R.M.P.).

The J-X processes and the renewal Markov processes have been introduced by J.Janssen [6] ,P.Levy [8] ,and W.L.Smith [18] . Many other authors studied these processes; we mention here only R.Pyke [14] , [15] ,R.Schaufele [16] , [17] ,E.Çinlar [1] , H.Kesten [7] ,L.Takács [19], [20],J.Yackel [21] , [22] , S.Grigorescu and G.Oprişan [2] , [10] .

The aim of this paper is to introduce a concept which generalizes the J-X processes, in the sense that the first component $(J_n)_{n\in N}$ is a chain with complete connections (C.C.C.).

The notion of a C.C.C. has been introduced by O.Onicescu and G.Mihoc [3] and generalized by M.Iosifescu [9] .

A random system with complete connections (R.S.C.C.) is defined by:

- two measurable spaces $(W, \mathcal{W})$, $(X, \mathcal{X})$;

- a transition probability function $P(\cdot,\cdot)$ from $(W, \mathcal{W})$ to $(X, \mathcal{X})$;

- a measurable function $u(\cdot,\cdot)$ from $W \times X$ to W.

It is known [5] that, for each $w \in W$, there exist a probability space $(\Omega, \mathcal{K}, \mathbb{P}_w)$ and two sequences of random variables $(\zeta_n)_{n \in N}$, $(\xi_n)_{n \in N}$, such that

$$\mathbb{P}_w(\zeta_0 = w) = 1 \ , \ \mathbb{P}_w(\xi_0 \in A) = P(w,A) \ , \ \mathbb{P}_w\text{- a.s.}$$

$$\zeta_{n+1} = u(\zeta_n, \xi_n) \ , \ n \in N^*$$

$$\mathbb{P}_w(\xi_{n+1} \in A \mid \xi_0, \ldots \xi_n) = P(\zeta_{n+1}, A) \quad \mathbb{P}_w\text{- a.s.}$$

for any $n \in N$, $A \in \mathcal{X}$.

The sequence $(\xi_n)_{n \in N}$ is called a C.C.C. Clearly, the sequence $(\zeta_n)_{n \in N}$ is a Markov chain.

Examples of random systems with complete connections pertain to mathematical learning theory, decision models and dynamic programming, stochastic approximation procedures, population genetics, mathematical economics, random automata.

In Section 2 we construct a process $(J_n, X_n)_{n \in N}$ in which $(J_n)_{n \in N}$ is a chain with complete connections and hence $(X_n)_{n \in N}$ is a chain with complete connections-dependent process. The process $(J_n, S_n)_{n \in N}$ will be called a Renewal process with complete connections (R.P.C.C)

In Section 3 we establish conditions that ensure the ergodicity of the process $(J_n, X_n)_{n \in N}$.

In Section 4 we examine the asymptotic behaviour of the process $(J_n, X_n)_{n \in N}$.

It can be noticed that the processes of the J-X type, in which the first component is Markovian, as well as those in which the first component is a C.C.C., are special cases of a more general type of process, which can be defined as follows:

Consider

- three measurable spaces $(L,\mathcal{L})$, $(X,\mathcal{X})$, $(Y,\mathcal{Y})$;
- $(\mathcal{X}^n,\mathcal{L})$ measurable functions $f_n : X^n \rightarrow L$, for each $n\in N$;
- a transition function $R(\cdot,\cdot)$ from $(L,\mathcal{L})$ to $(X\times Y,\mathcal{X}\times\mathcal{Y})$.

With these elements, for each probability μ on $\mathcal{X}\times\mathcal{Y}$, we can define an $X\times Y$-valued process $(J_n,X_n)_{n\in N}$ on a probability space $(\Omega,\mathcal{K},P_\mu)$, such that

$$P_\mu(J_o\in A, X_o\in B) = \mu(A\times B);\quad P_\mu(J_1\in A, X_1\in B \mid J_o,X_o) = R(f_1(J_o),A\times B);$$

$$P_\mu(J_{n+1}\in A, X_{n+1}\in B \mid (J_k,X_k),\ 0\leq k\leq n) = R(f_{n+1}(J_o,\dots,J_n),A\times B)$$

P_μ - a.s. for any $n\in N$, $A\in\mathcal{Y}$.

A J-X process is a process of this type, for which $L = X$, $Y = R_+$, $f_{n+1}(j_o,\dots,j_n) = j_n$; $\mu(A\times B) = \delta_{(w,o)}(A\times B)$; $R(\cdot, A\times B) = Q(\cdot,A\times B)$.

The process that will be defined in this paper is also a particular case of the above model.

For the J-X process, as well as for the process we shall define, the process $(\zeta_n)_{n\in N}$, defined by $\zeta_{n+1} = f_{n+1}(J_o,\dots,J_n), n\in N$, is a Markov chain.

2. Existence and properties of Renewal processes with complete connections

Let $((W,\mathcal{W}),(X,\mathcal{X}),u,P)$ be a random system with complete connections and, for each $w\in W$, $x,y\in X$ let $F_{w,x,y}(\cdot)$ be a probability distribution function on the real line, satisfying the following conditions:

(i) $F_{w,x,y}(0) = 0$, $(\forall)$ $w\in W$ and $(\forall)$ $x,y\in X$;

(ii) $F_{\cdot,\cdot,\cdot}(t)$ is a $\mathcal{W}\times\mathcal{X}\times\mathcal{X}$ -measurable function for each $t\in R_+$.

The system $((W,\mathcal{W}),(X,\mathcal{X}),u,P,F)$ will be called a Renewal system with complete connections (R.R.S.C.C.).

Theorem 1. Let $((W,\mathcal{W}),(X,\mathcal{X}),u,P,F)$ be a R.R.S.C.C. and $w_0\in W$. There exist a probability space $(\Omega,\mathcal{K},\mathbb{P}_{w_0})$ and three sequences of random variables $(\zeta_n)_{n\in N}$, $(J_n)_{n\in N}$, $(X_n)_{n\in N}$, taking values in W,X,R_+, respectively, such that

(1) $\mathbb{P}_{w_0}(J_0\in A) = P(w_0,A)$; $X_0 = 0$, $\mathbb{P}_{w_0}$ - a.s. , $\zeta_1 = u(w_0,J_0)$;

(2) $\zeta_{n+1} = u(\zeta_n,J_n)$; $P_{w_0}(J_{n+1}\in A,\ X_{n+1}\le t \mid J_0,\dots,J_n,X_1,\dots,X_n) =$

$= \int_A P(\zeta_{n+1},dx)F_{\zeta_n,J_n,x}(t)$, P_{w_0} -a.s.

Proof Let it be $\Omega = (X\times R_+)^N$, $\mathcal{K} = (\mathcal{X}\times\mathcal{B}_+)^N$ and let P_{w_0} be the probability given by the Ionescu-Tulcea theorem [9] , corresponding to the initial probability $P(w_0,\cdot)\times I_{\{0\}}(\cdot)$ on $\mathcal{X}\times\mathcal{B}_+$ and to the regular conditional distributions $P_{x_0,t_0,\dots,x_n,t_n}(A\times[0,t]) =$

$= \int_A P(w_{n+1},dx)F_{w_n,x_n,x}(t)$, $A\in\mathcal{X}$, $t\in R_+$, where $w_1 = u(w_0,x_0)$,

$w_n = u(w_{n-1}, x_{n-1})$.

For $\omega \in \Omega$, we set $J_n(\omega) = x_n$, $X_n(\omega) = t_n$ if $\omega = ((x_n, t_n))_{n\in N}$, $x_n \in X$, $t_n \in R_+$. According to the Ionescu-Tulcea theorem, equations (1) and (2) above hold.

The process $(J_n, X_n)_{n\in N}$ will be called a generalized J-X process (G.J-X.P).

In the following theorem we give some properties of a G.J-X.P.

<u>Theorem 2.</u> The following assertions hold true:

(A) The process $(J_n)_{n\in N}$ is a C.C.C. and $(\zeta_n)_{n\in N}$ is its associated Markov chain.

(B) The random variable X_{n+1} and the random vector $(X_1, \ldots, X_n)$ are conditionally independent, given $(J_k)_{0\le k\le n+1}$, $n \in N$.

(C) The random variable X_{n+1} and the random sequence $(J_k)_{k\ge n+2}$ are conditionally independent, given $(J_k)_{0\le k\le n+1}$, $n \in N$.

(D) The sequence of random variables $(J_n, X_{n+1})_{n\in N}$ is a C.C.C., whose associated Markov chain is $(\zeta_n)_{n\in N}$ and the transition probability is $R(w, A\times[0,t]) = \int_A P(w,dx)\int_{\mathcal{X}} P(u(w,x),dy)F_{w,x,y}(t)$ for all $w \in W$, $x, y \in X$, $t \in R_+$, $A \in \mathcal{X}$.

(E) The process $(\zeta_n, X_n)_{n\in N}$ is a J-X process.

<u>Proof</u> (A) $P_{w_0}(J_{n+1} \in A / J_0, \ldots, J_n) =$

$$= \mathbb{E}_{w_0}\left[\mathbb{P}_{w_0}(J_{n+1}\in A, X_{n+1}<\infty \mid J_0, \ldots, J_n, X_1, \ldots, X_n) \middle| J_0, \ldots, J_n\right] =$$

$$= \mathbb{E}_{w_0}\left[P(\zeta_{n+1}, A) \mid J_0, \ldots, J_n\right] = P(\zeta_{n+1}, A), \ \mathbb{P}_{w_0} \text{ - a.s.},$$

taking into account that $\zeta_{n+1} = u(\zeta_n, J_n) = u(w_0, J^{(n+1)})$, where $u(\cdot,(\cdot)^n): W\times X^n \longrightarrow W$ is defined by $u(w, x^{(n+1)}) = u(u(w, x^{(n)}), x_n)$

for $w \in W$, $n \in N^*$, $x^{(n)} = (x_0,\dots,x_{n-1}) \in X^n$, $x^{(n+1)} = (x_0,\dots,x_{n-1},x_n)$.

(B) We have

$$\mathbb{P}_{w_0}(X_{n+1} \leq t \mid J_0,\dots,J_n,X_1,\dots,X_n) = \int_X P(\zeta_{n+1},dx)F_{\zeta_n,J_n,x}(t),$$

$\mathbb{P}_{w_0}$ - a.s. Then

$$\mathbb{P}_{w_0}(X_{n+1} \leq t \mid J_0,\dots,J_n,J_{n+1},X_1,\dots,X_n) = F_{\zeta_n,J_n,J_{n+1}}(t), \mathbb{P}_{w_0} \text{-a.s.}$$

(C) For $x_0,x_1,\dots,x_{n+1} \in X$, $A_2,\dots,A_k \in \mathcal{X}$,$t \in R_+$,we have

$$\mathbb{P}_{w_0}(X_{n+1} \leq t, J_{n+2} \in A_2,\dots,J_{n+k} \in A_k \mid J_0 = x_0,\dots,J_{n+1} = x_{n+1}) =$$

$$\int_0^t \mathbb{P}_{w_0}(X_{n+1} \in ds \mid J_0 = x_0,\dots,J_{n+1} = x_{n+1})\mathbb{P}_{w_0}(J_{n+2} \in A_2,\dots,J_{n+k} \in A_k \mid J_0 =$$

$$= x_0,\dots,J_{n+1} = x_{n+1}; X_{n+1} = s) = F_{w_0,u(w_0,x^{(n+1)}),x_{n+1}}(t) \times$$

$$\times \mathbb{P}_{w_0}(J_{n+2} \in A_2,\dots,J_{n+k} \in A_k \mid J_0 = x_0,\dots,J_{n+1} = x_{n+1}) =$$

$$= \mathbb{P}_{w_0}(X_{n+1} \leq t \mid J_0 = x_0,\dots,J_{n+1} = x_{n+1})\mathbb{P}_{w_0}(J_{n+2} \in A_2,\dots,J_{n+k} \in A_k \mid J_0 =$$

$$= x_0,\dots,J_{n+1} = x_{n+1}).$$

Using a monotone class argument,the assertion is proved.

Moreover ,we have

$$(3) \quad \mathbb{P}_{w_0}(X_1 \leq t_1,\dots,X_{n+1} \leq t_{n+1} \mid (J_n)_{n \in N}) =$$

$$= \mathbb{P}_{w_0}(X_1 \leq t_1,\dots, X_{n+1} \leq t_{n+1} \mid J_0,\dots,J_{n+1}) =$$

$$= \prod_{i=1}^{n+1} \mathbb{P}_{w_0}(X_i \leq t_i \mid J_0,\dots,J_i) = \prod_{i=1}^{n+1} F_{\zeta_{i-1},J_{i-1},J_i}(t_i) \dots$$

(D) To prove this assertion,it is sufficient to note that,for each $A \in \mathcal{X}$, $t \in R_+$,

$$\mathbb{P}_{w_0}(J_n \in A, X_{n+1} \le t \mid J_0,\dots,J_{n-1}, X_1,\dots,X_n) = \int_A P(\zeta_n, dx) \times$$

$$\times \int_X P(u(\zeta_n,x),dy)\, F_{\zeta_n,x,y}(t) = R(\zeta_n, A \times [0,t]) =$$

$$= R(\tilde{u}(w_0,(J,X)^{(n)}), A \times [0,t])$$

where

$$\tilde{u}(w,(x,t)^{(n)}) = u(w,x^{(n)});\ (x,t)^{(n)} = ((x_0,t_1),(x_1,t_2),\dots,(x_{n-1},t_n))$$

(E) Obviously

$$\mathbb{P}_{w_0}(\zeta_0 = w_0, X_0 = 0) = 1 \ ; \ \mathbb{P}_{w_0}(\zeta_{n+1} \in B, X_{n+1} \le t \mid \zeta_0, X_0, \dots, \zeta_n, X_n) =$$

$$= \int_{\tilde{u}^{-1}(\zeta_n,\cdot)(B)} P(\zeta_n, dx) \int_X P(u(\zeta_n,x),dy)\, F_{\zeta_n,x,y}(t) = Q(\zeta_n, B \times [0,t]),$$

$\mathbb{P}_{w_0}$ - a.s., $B \in \mathcal{W}$, $t \in R_+$, $n \in N$, where

$$Q(w, B \times [0,t]) = \int_{\tilde{u}^{-1}(w,\cdot)(B)} P(w,dx) \int_X P(u(w,x),dy)\, F_{w,x,y}(t).$$

<u>Remark</u> We note that a generalized J-X process can be regarded as a particular case of the general model described in Section 1. We should take $L = W \times X$, $Y = R_+$, $f_{n+1}(j_0,\dots,j_n) = (u(w_0, j^{(n+1)}), j_n)$

$$\mu(A \times [0,t]) = P(w_0, A) \ ; \ R((w,x), A \times [0,t]) = \int_A P(w,dy)\, F_{w,x,y}(t).$$

As usual, we set $S_n = X_0 + \dots + X_n$, $n \in N$. To keep a similar terminology with R.M.P., the process $(J_n, S_n)_{n \in N}$ will be called a Renewal process with complete connections.

Obviously, we have

(4) $\mathbb{P}_{w_0}(J_n \in A, S_n \le t \mid J_0 = x_0, \dots, J_{n-1} = x_{n-1}, S_1 = t_1, \dots, S_{n-1} = t_{n-1}) =$

$= \mathbb{P}_{w_0}(J_n \in A, X_n \le t - t_{n-1} \mid J_0 = x_0, \dots, J_{n-1} = x_{n-1}, X_1 = t_1, X_2 = t_2 - t_1, \dots$

$\ldots, X_{n-1} = t_{n-1} - t_{n-2}) = \int_A P(u(w_0, x^{(n)}), dx) F_{u(w_0, x^{(n-1)}), x_{n-1}, x}(t - t_{n-1}),$

$A \in \mathcal{X},\ 0 < t_1 < \ldots < t_n,\ x_0, x_1, \ldots, x_{n-1} \in X,\ n \in N.$

Hence

$$(5)\quad \mathbb{P}_{w_0}(S_n \leq t \,/\, J_0 = x_0, \ldots, J_n = x_n) = \mathop{\ast}_{k=0}^{n-1} F_{u(w_0, x^{(k)}), x_k, x_{k+1}}(t)$$

and

$$(6)\quad \mathbb{P}_{w_0}(S_n \leq t) = \int_{X^{n+1}} \Big(\mathop{\ast}_{k=0}^{n-1} F_{u(w_0, x^{(k)}), x_k, x_{k+1}}(t) P_{n+1}(w_0, dx^{(n+1)})$$

where

$$P_{n+1}(w_0, A_0 \times \ldots \times A_n) = \int_{A_0} P(w_0, dx_0) \int_{A_1} P(u(w_0, x_0), dx_1) \ldots \int_{A_{n-1}} P(u(w_0, x^{(n-1)}), A_n)$$

It follows from Th.2 (E) that the process $(\zeta_n, S_n)_{n \in N}$ is a R.M.P.

As for the J-X process, we shall associate with the G.J-X.P. a continuous time process, by setting

$N(t) = \max\{n \mid S_n \leq t\}$ (the number of "renewals" in the interval $[0,t]$) and $Z_t = J_{N(t)},\ t \in R_+$.

The process $(Z_t)_{t \in R_+}$ generalizes the notion of a semi-Markov process. This process is called w_0-regular if $\mathbb{P}_{w_0}(N(t) < \infty) = 0$ for all $t \in R_+$. The process (Z_t) will be studied elsewhere. We only mention here

<u>Proposition 3</u> Suppose that there exists a distribution function $F(t)$, $t \in R_+$, with $F(0) = 0$, such that $F_{w,x,y}(t) \leq F(t)$, $(\forall)\ w \in W$, $x, y \in X$. Then the process $(Z_t)_{t \in R_+}$ is w - regular for all $w \in W$.

<u>Proof</u> It follows easily from equation (6) that

$\mathbb{P}_w(S_n \leq t) \leq (F * F * \ldots * F)(t) = F^{(n)}(t)$

Hence

$$\lim_{n\to\infty} \mathbb{P}_w(S_n \leq t) \leq \lim_{n\to\infty} F^{(n)}(t) = 0 , \ (\forall)\ t \in R_+ .$$

Therefore

$$\mathbb{P}_w(\lim_{n\to\infty} S_n \leq t) = 1 , \ (\forall)\ t \in R_+ .$$

On the other hand

$$\mathbb{P}_w(N(t) < \infty) = \mathbb{P}_w(\lim_{n\to\infty} S_n \geq t)$$

which ends the proof.

Remark If X is a finite set and $F_{w,x,y}(t)$ does not depend on w, the hypothesis of Proposition 3 is fulfilled. This result is known for a R.M.P. [14] .

It is known [14] , [15] , that for a J-X process, the processes $(J_n, X_n)_{n\in N}$ and $(J_n, S_n)_{n\in N}$ are Markov chains. In the following two theorems, we give similar results for a G.J-X.P.

Theorem 4 There exists a R.S.C.C. $((\widetilde{W},\widetilde{\mathcal{W}}),(\widetilde{X},\widetilde{\mathcal{X}}),\widetilde{u},\widetilde{P})$ whose C.C.C. $(\widetilde{J}_n,\widetilde{X}_n)$ has the same finite-dimensional distributuions as (J_n, X_n).

Proof Set $\widetilde{W} = W \cup (W \times X)$, $\widetilde{\mathcal{W}} = \sigma(\mathcal{W} \cup (\mathcal{W} \times \mathcal{X}))$, $\widetilde{X} = X \times R_+$, and define $\widetilde{u} : \widetilde{W} \times \widetilde{X} \to \widetilde{W}$ by

$$\widetilde{u}(\widetilde{w},\widetilde{y}) = \begin{cases} (w,y) \ \text{if}\ \widetilde{w} = w \in W , \ \widetilde{y} = (y,t) \in X \times R_+ \\ (u(w,x),y) \ \ \text{if}\ \widetilde{w} = (w,x) \in W \times X, \widetilde{y} = (y,t) \in X \times R_+ \end{cases}$$

If we set

$$\widetilde{P}(w, A \times [0,t]) = \begin{cases} P(w,A) I_{\{0\}}(t) \ \ \text{if}\ \widetilde{w} = w \in W, A \in \mathcal{W} \\ \int_A P(u(w,x),dy) F_{w,x,y}(t) \ \ \text{if}\ \widetilde{w} = (w,x),\ A \in \mathcal{X} \end{cases}$$

it is easy to check that $((\widetilde{W},\widetilde{\mathcal{W}}),(\widetilde{X},\widetilde{\mathcal{X}}),\widetilde{u},\widetilde{P})$ is a R.S.C.C. whose associated C.C.C. $(\widetilde{J}_n,\widetilde{X}_n)$, defined on $(\widetilde{\Omega},\widetilde{\mathcal{K}},\widetilde{P}_{\widetilde{w}_o})$ $(\widetilde{w}_o = w_o)$, satisfies the following equations

$$\widetilde{P}_{\widetilde{w}_o}(\widetilde{J}_o \in A) = \widetilde{P}(\widetilde{w}_o, A\times R_+) = P(w_o,A) = \mathbb{P}_{w_o}(J_o \in A);$$

$$\widetilde{\mathbb{P}}_{\widetilde{w}_o}(\widetilde{X}_o = 0 \mid \widetilde{J}_o) = 1 ;$$

$$\widetilde{\mathbb{P}}_{\widetilde{w}_o}(\widetilde{J}_1 \in A, \widetilde{X}_1 \le t \mid \widetilde{J}_o, \widetilde{X}_o) = \widetilde{P}(\widetilde{u}(\widetilde{w}_o,(\widetilde{J}_o,\widetilde{X}_o)), A\times[0,t]) =$$

$$\mathbb{P}_{w_o}(J_1 \in A, X_1 \le t \mid J_o, X_o), \quad A \in \mathcal{X};$$

$$\widetilde{\mathbb{P}}_{w_o}(\widetilde{J}_{n+1} \in A, \widetilde{X}_{n+1} \le t \mid \widetilde{J}_o,\widetilde{X}_o,\dots,\widetilde{J}_n,\widetilde{X}_n) = \widetilde{P}(\widetilde{\zeta}_{n+1}, A\times[0,t]) =$$

$$\int_A P(\zeta_{n+1},dx)F_{\zeta_n,J_n,x}(t) = \mathbb{P}_{w_o}(J_{n+1} \in A, X_{n+1} \le t \mid J_o,X_o,\dots,J_n,X_n),$$

where $\widetilde{\zeta}_o = \widetilde{w}_o = w_o$, $\widetilde{\zeta}_{n+1} = \widetilde{u}(\widetilde{\zeta}_n,(\widetilde{J}_n,\widetilde{X}_n))$.

<u>Theorem 5</u> There exists a R.S.C.C. $((\overline{W},\overline{\mathcal{W}}),(\overline{X},\overline{\mathcal{X}}),\overline{u},\overline{P})$ whose C.C.C. $(\overline{J}_n,\overline{S}_n)_{n\in N}$ has the same finite-dimensional distributions as $(J_n,S_n)_{n\in N}$

<u>**Proof**</u> Set $\overline{W} = W \cup (W\times X\times R)$, $\overline{\mathcal{W}} = \sigma(\mathcal{W} \cup \mathcal{X}\times\mathcal{B}_+)$, $\overline{X} = X\times R_+$, $\overline{\mathcal{X}} = \mathcal{X}\times\mathcal{B}_+$. Define $\overline{u}:\overline{W}\times\overline{X} \longrightarrow \overline{W}$ by

$$\overline{u}(\overline{w},\overline{y}) = \begin{cases} (w,y,t) \text{ if } \overline{w} = w, \ \overline{y} = (y,t) \in X\times R_+ \\ \\ (u(w,x),y,t) \text{ if } \overline{w} = (w,x,s) \in W\times X\times R_+, \ \overline{y} = (y,t) \end{cases}$$

and

$$\overline{P}(\overline{w}, A\times[0,t]) = \begin{cases} P(w,A)I_{\{o\}}(t) \quad \text{if } \overline{w} = w \in W, \ A \in \mathcal{X} \\ \\ \int_A P(u(w,x),dy)F_{w,x,y}(t-s) \text{ if } \overline{w} = (w,x,s) \in W\times X\times R_+ \end{cases}$$

The proof carries on the manner of the previous theorem.

3. Ergodicity

A C.C.C. $(J_n)_{n\in N}$ is called ergodic [5] if for every $\ell \in N^*$ there exists a probability $P_\ell^\infty(\cdot)$, defined on $\mathcal{X}^{(\ell)}$, such that

$$|P_w((J_k)_{n\leq k\leq n+\ell-1} \in A^{(\ell)}) - P_\ell^\infty(A^{(\ell)})| \leq \varepsilon_n$$

(where $\varepsilon_n \to 0$), for all $\ell \in N^*$, $w \in W$, $A^{(\ell)} \in \mathcal{X}^{(\ell)}$.

In this section we shall prove that, under certain conditions, the ergodicity of the C.C.C. $(J_n)_{n\in N}$ entails the ergodicity of the C.C.C. $(J_n, X_{n+1})_{n\in N}$.

Let us denote by $B_{(\ell_n);A_0}$ the Banach algebra of the bounded measurable functions ψ for which

$$|\psi(u(w', x^{(r)})) - \psi(u(w'', x^{(r)}))| \leq \ell_n$$

for any $r, n \in N$, $w', w'' \in W$ and the vector $x^{(r)} \in X^{(r)}$ has at least n components in $A_0 \subset X$; here (ℓ_n) is a sequence convergent to zero. It is known (see [5], p.34) that conditions $M(n_0)$ and $FLS(A_0, \nu)$ ensure the regularity of the associated Markov chain $(\zeta_n)_{n\in N}$ with respect to $B_{(\ell_n);A_0}$, i.e. for each $\psi \in B_{(\ell_n);A_0}$, there exists a constant $U^\infty\psi$ such that

$$|U^n\psi - U^\infty\psi| \to 0 \quad \text{as } n \to \infty$$

where U is the transition operator of the Markov chain $(\zeta_n)_{n\in N}$.

Theorem 6 Suppose that conditions $M(n_0)$ and $FLS(A_0, \nu)$ are fulfilled for the C.C.C. $(J_n)_{n\in N}$. If the function $F_{\cdot, x, y}(t)$ belongs to $B_{(\ell_n);A_0}$ for each $x, y \in X$ and $t \in R_+$, then the C.C.C. $(J_n, X_{n+1})_{n\in N}$ is ergodic.

Proof We have

$$\mathbb{P}_{w_0}(J_n\in A, X_{n+1}\le t) = \int_W \mathbb{P}_{w_0}(\zeta_n\in dw)\int_X \mathbb{P}_{w_0}(J_n\in A, X_{n+1}\le t \mid \zeta_n = w,$$

$$J_n = x)\mathbb{P}_{w_0}(J_n\in dx \mid \zeta_n = w) = [U^n\psi](w_0),$$

where $\psi(w) = \int_X P(w,dx)\int_A P(u(w,x),dy)F_{w,x,y}(t).$

We shall prove that $\psi\in B_{(\ell_n);A_0}$. Indeed, using the ergodicity of the chain (J_n) and L.1.2.1. [5] ,p.40, we have, for $r\ge n$, $x^{(r)}$ having at least n components in A_0, $w', w''\in W$

$$\left|\psi(u(w',x^{(r)})) - \psi(u(w'',x^{(r)}))\right| \le$$

$$\int_X P(u(w',x^{(r)}),dx)\int_A \left|F_{u(w',x^{(r)}),x,y}(t) - F_{u(w'',x^{(r)}),x,y}(t)\right| \times$$

$$\times P(u(u(w',x^{(r)}),x),dy) + \int_X P(u(w',x^{(r)}),dx)\int_A \left|P(u(u(w',x^{(r)}),x),dy)\right.$$

$$\left.- P(u(u(w'',x^{(r)}),x),dy)\right| F_{u(w'',x^{(r)}),x,y}(t) + \int_X \left|P(u(w',x^{(r)}),dx) -\right.$$

$$\left.- P(u(w'',x^{(r)}),dx)\right| \int_A P(u(u(w'',x^{(r)}),x),dy)\; F_{u(w'',x^{(r)}),x,y}(t) \le 3\ell_n.$$

In the same way, we can see that

$\mathbb{P}_{w_0}(J_{n+\ell-1}\in A_\ell, X_{n+\ell}\le t_\ell,\ l = 1,2,\dots,k) = [U^n\psi](w\ )$, where

$$\psi(w) = \int_X P(w,dx_0)\int_{A_1} P(u(w,x_0),dx_1)F_{w,x_0,x_1}(t_1)\dots\int_{A_k} P(u(w,x^{(k)}),dx_k)\times$$

$$\times F_{u(w,x^{(k-1)}),x_{k-1},x_k}(t\ ).$$

Using similar arguments as in P.2.1.13. [5] ,p.73, one can prove that the function ψ above belongs to $B_{(\ell_n);A_0}$.

According to Th.2.1.24 [5] , p.84, for each $A_1,\dots,A_k\in\mathcal{X}$ $t_1,\dots,t_k\in R_+$, there exists a constant $U^\infty\psi$, such that $|U^n\psi - U^\infty\psi| \to 0$ as $n\to\infty$ and therefore

$$\mathbb{P}_{w_0}(J_{n+\ell-1}\in A_\ell, X_{n+\ell}\le t_\ell,\ l = 1,2,\dots,k) \to P_k^\infty\left(\prod_{\ell=1}^k A_\ell\times[0,t_\ell]\right)$$

where $P_k^\infty\left(\prod_{\ell=1}^k A_\ell\times[0,t_\ell]\right) = U^\infty\psi$.

Similarly, one can prove

Theorem 7 Under the conditions of the previous theorem, the C.C.C. $(\tilde{J}_n, \tilde{X}_n)_{n\in N}$ is ergodic.

The ergodicity of the C.C.C. (J_n, X_{n+1}) can also be obtained as a consequence of the regularity of the operator U with respect to B(W) (the space of all bounded measurable functions $f:W \longrightarrow R$) or to L(W) (the space of all Lipschitz bounded functions $f:W \longrightarrow R$).

Theorem 8 Assume that the associated Markov chain $(\zeta_n)_{n\in N}$ of a R.S.C.C. $((W,\mathcal{W}),(X,\mathcal{X}),u,P)$ is regular with respect to B(W). Then the C.C.C. $(J_n, X_{n+1})_{n\in N}$ is ergodic.

Proof It follows from Th.2.1. [10], p.101 and Th.2 (D).

Certain conditions which ensure the regularity of the operator U are given in [10] for distance diminishing models. We give sufficient conditions for the R.S.C.C. $(J_n, X_{n+1})_{n\in N}$ to be a distance diminishing model in

Theorem 9 If the R.S.C.C. $((W,\mathcal{W}),(X,\mathcal{X}),u,P)$ is a distance diminishing model and the function $F_{\cdot,x,y}(t)$ is Lipschitz for all $x,y\in X$, $t\in R_+$, then the R.S.C.C. $((W,\mathcal{W}),(X\times R_+, \mathcal{X}\times \mathcal{B}_+),u,R)$ is a distance diminishing model.

Proof The assumptions made imply that

(i) (W,d) is a metric space;

(ii) If we denote $$r_j = \sup_{w'\neq w''} \int_X P_j(w', dx^{(j)}) \frac{d(u(w',x^{(j)}),u(w'',x^{(j)}))}{d(w',w'')}$$

then $r_1 < \infty$ and $(\exists)\ k\in N^*$ such that $r_k < 1$;

(iii) If we denote

$$R_1 = (1/2) \sup_{w' \neq w''} \frac{|P_1(w', \cdot) - P_1(w'', \cdot)|}{d(w', w'')}$$

then $R_1 < \infty$.

The R.S.C.C. $((W, \mathcal{W}), (X \times R_+, \mathcal{X} \times \mathcal{B}_+), u, R)$ is a distance diminishing model if

$$|R(w', A \times [0,t]) - R(w'', A \times [0,t])| \leq kd(w', w''), \ (\forall)\ w', w'' \in W,$$

$A \in \mathcal{X}, t \in R_+$.

We have

$$|R(w', A \times [0,t]) - R(w'', A \times [0,t])| =$$

$$|\int_A P(w', dx) \int_X P(u(w', x), dy) F_{w', x, y}(t) - \int_A P(w'', dx) \int_X P(u(w'', x) dy) \times$$

$$\times F_{w'', x, y}(t)| \leq |\int_A P(w', dx) \int_X P(u(w', x), dy) F_{w', x, y}(t) -$$

$$- \int_A P(w', dx) \int_X P(u(w', x), dy) F_{w'', x, y}(t)| + |\int_A P(w', dx) \int_X P(u(w', x), dy) \times$$

$$\times F_{w'', x, y}(t) - \int_A P(w', dx) \int_X P(u(w'', x), dy) F_{w'', x, y}(t)| +$$

$$+ |\int_A P(w', dx) \int_X P(u(w'', x), dy) F_{w'', x, y}(t) - \int_A P(w'', dx) \int_X P(u(w'', x), dy) \times$$

$$\times F_{w'', x, y}(t)| \leq \lambda\, d(w', w'') P(w', A) + \int_A P(w', dx) \sup_{B \in \mathcal{X}} |P(u(w', x), B) -$$

$$- P(u(w'', B)| + \sup_{B \in \mathcal{X}} |P(w', B) - P(w'', B)| \leq \lambda \cdot d(w', w'') +$$

$$+ R_1 d(w', w'') \int_A P(w\ , dx) \frac{d(u(w', x), u(w'', x))}{d(w', w'')} + R_1 d(w\ , w\ ) \leq$$

$$\leq kd(w\ , w\ ) \quad , \text{ where } k = \lambda + R_1(r_1 + 1).$$

4. Limit theorems

The ergodicity of the C.C.C. $(J_n, X_{n+1})_{n\in N}$ entails the following consequences [5]

(1) There exists a probability $\mathbb{P}_\infty$ on $(\Omega, \mathcal{K})$, such that the sequence $(J_n, X_{n+1})_{n\in N}$ is strictly stationary with respect to $\mathbb{P}_\infty$;

(2) $|\mathbb{P}_w(A) - \mathbb{P}_\infty(A)| \leq \varepsilon_n \to 0$; for all $A \in \sigma(J_m, X_{m+1}, m \geq n)$

(3) $\mathbb{P}_w(A) = \mathbb{P}_\infty(A)$ for all A belonging to the tail σ-algebra of the sequence $(J_n, X_{n+1})_{m\in N}$;

(4) The sequence $(J_n, X_{n+1})_{n\in N}$ is φ-mixing with respect to $\mathbb{P}_w$ and $\mathbb{P}_\infty$ and hence its tail σ-algebra is trivial.

Let $h: X \times R_+ \to R$ be a Borel measurable function and $Y_n = h(J_n, X_{n+1})_{n\in N}$. For the sake of simplicity, we assume that $\mathbb{E}_\infty(Y_1) = 0$. In the sequel, $\mathbb{W}$ is the Wiener measure on $\mathcal{D}[0,1]$, endowed with the Skorohod topology and " $\Rightarrow$ " means weak convergebce.

The following theorems can be proved using properties (1)-(4), [3], [4], [13].

Theorem 10 (Strong law of large numbers). For any $w \in W$, we have

$$(1/n)\sum_{i=1}^{n} Y_i \to 0,\ \mathbb{P}_w \text{ - a.s. as } n \to \infty$$

Theorem 11 Assume that $\mathbb{E}_w(|Y_1|^{2+\delta}) < \infty$ for some $\delta > 0$. Then the process $(Y_n)_{n\in N}$ obeys Strassen's version of the loglog law with respect to $\mathbb{P}_w$ for any $w \in W$.

Theorem 12 (Functional central limit theorem). Assume $\mathbb{E}(Y_1^2) < \infty$

and set $\sigma^2 = \mathbb{E}_\infty(Y_1^2) + 2\sum_{k=2}^{\infty} E_\infty(Y_1 Y_k)$. Then

(i) $0 \leq \sigma^2 < \infty$

(ii) If $\sigma^2 > 0$ then $V_n \circ \mathbb{P}_w^{-1} \Rightarrow \mathbb{W}$ for each $w \in W$.

Here $V_n(t) = (1/\sigma \cdot n^{1/2}) \sum_{0<k<[nt]} Y_k$.

In the following theorems we assume that $\mathbb{E}_w(|Y_1|) < \infty$,

$|h(x,t)| \leq 1$ for all $x \in X$, $t \in R_+$; we use the following notations:

$m = \mathbb{E}_\infty(Y_1)$, $F_n^*(t) = (1/n)\sum_{k=1}^{n} I_{[0,t]} \circ Y_k$, $t \in [0,1]$, $F_\infty(t) = \mathbb{P}_\infty(Y_1 \leq t)$.

Theorem 13 (Renewal theorem) If we denote

$$Z_n(t) = \frac{N(nt) - nt \cdot m^{-1}}{n^{1/2} \quad m^{-3/2}}$$

then $\mathbb{P}_w \circ Z_n^{-1} \Rightarrow \mathbb{W}$ for any $w \in W$.

Theorem 14 (Kolmogorov-Smirnov test) The process

$n^{1/2}(F_n^*(t) - F_\infty(t))$, $t \in [0,1]$,converges weakly to a Gaussian

random element in $\mathcal{D}[0,1]$.

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