SOME RESULTS ON MIXED MANIFOLDS

by

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The notion of mixed manifold (space) was considered by M.Jurchescu in [J,1] and [J,2] ,where he proves some foundamental results concerning the cohomology of coherent sheaves on such manifolds.

In this paper we extend some facts from the complex (resp. differentiable) case to the mixed case.

Mixed manifolds

Mixed manifolds are defined as follows (see [J,1]):

Consider the category $\mathcal{L}$ of local models whose objects are open subsets of spaces of type $\mathbb{R}^m x\mathbb{C}^n$ and where the morphisms are C^∞-maps holomorphic with respect to the complex variables. Localizing by standard procedures the category $\mathcal{L}$ one obtains the category $\mathcal{M}$ of mixed manifolds. Morphisms in $\mathcal{M}$ will often be called mixed morphisms. The type of a mixed manifold X in $x\in X$ is $(m,n) \overset{def}{\Longleftrightarrow}$ there exists an open neighbourhood of x in X isomorphic (in $\mathcal{M}$) to an open subset of $\mathbb{R}^m x\mathbb{C}^n$. The type is constant on connected components. The structure sheaf of a mixed manifold X is the sheaf of germs of complex valued mixed morphisms on X, and will be denoted by $\mathcal{O}_X(\mathbb{C})$. We shall denote by $\mathcal{O}_X(\mathbb{R})$ the sheaf of germs of real valued mixed morphisms on X. Mixed manifolds will be supposed with countable base and Hausdorff. Mixed differential forms on mixed manifolds are defined in [J,1] . Then:

Theorem 1 . (The mixed Dolbeault resolution). ([J,1])

Let X be a mixed manifold of type (m,n), let $p \geq 0$. Then

a) There exists an exact sequence of $\mathcal{O}_X(\mathbb{C})$-modules :

$$0 \longrightarrow \Omega^{(p)} \longrightarrow \mathcal{E}^{(p,0)} \xrightarrow{\bar{\partial}^{(p,0)}} \cdots \xrightarrow{\bar{\partial}^{(p,n-1)}} \mathcal{E}^{(p,n)} \longrightarrow 0$$

b) $H^q(X, \Omega^{(p)}) = \ker \bar{\partial}^{(p,q)} / \operatorname{Im} \bar{\partial}^{(p,q-1)}$ $\quad 1 \leq q \leq n$

and $H^q(X, \Omega^{(p)}) = 0$ for $q \geq n+1$. (See also [J,1] for notations.)

Definition 1.

An $\mathcal{O}_X(\mathbb{C})$-module $\mathcal{F}$ is r-coherent iff for each $x \in X$ there exist an open neighbourhood U of x in X and an exact sequence

$$\mathcal{L}_r \longrightarrow \cdots \longrightarrow \mathcal{L}_o \xrightarrow{\varepsilon} \mathcal{F}|_U \longrightarrow 0$$

with $\mathcal{L}_i = (\mathcal{O}_U(\mathbb{C}))^{q_i}$.

An $\mathcal{O}_X(\mathbb{C})$-module $\mathcal{F}$ which is r-coherent for each $r \in \mathbb{N}$ is called coherent. (We shall often write $\mathcal{F} \in \mathrm{Coh}(X)$.)

The structure sheaf $\mathcal{O}_X(\mathbb{C})$ has a canonical Frechet structure (see [J,1]).

A coherent $\mathcal{O}_X(C)$ module $\mathcal{F}$ is separate iff for each $x \in X$ there exist an open neighbourhood U of x and an exact sequence:

$$0 \longrightarrow \mathcal{F}_o \longrightarrow \mathcal{O}_U^{r_o}(C) \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

where $\mathcal{F}_o$ is a closed subsheaf of $\mathcal{O}_U^{r_o}(\mathbb{C})$ (i.e. $\Gamma(V, \mathcal{F}_o)$ is a closed subset of $\Gamma(V, \mathcal{O}_U^{r_o}(\mathbb{C}))$ with respect to the canonical Fréchet topology mentioned above for any open subset V in U).

<u>Proposition</u> 1 . Let X be a mixed manifold of type (m,n) let $\mathcal{F} \in \mathrm{Coh}(X)$. For each $x \in X$, there exist an open neighbourhood U of x in X and an exact sequence:

$$0 \longrightarrow \mathcal{L}_{m+n} \longrightarrow \cdots \longrightarrow \mathcal{L}_o \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

with $\mathcal{L}_i = (\mathcal{O}_U(\mathbb{C}))^{q_i}$.

We may prove now :

<u>Theorem</u> 2 . Let X be a mixed manifold of type (m,n) let $\mathcal{F} \in \mathrm{Coh}(X)$. Then $H^q(X,\mathcal{F}) = 0$ for $q \geqslant n+1$.

<u>Proof</u>. As in [R] it is sufficient to prove that for each $x \in X$, there exist an open neighbourhood U of x such that for each open subset $V \subseteq U$, $H^q(V, \mathcal{F}|_V) = 0$ for $q \geqslant n+1$. But, for any $x \in X$, we obtain from proposition 1 an exact sequence:

(1) $$0 \longrightarrow \mathcal{L}_{m+n} \xrightarrow{\delta^{m+n-1}} \cdots \xrightarrow{\delta^o} \mathcal{L}_o \xrightarrow{\varepsilon} \mathcal{F}|_U \longrightarrow 0$$

with $\mathcal{L}_i = (\mathcal{O}_U(\mathbb{C}))^{q_i}$.

For each open subset V of U, by restricting (1) to V we still get an exact sequence. We infer the existence of the following short exact sequences:

(2)
$$0 \longrightarrow \mathcal{F}_o \longrightarrow \mathcal{O}_V^{q_o}(\mathbb{C}) \xrightarrow{\varepsilon} \mathcal{F}|_V \longrightarrow 0$$
$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{O}_V^{q_1}(\mathbb{C}) \xrightarrow{\delta^o} \mathcal{F}_o \longrightarrow 0$$
$$0 \longrightarrow \mathcal{O}_V^{q_{m+n}}(\mathbb{C}) \xrightarrow{\delta^{m+n-1}} \mathcal{O}_V^{q_{m+n-1}}(\mathbb{C}) \xrightarrow{\delta^{m+n-2}} \mathcal{F}_{m+n-2} \longrightarrow 0$$

where $\mathcal{F}_o = \ker \varepsilon = \mathrm{Im}\, \delta^o$

$\mathcal{F}_1 = \ker \delta^o = \mathrm{Im}\, \delta^1$, etc.

By the mixed Dolbeault resolution, it follows that:

$$H^q(V, \mathcal{O}_V^{q_i}(\mathbb{C}))=0 \qquad q\geq n+1 \qquad i\geq 0$$

Then, by writing the long exact sequences of cohomology associated to (2) we get for $q\geq n+1$

$$H^q(V, \mathcal{F}_{|V})=H^{q+1}(V, \mathcal{F})= \ldots =H^{q+m+n}(V, \mathcal{O}_V^{q_{m+n-1}}(\mathbb{C}))=0$$

and the theorem is proved.

Cartan manifolds

Definition 2 . A mixed manifold will be called a Cartan manifold iff

a) X is $\mathcal{O}_X(\mathbb{C})$-convex i.e for any compact subset K of X, the set $\hat{K}_X:=\{x\in X \mid \sup_{y\in K} |f(y)|\geq |f(x)| \text{ for each } f\in\Gamma(X, \mathcal{O}_X(\mathbb{C}))\}$ is a compact subset in X (or, equivalently if for any discrete sequence $(x_n)_{n\in\mathbb{N}}$ in X, there exists $f\in\Gamma(X, \mathcal{O}_X(\mathbb{C}))$ such that $\sup_{n\in\mathbb{N}} |f(x_n)| =+\infty$).

b) $\Gamma(X, \mathcal{O}_X(\mathbb{C}))$ separates points, namely for $x\neq y$, there exists $f\in\Gamma(X, \mathcal{O}_X(\mathbb{C}))$ which satisfies $f(x)\neq f(y)$.

c) For any $x\in X$ there exists a coordinate system around x given by global sections.

The full subcategory of $\mathcal{M}$ whose objects are Cartan manifolds of finite type will be denoted by $\mathcal{C}$. The following result is proved in [J,1].

<u>Theorem</u> 3. (Theorems A and B for Cartan manifolds).

Let X be a Cartan manifold, let $\mathcal{F}$ be a coherent separate $\mathcal{O}_X(\mathbb{C})$-module. Then

A) For each $x \in X$, $\mathcal{F}_x$ is $\mathcal{O}_{X,x}(\mathbb{C})$-generated by the image of $\mathcal{F}(X)$ in $\mathcal{F}_x$.

B) $H^q(X,\mathcal{F})=0$ for each $q \geq 1$.

<u>Definition</u> 3. Let X, Y be two mixed manifolds, let $\varphi : X \longrightarrow Y$ be a mixed morphism. φ is a mixed embedding iff:

a) $\varphi(X)$ is a mixed submanifold of Y (for the definition of mixed submanifolds see [J,1]).

b) $\varphi : X \longrightarrow \varphi(X)$ is a mixed isomorphism.

The following result is proved in [J,1].

<u>Theorem</u> 4. (The embedding theorem).

Let X be a Cartan manifold of type (m,n). Then there exists a mixed embedding $\varphi : X \longrightarrow \mathbb{R}^M \times \mathbb{C}^N$ for M, N sufficiently large positive integers (one may take $M=2m, N=m+2n+1$). Note that the map $\alpha : \Gamma(\mathbb{R}^M \times \mathbb{C}^N, \mathcal{O}_{\mathbb{R}^M \times \mathbb{C}^N}(\mathbb{C})) \longrightarrow \Gamma(X, \mathcal{O}_X(\mathbb{C}))$ defined by $\alpha(H)=H \circ \varphi$ is onto.

We shall characterize by means of cohomology the open Cartan subsets of Cartan manifolds. One can notice that an open subset D of a Cartan manifold X is itself a Cartan manifold (with respect to the induced structure) iff D is $\mathcal{O}_D(\mathbb{C})$-convex. Then:

Theorem 5. Let X be a Cartan manifold of type (m,n) let D be an open subset in X. The following statements are equivalent:

(i) D is a Cartan manifold;

(ii) $H^q(D, \mathcal{O}_D(\mathbb{C}))=0$ for $q=1,2,\ldots,n$.

The proof goes along the same lines as a proof of S.Coen in the complex case using a theorem of Nagel [N].

It could be of interest to show at this stage that some "good" properties of the "real, resp. complex component" of a mixed manifold do not extend to this latter.

Definition 4 . A C^∞- family of complex manifolds is $p:X \to Y$ where X is a mixed manifold of type (m,n), Y is a C^∞- manifold of dimension m and p is a mixed morphism which is onto and which satisfies: For each $x \in X$, there exist an open neighbourhood U of x in X, an open neighbourhood V of $p(x)$ in Y, an open subset D of $\mathbb{C}^n$ and a mixed isomorphism $\psi : U \to V \times D$, such that $p(U) \subseteq V$ and the diagram

$$\begin{array}{ccc} U & \xrightarrow[\sim]{\psi} & V \times D \\ {\scriptstyle p}\downarrow & \swarrow {\scriptstyle \pi_1} & \\ V & & \end{array}$$

is commutative.

Remark 1 .

Let $p:X \longrightarrow Y$ be a C^∞-family of complex manifolds. Suppose that X is a Cartan manifold of type (m,n). Then for each $y \in Y$ the closed submanifold of type $(0,n)$ of X, $X_y := \{x \in X \mid p(x)=y\}$ is a Stein manifold of dimension n. The converse of this statement is not true.

Example 1 . Let $X=\mathbb{R}\times\mathbb{C}\setminus\{(0,0)\}$ (viewed as a family by means of the canonical projection on $\mathbb{R}$). We have:

$$X_t=\begin{cases}\mathbb{C} & t\neq 0\\ \mathbb{C}\setminus\{0\} & t=0\end{cases}$$

hence each fibre is Stein. But X is not a Cartan manifold because if

$$K=\{(t,z)\in X \mid |t|\leq 1,\ |z|=1\}$$

then

$$\hat{K}_X=\{(t,z)\in X \mid |t|\leq 1\ \ |z|\leq 1\}$$

and hence not a compact subset of X.

Let us note some further properties of X:

a) $H^1(X,\mathcal{O}_X(\mathbb{C}))\neq 0$ (immediate consequence of theorem 5)

b) The canonical restriction $\alpha: \Gamma(\mathbb{R}\times\mathbb{C},\mathcal{O}_{\mathbb{R}\times\mathbb{C}}(\mathbb{C}))\longrightarrow\Gamma(X,\mathcal{O}_X(\mathbb{C}))$ defined by $\alpha(F)=F|_X$ is onto.

Indeed, let us notice that if $f\in\Gamma(X,\mathcal{O}_X(\mathbb{C}))$ then $f|_{\mathbb{R}\times\mathbb{C}^*}$ is an element of $\Gamma(\mathbb{R}\times\mathbb{C}^*,\mathcal{O}_{\mathbb{R}\times\mathbb{C}^*}(\mathbb{C}))$, hence for each $t\in\mathbb{R}$ the functions $f_t:\mathbb{C}^*\longrightarrow\mathbb{C}$ $f_t(z):=f(t,z)$ have Laurent developments $f_t(z)=\sum_{\nu\in\mathbb{Z}}\alpha_\nu(t)\,z^\nu$. The coefficients α_ν are C^∞-functions with respect to t. But for $t\neq 0$ it follows $\alpha_\nu(t)=0$ for each $\nu<0$ as f_t are holomorphic. By continuity it results $\alpha_\nu(0)=0$, and b) follows.

It is not difficult to show that $X=\mathbb{R}\times\mathbb{C}\setminus D$ (where D is a real line) is an open Cartan subset of $\mathbb{R}\times\mathbb{C}$.

The following example shows how the "mixed continuation" works.

<u>Example</u> 2.

Let $X=\mathbb{R}\times\mathbb{C}-\{(t,0) \mid |t|\leq 1\}$.

It is an open subset of $\mathbb{R}\times\mathbb{C}$, but it is not Cartan (as we see noticing that for $K=\{(t,z)\in X \mid 1\leq|t|\leq 2 \quad |z|=1\}$, $\hat{K}_X=\{(t,z)\in X \mid 1\leq|t|\leq 2, |z|\leq 1\}$ and hence a noncompact subset of X).

Let $\eta:\mathbb{R}\longrightarrow\mathbb{R}$ be a C^{∞}- function with compact support: $\operatorname{supp}\eta=[-1,1]$. Define
$$f(t,z)=\begin{cases}\frac{\eta(t)}{z} & |t|<1\\ 0 & |t|\geq 1\end{cases}$$

f is an element of $\Gamma(X,\mathcal{O}_X(\mathbb{C}))$. One cannot find any open subset U of $\mathbb{R}\times\mathbb{C}$, $U\neq\emptyset$, $U\not\subset X$, for which there should exist $F\in\Gamma(U,\mathcal{O}_U(\mathbb{C}))$ such that $F|_{X\cap U}=f|_{X\cap U}$ (because such an open subset contains points from $(-1,1)\times\{0\}$ where f is unbounded).

<u>A theorem of equivalence</u>

<u>Definition</u> 5. A $\mathbb{C}$-algebra (algebraically) isomorphic to $\Gamma(X,\mathcal{O}_X(\mathbb{C}))$ for a certain Cartan manifold X of type (m,n) will be called a smooth Cartan algebra.

Smooth Cartan algebras and smooth Cartan algebra morphisms (i.e. unitary nonzero $\mathbb{C}$-algebra morphisms) form a category, denoted by $\mathcal{AC}$. We shall usually omit "smooth", if no confusion is possible.

<u>Definition</u> 6. Let A be a Cartan algebra. An element of the set $\operatorname{Hom}_{\mathcal{AC}}(A,\mathbb{C})$ is called a character of A.

<u>Example</u> 4. For a Cartan manifold X of type (m,n) and for $x\in X$, the morphism $\chi_x:\Gamma(X,\mathcal{O}_X(\mathbb{C}))\longrightarrow\mathbb{C}$ given by $\chi_x(f)=f(x)$ for each $f\in\Gamma(X,\mathcal{O}_X(\mathbb{C}))$, is a character of $\Gamma(X,\mathcal{O}_X(\mathbb{C}))$. Such a character will be called "point-character".

Remark 2. Due to the embedding theorem, to theorems A and B for Cartan manifolds and to theorem 1 in [N], each character of a Cartan algebra $\Gamma(X, \mathcal{O}_X(\mathbb{C}))$ is a point character. Moreover it follows that, with respect to the cannonical topology, characters are continuous; they carry $\Gamma(X, \mathcal{O}_X(\mathbb{R}))$ in $\mathbb{R}$.

Lemma 1 . Let X, Y be Cartan manifolds. Let $u \in \mathrm{Hom}_{\mathcal{A}\mathcal{C}}(\Gamma(X, \mathcal{O}_X(\mathbb{C})), \Gamma(Y, \mathcal{O}_Y(\mathbb{C}))$. Then $u(\Gamma(X, \mathcal{O}_X(\mathbb{R}))) \subseteq \Gamma(Y, \mathcal{O}_Y(\mathbb{R}))$

Proof. Let $f \in \Gamma(X, \mathcal{O}_X(\mathbb{R}))$, $y \in Y$. The point character associated to y on $\Gamma(Y, \mathcal{O}_Y(\mathbb{C}))$ will be denoted by χ_y. It follows that $\chi_y \circ u$ is a character on $\Gamma(X, \mathcal{O}_X(\mathbb{C}))$ and hence, due to remark 1, it is a point character χ_x associated to a certain point $x \in X$.

We get then

$$(u(f))(y) = \chi_y(u(f)) = (\chi_y \circ u)(f) = \chi_x(f) = f(x) \in \mathbb{R}$$

which concludes the proof.

Lemma 2.

Let χ be a character on $\Gamma(\mathbb{R}^m x \mathbb{C}^n, \mathcal{O}_{\mathbb{R}^m x \mathbb{C}^n}(\mathbb{C}))'$. Then χ is the point character associated to the point $x = (\chi(\pi_1), \ldots \chi(\pi_{m+n}))$ where π_i denotes the canonical projection from $\mathbb{R}^m x \mathbb{C}^n$ on the i^{th} factor).

Proof. By remark 2, let $x \in \mathbb{R}^m x \mathbb{C}^n$ be the point such that $\chi = \chi_x$. We obtain, for each i=1,2,....,m+n

$$\chi(\pi_i) = \chi_x(\pi_i) = \pi_i(x) = x_i$$

Corollary 1 . Two characters on $\Gamma(\mathbb{R}^m x \mathbb{C}^n, \mathcal{O}_{\mathbb{R}^m x \mathbb{C}^n}(\mathbb{C}))$ which coincide on the coordinate functions (i.e. canonical projections) coincide.

We remind that $\mathcal{C}$ denotes the category of Cartan manifolds of finite type.

We define the contravariant functor

$$\mathcal{G} : \mathcal{C} \longrightarrow \mathcal{A}\mathcal{C}$$

as follows:

On the objects: For $X \in Ob(\mathcal{C})$, $\mathcal{G}(X) = \Gamma(X, \mathcal{O}_X(\mathbb{C}))$

On the morphisms: For $f \in Hom_{\mathcal{C}}(X,Y)$, $\mathcal{G}_{XY}(f) : \Gamma(Y, \mathcal{O}_Y(\mathbb{C})) \to \Gamma(X, \mathcal{O}_X(\mathbb{C}))$ is given by :

$$\mathcal{G}_{XY}(f)(h) = h \circ f, \quad \text{for each } h \in \Gamma(Y, \mathcal{O}_Y(\mathbb{C}))$$

The fact that $\mathcal{G}$ is a functor is then an obvious consequence of standard computations.

Then, one gets:

<u>Theorem</u> 6. $\mathcal{G}$ is an (anti) equivalence of categories.

<u>Proof</u>. We have to prove that for each $X \in Ob(\mathcal{C})$, $Y \in Ob(\mathcal{C})$

$$\mathcal{G}_{XY} : Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{A}\mathcal{C}}(\Gamma(Y, \mathcal{O}_Y(\mathbb{C})), \Gamma(X, \mathcal{O}_X(\mathbb{C})))$$

is bijective

Let $f, g \in Hom_{\mathcal{C}}(X,Y)$. Suppose $\mathcal{G}_{XY}(f) = \mathcal{G}_{XY}(g)$. This means that for each $h \in \Gamma(Y, \mathcal{O}_Y(\mathbb{C}))$ the following relation holds:

$$h \circ f = h \circ g$$

As Y is a Cartan manifold, $\Gamma(Y, \mathcal{O}_Y(\mathbb{C}))$ separates points and hence $f=g$.

Whence $\mathcal{G}_{XY}$ is one to one.

Suppose now $Y = \mathbb{R}^p \times \mathbb{C}^q$ and let

$$u: \Gamma(\mathbb{R}^p x \mathbb{C}^q, \mathcal{O}_{\mathbb{R}^p x \mathbb{C}^q}(\mathbb{C})) \longrightarrow \Gamma(X, \mathcal{O}_X(\mathbb{C}))$$

be a Cartan algebra morphism.

For each $i=1,2,\ldots, p+q$, we denote by π_i the canonical projection of $\mathbb{R}^p x \mathbb{C}^q$ on the i^{th} factor, and by $f_i = u(\pi_i)$

$f_i \in \Gamma(X, \mathcal{O}_X(\mathbb{C}))$ for $i=1\ldots p+q$ and by lemma 1, $f_i \in \Gamma(X, \mathcal{O}_X(\mathbb{R}))$ for $i=1,\ldots,p$.

Let $f: X \longrightarrow \mathbb{R}^p x \mathbb{C}^q$, be the mixed morphism defined by $f(x)=(f_1(x),\ldots,f_{p+q}(x))$.

Let $h \in \Gamma(\mathbb{R}^p x \mathbb{C}^q, \mathcal{O}_{\mathbb{R}^p x \mathbb{C}^q}(\mathbb{C}))$, let $x \in X$, let χ_x be the point character on $\Gamma(X, \mathcal{O}_X(\mathbb{C}))$ associated to x.

Due to lemma 2 , the following relations hold:

$$\chi_x \circ u = \chi_{(\chi_x \circ u(\pi_1), \ldots \chi_x \circ u(\pi_{p+q}))} = \chi_{(f_1(x)\ldots f_{p+q}(x))} = \chi_{f(x)}$$

Then, it follows

$$(u(h))(x) = \chi_x(u(h)) = (\chi_x \circ u)(h) = \chi_{f(x)}(h) = h(f(x)) = h \circ f(x)$$

This shows

$$\mathcal{G}_{X,\mathbb{R}^p x \mathbb{C}^q}(f) = u$$

and hence $\mathcal{G}_{X,\mathbb{R}^p x \mathbb{C}^q}$ is onto for each $(p,q) \in \mathbb{N} x \mathbb{N}$

Let now Y be a Cartan manifold and let $u \in Hom_{\mathcal{AB}}(\Gamma(Y, \mathcal{O}_Y(\mathbb{C})), \Gamma(X, \mathcal{O}_X(\mathbb{C})))$

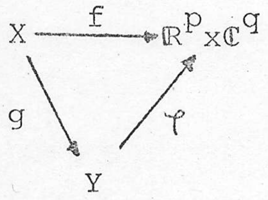
Due to the embedding theorem, there exists a mixed embedding

$$\varphi: Y \longrightarrow \mathbb{R}^p x \mathbb{C}^q$$

By the above mentioned facts, there exists $f \in Hom_{\mathcal{C}}(X, \mathbb{R}^p x \mathbb{C}^q)$ such that :

(1) $\mathcal{G}_{X,\mathbb{R}^p x\mathbb{C}^q}(f)=u$

We prove now that there exists a mixed morphism $g:X \longrightarrow Y$ such that the following diagram



is commutative. In order to prove this, it is sufficient to show that $f(X)\subseteq\varphi(Y)$.

In fact, let $x\in X$. For $h\in\Gamma(\mathbb{R}^p x\mathbb{C}^q, \mathcal{O}_{\mathbb{R}^p x\mathbb{C}^q}(\mathbb{C}))$ by (1) we obtain

(2) $h(f(x))=(u(h\circ\varphi))(x)=\chi_x(u(h\circ\varphi))=(\chi_x\circ u)(h\circ\varphi)$

As the characters on $\Gamma(Y,\mathcal{O}_Y(\mathbb{C}))$ are point characters, it follows that there exists $y\in Y$ with

$$\chi_x\circ u=\chi_y$$

Then, by (2):

(3) $h(f(x))=(\chi_x\circ u)(h\circ\varphi)=\chi_y(h\circ\varphi)=(h\circ\varphi)(y)=h(\varphi(y))$

As $\mathbb{R}^p x\mathbb{C}^q$ is a Cartan manifold and the global sections of $\mathcal{O}_{\mathbb{R}^p x\mathbb{C}^q}(\mathbb{C})$ separate points, we get

$$f(x)=\varphi(y) \quad ,$$

hence $f(X)\subseteq\varphi(Y)$.

As Y is (isomorphic to) a closed Cartan submanifold of $\mathbb{R}^p x\mathbb{C}^q$ it follows that for any $h\in\Gamma(Y,\mathcal{O}_Y(\mathbb{C}))$, there exists

$\tilde{h}\in\Gamma(\mathbb{R}^p\times\mathbb{C}^q, \mathcal{O}_{\mathbb{R}^p\times\mathbb{C}^q}(\mathbb{C}))$ such that $\tilde{h}\circ\varphi=h$ (simple consequence of theorem A and B for Cartan manifolds).

Then, for $h\in\Gamma(Y,\mathcal{O}_Y(\mathbb{C}))$ the following relations hold:

$$u(h)=u(\tilde{h}\circ\varphi)=u(\mathcal{G}_{Y,\mathbb{R}^p\times\mathbb{C}^q}(\varphi)(\tilde{h}))=(u\circ\mathcal{G}_{Y,\mathbb{R}^p\times\mathbb{C}^q}(\varphi))(\tilde{h})=$$

$$=\mathcal{G}_{X,\mathbb{R}^p\times\mathbb{C}^q}(f)(\tilde{h})=\tilde{h}\circ f=\tilde{h}\circ(\varphi\circ g)=(\tilde{h}\circ\varphi)\circ g=h\circ g=\mathcal{G}_{XY}(g)(h)$$

which shows that

$$\mathcal{G}_{XY}(g)=u$$

hence $\mathcal{G}_{XY}$ is onto, and the theorem is completely proved.

Let $\mathcal{X}$:= the set of characters of $\Gamma(X,\mathcal{O}_X(\mathbb{C}))$. We endow $\mathcal{X}$ with the trace of the weak topology (on the dual of $\Gamma(X,\mathcal{O}_X(\mathbb{C}))$. We remind that, with respect to this topology, a foundamental system of neighbourhoods of a character $\chi\in\mathcal{X}$ is given by the neighbourhoods of the following type:

$$V=V_{\chi,f_1\ldots f_s,\varepsilon}=\{\psi\in\mathcal{X} \mid |\psi(f_i)-\chi(f_i)|<\varepsilon\ ,\ i=1,\ldots s\}$$

for all finite systems $f_1,\ldots,f_s$ of elements in $\Gamma(X,\mathcal{O}_X(\mathbb{C}))$ and for all $\varepsilon>0$.

Then as in the complex case we get:

<u>Proposition</u> 3.

The map $\theta:X\longrightarrow\mathcal{X}$ given by $\theta(x)=\chi_x$ is a homeomorphism.

Complex (locally trivial) vector bundles of finite rank over mixed manifolds

Definition 7.

A trivial complex vector bundle of rank k over the mixed manifold X (in short a "tvb") is the mixed manifold $X\times\mathbb{C}^k$. A morphism between the tvb $X\times\mathbb{C}^k$ and $X\times\mathbb{C}^j$ is given by a mixed morphism

$$H=(H_1,H_2):X\times\mathbb{C}^k \longrightarrow X\times\mathbb{C}^j$$

which satisfies:

a) H_1 is the canonical projection on X.

b) There exists a (uniquely determined) mixed morphism $h:X\longrightarrow \mathcal{L}(\mathbb{C}^k,\mathbb{C}^j)$ such that the diagram

$$\begin{array}{ccc} X\times\mathbb{C}^k & \xrightarrow{\;h\times 1_{\mathbb{C}^k}\;} & \mathcal{L}(\mathbb{C}^k,\mathbb{C}^j)\times\mathbb{C}^k \\ & {\scriptstyle H_2}\searrow \quad \swarrow{\scriptstyle \delta} & \\ & \mathbb{C}^j & \end{array}$$

is commutative, where $\delta(A,z)=A\cdot z$.

We shall often refer to H and h by saying that they are associated to each other.

The composition of two morphisms of tvb

$$F:X\times\mathbb{C}^k \longrightarrow X\times\mathbb{C}^j$$

and

$$G:X\times\mathbb{C}^j \longrightarrow X\times\mathbb{C}^\ell$$

is defined as $H:X\times\mathbb{C}^k \longrightarrow X\times\mathbb{C}^\ell$

$$H=G\circ F$$

Note that H is a morphism of trivial vector bundles over X as

H_1 is the canonical projection and H is associated to $h:X \longrightarrow \mathcal{L}(\mathbb{C}^k, \mathbb{C}^\ell)$ given by the commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{(g,f)} & \mathcal{L}(\mathbb{C}^j,\mathbb{C}^\ell) \times \mathcal{L}(\mathbb{C}^k,\mathbb{C}^j) \\ & {\scriptstyle h}\searrow & \swarrow{\scriptstyle \delta} \\ & \mathcal{L}(\mathbb{C}^k,\mathbb{C}^j) & \end{array}$$

where $\delta(A,B)=A\cdot B$.

It follows that tvb of finite rank over a mixed manifold X form a category. Note that a morphism $H:X \times \mathbb{C}^k \longrightarrow X \times \mathbb{C}^\ell$ in this category is an isomorphism iff $k = \ell$ and the values of the associated morphism h are in $GL(k, \mathbb{C})$.

<u>Definition</u> 8. A (locally trivial) complex vector bundle (in short v.b.) of rank k over the mixed manifold X is a mixed manifold E together with a mixed morphism $\pi_E : E \longrightarrow X$ which is onto, such that there exists an open covering $\mathcal{U} = (U_i)_{i\in I}$ of X and the following properties are satisfied.

1) There exist mixed isomorphisms

$$f_i : \pi_E^{-1}(U_i) \longrightarrow U_i \times \mathbb{C}^k$$

and the diagram

$$\begin{array}{ccc} \pi_E^{-1}(U_i) & \xrightarrow{f_i} & U_i \times \mathbb{C}^k \\ & {\scriptstyle \pi_E}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & U_i & \end{array}$$

is commutative.

2) $F_{ij}=f_i \circ f_j^{-1}\Big|_{U_i\cap U_j \times \mathbb{C}^k} : U_i\cap U_j\times\mathbb{C}^k \longrightarrow U_i\cap U_j\times\mathbb{C}^k$

are isomorphisms of trivial complex vector bundles of rank k over $U_i \cap U_j$; for each pair $(i,j)\in I \times I$ for which $U_i\cap U_j \neq \phi$

$(U_i,f_i)_{i\in I}$ will be called a trivialization for E. As $f_{ij}:U_i\cap U_j \longrightarrow GL(k,\mathbb{C})$, associated to F_{ij} in the definition above satisfy $f_{ij}f_{j\ell}=f_{i\ell}$ on $U_i\cap U_j\cap U_\ell$ they determine a cocycle $(f_{ij})_{ij}\in \mathcal{Z}^1(\mathcal{U}, GL(k, \mathcal{O}_X(\mathbb{C})))$.

A morphism between the v.b. E and F (rank E=k rank F=ℓ) is a mixed morphism $h:E\longrightarrow F$ such that

1) The diagram

$$\begin{array}{ccc} E & \xrightarrow{h} & F \\ & {\scriptstyle \pi_E}\searrow \quad \swarrow{\scriptstyle \pi_F} & \\ & X & \end{array}$$

is commutative.

2) For each trivialization $(U_i,f_i)_{i\in I}$ for E and for each trivialization $(V_i,g_j)_{j\in J}$ for F the maps H_{ij} defined by the commutative diagram

$$\begin{array}{ccc} U_i\cap V_j\times\mathbb{C}^k & \xrightarrow{\quad H_{ij}\quad} & U_i\cap V_j\times\mathbb{C}^\ell \\ \Big\uparrow {\scriptstyle f_i|\pi_E^{-1}(U_i\cap V_j)} & & \Big\uparrow {\scriptstyle g_j|\pi_F^{-1}(U_i\cap V_j)} \\ \pi_E^{-1}(U_i\cap V_j) & \xrightarrow{\ h|\pi_E^{-1}(U_i\cap V_j)\ } & \pi_F^{-1}(U_i\cap V_j) \end{array}$$

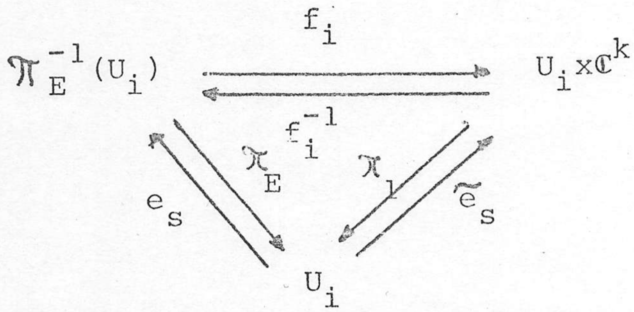
are morphisms of t v b over $U_i\cap V_j$.

By defining the composition of the morphisms as the usu composition of maps, it follows immediately that v.b.of finite rank form a category $\mathcal{VB}$

It is not difficult to notice that two such v.b. are isomorphic in $\mathcal{VB}$ iff their associated cocycles give the same class in $H^1(X, GL(k, \mathcal{O}_X(\mathbb{C}))$. One may find the v.b. starting by cocycles by standard procedures see [W] , and it is straightforward to prove that the association between (classes of isomorphism of) v.b. of rank k over X and elements of $H^1(X, GL(k, \mathcal{O}_X(\mathbb{C}))$ is a bijection.

To each v.b. E of rank k over X one can associate a locally free sheaf $\mathcal{E}$ of rank k on X namely the sheaf of cross sections of E: $\mathcal{E}(u)=\{s:U\to E\}$ s is a mixed morphism, $\pi_E \circ s = 1_U \}$ for each open subset U of X, the restrictions being the obvious ones.

The fact that $\mathcal{E}$ is locally free follows immediately by seeing that for a trivialization $(U_i, f_i)_{i\in I}$ for E, for each $i\in I$ the maps $(e_s)_{s=1..k}: U_i \longrightarrow \pi_E^{-1}(U_i)$ defined by the commutative diagram



where $\tilde{e}_s(x)=(0,\ldots,\overset{s}{1},\ldots,0)$ for each $x\in X$ form a local frame for E, and $\mathcal{E}(U_i)$ is therefore canonically isomorphic to $\Gamma(U_i, \mathcal{O}_X(\mathbb{C}))^k$, whence $\mathcal{E}|_{U_i} \cong \mathcal{O}^k_{U_i}(\mathbb{C})$.

It may be shown like in the case where X is a C^∞ or a complex manifold that the association between (classes of isomorphism of) v.b. of rank k over X and (classes of isomorphism of) locally free sheaves of rank k over X is a bijection.

The usual operations on bundles may be performed in $\mathcal{VB}$. All these operations are canonical and we shall not insist on them. We shall only mention the existence of the dual E^* of a v.b. E which is also a v.b. of rank k on X and whose associated sheaf will be denoted by $\mathcal{E}^*$. The following property of E^* may be proved by a straightforward computation and we omit the details.

<u>Lemma</u> 3 . Let $E \xrightarrow{\pi_E} X$ be a v.b. of rank k over X. Let $E^* \xrightarrow{\pi_{E^*}} X$ be the dual v.b. and let $s^*: X \longrightarrow E^*$ be a global section in E^*. Then there exists a mixed morphism $s: E \longrightarrow \mathbb{C}$ such that for each $x \in X$: $s\big|_{\pi_E^{-1}(x)} = s^*(x)$.

We shall prove now

<u>Theorem</u> 7. Let X be a Cartan manifold of type (m,n) let $E \xrightarrow{\pi_E} X$ be a v.b. of rank k on X. Then E is a Cartan manifold.

<u>Proof</u>

a) $\Gamma(E, \mathcal{O}_E(\mathbb{C}))$ separates points.

Indeed, if $e_1 \neq e_2$ and $\pi_E(e_1) \neq \pi_E(e_2)$ there exists $f \in \Gamma(X, \mathcal{O}_X(\mathbb{C}))$ such $f(\pi_E(e_1) \neq f(\pi_E(e_2))$ as $\Gamma(X, \mathcal{O}_X(\mathbb{C}))$ separates points.

Then $f \circ \pi_E \in \Gamma(E, \mathcal{O}_E(\mathbb{C}))$ and it separates e_1 from e_2 . If $\pi_E(e_1) = \pi_E(e_2) = x$, for a trivialization $(U_i, f_i)_{i \in I}$ for E we have $f_{i_o}(e_1) = (x, z_1)$, $f_{i_o}(e_2) = (x, z_2)$ with $z_1 \neq z_2$ for $f_{i_o}: \pi_E^{-1}(U_{i_o}) \longrightarrow U_{i_o} \times \mathbb{C}^k$.

There exists a linear functional $L: \mathbb{C}^k \longrightarrow \mathbb{C}$ such that $L(z_1) \neq L(z_2)$. L defines a germ of cross section of E^* around x hence $\ell_x \in \mathcal{E}^*_x$. As $\mathcal{E}^*$ is an $\mathcal{O}_X(\mathbb{C})$-locally free module (by the discussion above) we may apply to it theorem A for Cartan manifolds. It follows that there exists $s_1^*,\dots,s_p^* \in \Gamma(X,\mathcal{E}^*)$ and $\alpha_1\dots\alpha_p \in \Gamma(U,\mathcal{O}_U(\mathbb{C}))$ for a suitable open neighbourhood U of x in X, such that $\ell_x(y)=\sum_{j=1}^{p}\alpha_j(y)s_j^*(y)$ for each $y\in U$

Then $s^*:X \longrightarrow E^*$ given by $s^*(y)=\sum_{j=1}^{p}\alpha_j(x)s_j^*(y)$ is a global cross section in E^* and it satisfies $s^*(x)=\ell_x(x)=L$. Due to lemma 3 there exists $s\in\Gamma(E,\mathcal{O}_E(\mathbb{C}))$ such that $s\big|_{\pi_E^{-1}(x)}=s^*(x)$

Hence:

$$s(e_1)=L(z_1)\neq L(z_2)=s(e_2)$$

b) There are local coordinates given by global sections Indeed, let $e\in E$ and $\pi_E(e)=x\in X$. Take a trivialization $(U_i,f_i)_{i\in I}$ for E. Consider a sufficiently small open neighbourhood U for x such that $U\subseteq U_{i_o}$ and there exists a system of local coordinates on U around x given by global sections in $\mathcal{O}_X(\mathbb{C})$ (remember X is a Cartan manifold).

Take now a frame $s_1^*\dots s_k^*$ for E^* on U (as above). By applying again theorem A we obtain global sections $h_1^*\dots h_k^*$ from $\Gamma(X,\mathcal{E}^*)$ such that (on an eventually restricted U) the following holds $h_i^*\big|_U=s_i^*\big|_U$ for each $i=1,\dots,k$.

By lemma 3 we obtain $h_i\in\Gamma(E,\mathcal{O}_E(\mathbb{C}))$ such that for each $y\in U$ $h_i\big|_{\pi_E^{-1}(y)}=h_i^*(y)$. It is not hard to prove now that

$$(f_1\circ\pi_E\dots f_{m+n}\circ\pi_E,\ h_1,\dots,h_k)$$

give local coordinates in $e\in E$.

c) Finally we prove that E is $\mathcal{O}_E(\mathbb{C})$-convex.

Let then $(e_n)_{n\in\mathbb{N}}$ be a discrete sequence in E; if $(\pi_E(e_n))_n=(x_n)_n$ is a discrete sequence in X, there exists $f\in\Gamma(X,\mathcal{O}_X(\mathbb{C}))$ with $\sup_n |f(x_n)| =+\infty$; then $f\circ\pi \in\Gamma(E,\mathcal{O}_E(\mathbb{C}))$ and $\sup |f\circ\pi(e_n)| =+\infty$.

Now if $x_n \longrightarrow x$, take (U_{i_o}, f_{i_o}) an element of a trivialization for E with $U_{i_o} \ni x$; for n sufficiently great it follows $x_n \in U_{i_o}$. If $f_{io}(e_n)=(x_n,z_n)$, z_n has to be a discrete sequence in $\mathbb{C}^k$. There exists then a linear functional $L:\mathbb{C}^k \longrightarrow \mathbb{C}$ such that

$$|L(z_n)| \xrightarrow{n\to+\infty} +\infty$$

By a similar reasoning to that in a) we find $s_1^*\dots s_p^*$ from $\Gamma(X,\mathcal{E}^*)$ and $\alpha_1\dots\alpha_p \in\Gamma(U,\mathcal{O}_U(\mathbb{C}))$ (for a sufficiently small $U \ni x$) such that:

$$L=\sum_{i=1}^{p}\alpha_i(y)s_i^*(y) \qquad \text{for each } y\in U .$$

Let now s_i be associated to s_i^* as in lemma 3; then

$$L(z_n)=\sum_{i=1}^{p}\alpha_i(x_n)s_i(e_n) .$$

As

$$\alpha(x_n)\xrightarrow{n\to\infty}\alpha(x) \text{ and } |L(z_n)| \xrightarrow{n\to+\infty} +\infty$$

it follows that at least one of the sections s_i is unbounded on $(e_n)_{n\in\mathbb{N}}$.

q.e.d.

These results have been announced on the 5th Romanian Finish Seminar on Complex Analysis. Some of these results will be extended to the case of mixed spaces and published elsewhere.

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