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*) Department of Mathematics, National Institute for Scientific and Technical Creation, Bd.Pacii 220, 79622 Bucharest, Romania.

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Summary

On the base of the result obtained by R.E.Caflisch [1] we prove that the macroscopic fields associate to a rarefied gas are uniformly bounded by the initial value of the molecular density.

1. Introduction

We know that a rarefied gas, looked from the point of view of the kinetic theory, is completly described once the molecular density is known. The molecular density is, in general, a fonction

$$f : \mathcal{T} \times \mathcal{E} \times \mathcal{V} \longrightarrow \mathbb{R}_+$$

($\mathcal{T}$ being an Euclidian 1-dimensional space, $\mathcal{E}$ an Euclidian 3-dimensional space and $\mathcal{V}$ a linear 3-dimesnional space with scalar product) measurable and integrable over $\mathcal{V}$. A molecular density f must be a solution of the integro-differential Boltzmann equation, that, in the absence of the external body forces, can be written in the form:

$$\frac{\partial f}{\partial t} + \xi \cdot \frac{\partial f}{\partial x} + Q(f,f) = 0 \tag{1.1}$$

where the nonlinear operator Q is defined by:

$$Q(f,f) \equiv \int_{\mathcal{V}}\int_{\mathcal{S}} (f f_* - f' f'_*)\, |w|\, d\mathcal{S}\, d\xi_* \tag{1.2}$$

and where $f' = f(t,x,\xi')$; $f'_* = f(t,x,\xi'_*)$; $f = f(t,x,\xi)$; $f_* = f(t,x,\xi_*)$
the variables (ξ', ξ'_*), (ξ, ξ_*) verifying the laws of balance of the impuls and of the kineric energy,

$w = \xi_* - \xi$ is the relative velocity of two particles system.

$\mathcal{S}$ is the cross-section of collision and together with the molecular mass and the operator that give the solution of the "two body problem" are the constitutive quantities of the kinetic theory of monatomic gases. For $\mathcal{S}$'s definiteness we can determine the intermolecular forces. We consider intermolecular forces with soft potential, that is

$$X(r) = r^{-s} \tag{1.3}$$

with $3 < s < 5$. For this type of potentials R.E. Caflisch [1] has proved the global existence and uniqueness of special periodic solution for the initial values problem

$$f(0,x,\xi) = f_0(x,\xi) \tag{1.4}$$

attached to the whole nonlinear Boltzmann equation (1.1). The analyse of the linear collisional operator has been performed by H.Grad [2].

We denote by D^3 the periodicity cube in $\mathcal{E} \equiv \mathbb{R}^3$ and let $\mathcal{N} = \{ h(x,\xi) \mid \int_{D^3}\int_{V} \psi(\xi) h(x,\xi)\, d\xi\, dx = 0,\ \psi(\xi) = 1, \xi_i, \xi^2 \}$

We take the initial data $f_0 \in \mathcal{N}$, that is, we suppose that we perturb an equilibrium state governed by a Maxwellian molecular density. Now we are ready to write the existence and uniqueness theorem for the solution of the problem (1.1), (1.4) (the theorem 4.1 in the paper of Caflisch [1]):

Theorem

Let $0 < \alpha < 1/4$. There exist a positive constant δ such that if $\| f_0 \|_\alpha < \delta$, the problem (1.1), (1.4) has a unique solution in $\mathcal{N}_\alpha$ and:

(1.5) $$\| f(t) \| < c \| f_0 \|_\alpha e^{-\lambda t^\beta}$$

(1.6) $$\| f(t) \|_\infty < c \| f_0 \|_\alpha e^{-\lambda t^\beta}$$

(1.7) $$\| f(t) \|_\alpha < c \| f_0 \|_\alpha$$

where $\beta = \frac{2}{2+\gamma}$, $\lambda = (1-2\varepsilon)\left(\frac{\alpha}{2}\right)^{1-\beta}\left(\frac{c_0}{\beta}\right)^{\beta}$, for an arbitrary positive ε. The constant c may depend on ε.

For this model $\gamma = -\frac{s-5}{s-1}$ and c_0 is given by the formula (2.20) in [1]. The norms that appear in preceding formulas are:

(1.8) $$\| f(t,\cdot,\xi) \|_{H^4(x)} = \sum_{s=1}^{4} \left(\int_{D^3} | \nabla_x^s f(t,x,\xi) |^2 dx \right)^{1/2}$$

(1.9) $$\| f(t,\cdot,\xi) \| = \left(\int_{V} \| f(t,\cdot,\xi) \|^2_{H^4(x)} d\xi \right)^{1/2}$$

(1.10) $$\|f(t,\cdot,\xi)\|_{\alpha} = \sup_{|\xi|} e^{\alpha\xi^2} \|f(t,\cdot,\xi)\|_{H^4(x)}$$

(1.11) $$\|f(t,\cdot,\xi)\|_{\infty} = \sup_{|\xi|} \|f(t,\cdot,\xi)\|_{H^4(x)}$$

and the space $\mathcal{N}_{\alpha}$ is:

(1.12) $$\mathcal{N}_{\alpha} = \{ f(t,x,\xi) \,|\, \|f(t)\|_{\alpha} < \infty \text{ and } f \text{ periodic in } x$$

2. Macroscopic fields

The kinetic gas may be thought as a continuum body. The existence of the solution of the problem (1.1), (1.4) in the $\mathcal{N}_{\alpha}$ space (stated by the preceding Theorem) assure the integrability of fonctions $|\xi|^r f(t,x,\xi)$. From the point of view of continuum mechanics it is necessary to define the macroscopic fields and for this we define first the numerical density

(2.1) $$n(t,x) = \int_{v} f(t,x,\xi)\, d\xi$$

and then all the fields which are necessary for a continuum description of the gas: ρ (mass-density), v (mean velocity field), T (mean pressure tensor field), p (mean normal pressure) q (mean energy flux), ε (mean internal energy), that is:

(2.2) $$\rho(t,x) = m\int_{v} f(t,x,\xi)\, d\xi = m\, n(t,x)$$

(2.3) $$v(t,x) = \frac{1}{n(t,x)} \int_{v} \xi f(t,x,\xi)\, d\xi$$

(2.4) $$T_{ij}(t,x) = m\int_{v} c_i c_j f(t,x,\xi)\, d\xi$$

(2.5) $$p(t,x) = \frac{m}{3}\int_{v} c^2 f(t,x,\xi)\, d\xi$$

(2.6) $$q_i(t,x) = \frac{m}{2}\int_{v} c_i c^2 f(t,x,\xi)\, d\xi$$

(2.7) $$\varepsilon(t,x) = \frac{1}{2n}\int_{v} c^2 f(t,x,\xi)\, d\xi$$

where $c(t,x,\xi) = \xi - v(t,x)$ is the deviation of the microscopic velocity from the mean velocity. Because the kinetic theory assures that the fields defined by (2.2) to (2.7) verify the balance equations of the continuum mechanics

(2.8) $$\frac{d\rho}{dt} + \rho \operatorname{div} \bar{v} = 0$$

(2.9) $$\rho \frac{du_i}{dt} + \frac{\partial T_{ij}}{\partial x_j} - \rho b_i = 0$$

(2.10) $$\rho \frac{d\varepsilon}{dt} + \operatorname{div} q + T_{ij}\frac{\partial u_j}{\partial x_i} = 0$$

and the symmetry of the pressure-tensor, the existence and uniqueness of the molecular density stated by the Theorem quoted in the first chapter imply that this fields describe the corresponding solution of the problem of gas motion when we perturb the repose state of the gas by macroscopic fields that correspond to the initial molecular density $f_0(x,\xi)$.

3. Global behaviour of macroscopic fields

In connection with the macroscopic description of the gas appears, as a very important problem, the global behaviour (in time) of macroscopic fields. But as a simple corollary of the theorem stated in the first Chapter the fields $\rho, v_i, T_{ij}, p, q_i$ and ε defined by (2.2) to (2.7) exist and verify the balance equations of continuum mechanics (2.8) to (2.10), in the $H^4(x)$ space. The main result of this paper is stated in the following theorem (that asserts the global time behaviour of macroscopic fields):

Theorem 3.1

Let $0 < \alpha < 1/4$. There exist a positive constant δ such that if $\| f_0 \|_\alpha < \delta$ the macroscopic fields defined by (2.2) to (2.7) exist, are unique in $H^4(x)$ and verify the following inequalities:

(3.1) $$\| \rho(t,\cdot) \|_{H^4(x)} \leq m A \| f_0 \|_\alpha$$

(3.2) $$\| v_i(t,\cdot) \|_{H^4(x)} \leq B \| f_0 \|_\alpha^2$$

(3.3) $$\| T_{ij}(t,\cdot) \|_{H^4(x)} \leq m C \| f_0 \|_\alpha$$

(3.4) $$\| p(t,\cdot) \|_{H^4(x)} \leq m D \| f_0 \|_\alpha$$

(3.5) $$\| q_i(t,\cdot) \|_{H^4(x)} \leq m E \| f_0 \|_\alpha$$

(3.6) $$\| \varepsilon(t,\cdot) \|_{H^4(x)} \leq F \| f_0 \|_\alpha^2$$

A, B, C, D, E, F being constants that may depend on the constant c (of the formula (1.7)) of the domain and possibly on the norm of f_0.

We base our proof on the estimations of moments of 0, 1, 2 and 3 order of f, that we achieve to the aide of following Lemmas (and the Sobolev inequality of norms): $\| f h \|_{H^4(x)} \leq C_1 \| f \|_{H^4(x)} \| h \|_{H^4(x)}$). We identify $\vec{\mathcal{C}} \equiv \mathbb{R}^3$ and $\mathcal{V} \equiv \mathbb{R}^3$.

Lemma 3.1

The following formula is obvious

$$(3.7) \qquad \int_{\mathbb{R}} \varsigma^{2r} e^{-\alpha \varsigma^2} d\varsigma = \frac{1}{2} \cdot \frac{3}{2} \cdot \ldots \cdot \frac{2r-1}{2} \frac{\sqrt{\pi}}{2} \alpha^{-\frac{2r+1}{2}}$$

Lemma 3.2

If $f \in \mathcal{N}_\alpha$ then:

$$(3.8) \qquad \| \int_{\mathbb{R}^3} f(t,x,\xi)\, d\xi \|_{H^4(x)} \leq A \| f_0 \|_\alpha$$

Proof. We apply the Schwartz inequality and we obtain

$$\| \int_{\mathbb{R}^3} f(t,\cdot,\xi)\, d\xi \|^2_{H^4(x)} \leq \left\{ \sum_{j=1}^{4} \left[\int_{D^3} \left(\int_{\mathbb{R}^3} e^{-\alpha\xi^2} d\xi \right) \left(\int_{\mathbb{R}^3} e^{\alpha\xi^2} |\nabla_x^j f(t,x,\xi)|^2 d\xi \right) dx \right]^{1/2} \right\}^2$$

Now we make use of Lemma 3.1, Fubini's theorem and the fact that $f \in \mathcal{N}_\alpha$ and we have:

$$\| \int_{\mathbb{R}^3} f(t,\cdot,\xi)\, d\xi \|^2_{H^4(x)} \leq \left(\frac{\pi}{\alpha}\right)^{3/2} \left\{ \int_{\mathbb{R}^3} e^{\alpha\xi^2} \left[\sum_{j=1}^{4} \left(\int_{D^3} |\nabla_x^j f(t,x,\xi)|^2 dx \right)^{1/2} \right]^2 d\xi \right.$$

$$+ \sum_{i \neq j = 1}^{4} \left[\int_{\mathbb{R}^3} e^{\alpha\xi^2} \left(\int_{D^3} |\nabla_x^i f(t,x,\xi)|^2 dx \right) d\xi \right]^{1/2} \times$$

$$\left. \times \left[\int_{D^3} e^{\alpha\xi^2} \left(\int_{D^3} |\nabla_x^j f(t,x,\xi)|^2 dx \right) d\xi \right]^{1/2} \right\} \leq$$

$$\leq \left(\frac{\pi}{\alpha}\right)^{3/2} \int_{\mathbb{R}^3} e^{-\alpha\xi^2} \| f(t,\cdot,\xi)\|_\alpha^2 d\xi \; +$$

$$+ 12\left(\frac{\pi}{\alpha}\right)^{3/2} \int_{\mathbb{R}^3} e^{-\alpha\xi^2} \left[e^{\alpha\xi^2} \| f(t,\cdot,\xi)\|_{H^4(x)} \right]^2 d\xi \leq$$

$$\leq 13\left(\frac{\pi}{\alpha}\right)^3 \| f(t,\cdot,\xi)\|_\alpha^2$$

Therefore, denoting $A = \sqrt{13}\left(\frac{\pi}{\alpha}\right)^{3/2} c$, where c is the constant of (1.7) we obtain (3.8).

Lemma 3.3

If $f \in \mathcal{N}_\alpha$ and the estimation (1.7) is true, we have

(3.9) $$\left\| \int_{\mathbb{R}^3} \xi_i f(t,x,\xi) d\xi \right\|_{H^4(x)} \leq A' \| f_0 \|_\alpha$$

Proof. By a similar way as in the proof of Lemma 3.2 we have

$$\left\| \int_{\mathbb{R}^3} \xi_i f(t,\cdot,\xi) d\xi \right\|_{H^4(x)}^2 \leq \frac{3}{2}\left(\frac{\pi}{\alpha}\right)^{3/2} \alpha^{-1} \left\{ \sum_{s=1}^{4} \left[\int_{D^3} \left(\int_{\mathbb{R}^3} e^{\alpha\xi^2} |\nabla_x^s f(t,x,\xi)|^2 d\xi \right) dx \right]^{1/2} \right\}^2$$

and then we appeal the Fubini's theorem, by the aid of which we arrive to the calculation of some similar terms as in Lemma 3.2. Thus we obtain

$$\left\| \int_{\mathbb{R}^3} \xi_i f(t,\cdot,\xi) d\xi \right\|_{H^4(x)}^2 \leq \frac{39}{2}\left(\frac{\pi}{\alpha}\right)^3 \alpha^{-1} \| f(t,\cdot,\cdot)\|_\alpha^2$$

Now we make use of (1.7) and denoting $A' = \sqrt{\frac{39}{2}}\left(\frac{\pi}{\alpha}\right)^{3/2} \alpha^{-1/2} c$ we obtain (3.9).

Lemma 3.4

If $f \in \mathcal{N}_\alpha$ and the estimation (1.7) is true, we have:

$$(3.10) \qquad \Big\| \int_{\mathbb{R}^3} \xi_i \xi_k f(t_0, \xi)\, d\xi \Big\|_{H^4(x)} \le A'' \| f_0 \|_\alpha$$

Proof. We proceed analogously as in the proofs of Lemmas 3.2 and 3.3 and we arrive to:

$$\Big\| \int_{v} \xi_i \xi_k f(t_0, \xi)\, d\xi \Big\|^2_{H^4(x)} \le \frac{3\cdot 5}{4\alpha^2} 13 \left(\frac{\pi}{\alpha}\right)^3 c^2 \| f_0 \|^2_\alpha$$

Denoting $A'' = \frac{\sqrt{15}}{2\alpha} \sqrt{13} \left(\frac{\pi}{\alpha}\right)^{3/2} c = \sqrt{\frac{3\cdot 5}{4\alpha^2}}\, A$ we obtain (3.10).

Finally we have:

Lemma 3.5

If $f \in \mathcal{N}_\alpha$ and the estimation (1.7) is true:

$$(3.11) \qquad \Big\| \int_{\mathbb{R}^3} \xi_i \xi^2 f(t_0, \xi)\, d\xi \Big\|_{H^4(x)} \le A''' \| f_0 \|_\alpha$$

Proof. The proof is similar to the previous one and we denote $A''' = \sqrt{\frac{3\cdot 5\cdot 7}{8\alpha^3}}\, A$.

Now we are ready to perform the proof of the Theorem 3.1.

Proof. of Theorem 3.1. First we observe that due to the definitions (2.1), (2.2) and to the result of Lemma 3.2 the evaluation of the mass-density field (3.1) is obvious. For the evaluation of the behaviour of the velocity field $v_i(t,x)$ we observe that due to the imbending theorems Sobolev-Kondrașev the fact that the field $n(t,\cdot) \in H^4(x)$ implies that $n(t,\cdot) \in \mathcal{C}^2(x)$ and his derivatives of higher orders are square integrables.

We this we evaluate the generalized derivatives of the fonction $\frac{1}{n(t,\cdot)}$ and finally we obtain (M_0 is the minimum of n, and M_1, M_2 the maximum of his first and second derivateves).

(3.12) $$\left\| \frac{1}{n(t,\cdot)} \right\|_{H^4(x)} \leq M \, \| f_0 \|_\alpha$$

where $$M = A M_0^{-2} \left[1 + \left(1 + 2\frac{M_1}{M_0}\right) + \left(1 + 6\frac{M_1}{M_0} + 6\frac{M_1^2}{M_0^2}\right) + \left(1 + 8\frac{M_1}{M_0} + 6\frac{M_2}{M_0} + 36\frac{M_1 M_2}{M_0^2} + 24\frac{M_1^3}{M_0^3}\right) \right]$$

Then by the use of Sobolev inequality of norms we have

$$\| v_i(t,\cdot) \|_{H^4(x)} \leq C_1 \left\| \frac{1}{n(t,\cdot)} \right\|_{H^4(x)} \left\| \int_{\mathbb{R}^3} \xi_i f(t,\cdot,\xi) \, d\xi \right\|_{H^4(x)}$$

With these using Lemma 3.3 we obtain the estimation (3.2), where the constant B is $C_1 M A'$.

For the estimation of the components of the pressure tensor field we start from the definition (2.4) and using the general properties of the norm and the Sobolev inequality of norms we arrive to:

(3.13) $$\| T_{ij}(t,\cdot) \|_{H^4(x)} \leq m \left\{ \left\| \int_{\mathbb{R}^3} \xi_i \xi_j f(t,\cdot,\xi) \, d\xi \right\|_{H^4(x)} + \right.$$

$$+ 2c_1 \| v_i(t,\cdot) \|_{H^4(x)} \| \int_{\mathbb{R}^3} \xi_j f(t,\cdot,\xi)\, d\xi \|_{H^4(x)} +$$

$$+ c_i^2 \| v_i(t,\cdot) \|_{H^4(x)} \| v_j(t,\cdot) \|_{H^4(x)} \| \int_{\mathbb{R}^3} f(t,\cdot,\xi)\, d\xi \|_{H^4(x)} \}$$

Using the Lemmas 3.2, 3.3 and 3.4 and also the estimation (3.2) we have:

(3.14) $$\| T_{ij}(t,\cdot) \|_{H^4(x)} \leq m C \| f_0 \|_\alpha ,$$

where: $$C = A'' + 2c_1 B A' \| f_0 \|_\alpha^2 + c_i^2 B^2 A \| f_0 \|_\alpha^4$$

For the normal-mean pressure field estimation we use directly the previous result and the definition (2.5) and we obtain an absolute analogous formula to (3.14). For the estimation of the mean energy flux components we start from the definition (2.6). By a similar reason with the one that has been performed for the formula (3.13), we obtain:

(3.15) $$\| q_i(t,\cdot) \|_{H^4(x)} \leq \frac{3}{2} m \{ \| \int_{\mathbb{R}^3} \xi_i \xi_k^2 f(t,\cdot,\xi)\, d\xi \|_{H^4(x)} +$$

$$+ 2c_1 \| v_k(t,\cdot) \|_{H^4(x)} \| \int_{\mathbb{R}^3} \xi_i \xi_k f(t,\cdot,\xi)\, d\xi \|_{H^4(x)} +$$

$$+ c_i^2 \| v_k(t,\cdot) \|_{H^4(x)}^2 \| \int_{\mathbb{R}^3} \xi_i f(t,\cdot,\xi)\, d\xi \|_{H^4(x)} \}$$

We employ the estimations of Lemmas 3.3, 3.4 and 3.5 and the formula (3.2) and we obtain (3.5), denoting $E = \frac{3}{2}\{A''' + 2c_1 B A'' \|f_0\|_\alpha^2 + c_1^2 B^2 A' \|f_0\|_\alpha^4\}$.

The behaviour of the mean internal energy of the gas is obtained starting with the definition (2.7) and emploing the Sobolev inequality of norms and the result just obtained in (3.12):

$$(3.16) \qquad \|\varepsilon(t,\cdot)\|_{H^4(x)} \leq \frac{1}{2} c_1 \left\| \frac{1}{n(t,\cdot)} \right\|_{H^4(x)} \left\| \int_{\mathbb{R}^3} c_i^2 f(t,\cdot,\xi)\, d\xi \right\|_{H^4(x)}$$

where the last norm is estimated as in (3.13). Then we obtain the estimation (3.6), where $F = \frac{3}{2} c_1 C M$.

4. The problem just solved is relevant by itself and also by the important contribution that it can bring to the comparative study of the behaviour of solutions that we obtain (by Chapman-Enskog expansion) for Navier-Stokes fluid and the macroscopic fields. We will analyse this problem in a future paper.

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