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INTRODUCTION

Let $i: X \hookrightarrow X^*$ be an open immersion of integral schemes over the field $\mathbb{C}$ of complex numbers, where X is a scheme of finite type over $\mathbb{C}$.

In §1 we show that for every closed point $x \in X^*$, the residue field $k(x)$ of x in X^* is $\mathbb{C}$. This fact permits to associate $\mathbb{C}$-valued functions on (Zariski) open subsets of X^*_{cl} (the subset of all closed points of X^*) to the sections of structure sheave $\mathcal{O}_{X^*}$. We call the <u>fine topology</u> of X^*, the weakest topology on X^*_{cl} containing all Zariski subsets of X^*_{cl} and making continuous all above $\mathbb{C}$-valued functions, where $\mathbb{C}$ is considered with the natural Haussdorf topology. The restriction of the fine topology of X^* to X is the usual fine topology of the complex algebraic prevariety X_{cl}.

The main result of this paper is the following: <u>if $x \in X^*$ is closed and $\mathcal{O}_{X^*,x}$ is noetherian, then x has a Zariski neighbourhood in X^* of finite type over $\mathbb{C}$, iff x has a compact neighbourhood in the fine topology of X^*_{cl}</u> (§3, Theorem).

In particular if X^* is noetherian, X^* <u>is of finite type over $\mathbb{C}$ if the fine topology of X^* is locally compact</u>. An equivalent form of this assertion is the following: <u>a noetherian subalgebra A of a $\mathbb{C}$-algebra of finite type is finitely generated iff the "Gel'fand topology" on the set of all maximal ideals of A is locally compact</u> (The Gel'fand topology on the set Spec max A of all maximal ideals of A is the weakest topology making continuous all functions $\tilde{f}$: Spec max $A \to \mathbb{C}$, $\tilde{f}(\underline{m}) =$ (residue of f in $\underline{m}$) $\in A/\underline{m} = \mathbb{C}$, where $f \in A$) (§3, Corollary 5).

The leading idea of the proof of the main result of the paper is the following: in [5], Prop.1, we have shown that the obstructions to the algebrization of a noetherian scheme X^* dominated by an algebraic variety consist in the existence of some schemes of finite type over X^*, of dimension ≥ 2 and with closed 1-codimensional points. Giving a "local form" to this fact, as in §3, Lemma 5, we can reduce the question of the existence of a Zariski open neighbourhood of finite type over $\mathbb{C}$ of a "noetherian" point $x \in X^*$ to the problem of proving that for an open embedding of a complex "algebraic variety" X in a complex scheme X^*, which has a closed 1-codimensional point $x \in X^*$, the fine topology of X^* around x is not locally compact, excepting the case when $\dim X^* = 1$. This last question is treated in §2: in virtue of Lemma 4 from §2, it suffices to consider the question for the case where X^* is normal in the closed 1-codimensional point $x \in X^*$ for which we can describe completely the local algebraic structure of X^* around x (see §2, Lemma 1); this description permits, via an étale morphism defined in a Zariski open neighbourhood of x in X^* and constructed in §2, Lemma 2, to reduce the question (§2, Lemma 3) to an elementary analysis of the fine topology of X^* around x in the case when X is a Zariski open subset of a complex affine space (§2, the last part of the proof of Proposition 2).

§1. THE FINE TOPOLOGY OF SOME COMPLEX SCHEMES

Firstly we shall establish an extension of a well-known form of Hilbert Nullstellensatz ever $\mathbb{C}$:

Proposition 1. Let A be a subalgebra of an algebra of finite type A' over the field $\mathbb{C}$ and $\underline{m} \subset A$ a maximal ideal. Then $A/_{\underline{m}} = \mathbb{C}$.

Proof. The $\mathbb{C}$ - vector space A' has a basis which is at most countable. Then A and $A/_{\underline{m}}$ have the same property.

Let us suppose, that there exists $x \in A/_{\underline{m}}$ which is transcendental over $\mathbb{C}$. Then the subfield $\mathbb{C}(x) \subseteq A/_{\underline{m}}$ is a $\mathbb{C}$-vector space having a basis which is countable. This implies that the set of all poles of all complex rational functions in one indeterminate is a subset of $\mathbb{C}$ which is at most countable (it is the set of all poles of all rational functions of a basis), which is not true.

Therefore $A/_{\underline{m}}$ is an algebraic extension of $\mathbb{C}$.

Q.E.D.

COROLLARY 1 (see [15]) - A subalgebra of an algebra of finite type over $\mathbb{C}$ is a Jacobson ring

PROOF. Let A be a subalgebra of A', where A' is of finite type over $\mathbb{C}$ and $\underline{p} \in \operatorname{Spec} A$. If $\underline{n}$ is the ideal of all nilpotent elements of A', we have $\underline{p} \supseteq \underline{n} \cap A$ and so $\underline{p}$ includes $\underline{q} \cap A$, where $\underline{q}$ is a minimal prime ideal of A'. Then $A/_{\underline{q} \cap A} \subseteq A'/_{\underline{q}}$ and it suffices to prove Corollary 1, when A is a domain.

If A is not a Jacobson ring, we can find $\underline{p} \in \operatorname{Spec} A$ and $x \in A$, $x \notin \underline{p}$, such that x is contained in all maximal ideals of A including $\underline{p}$.

Since the ring of fractions $A[\frac{1}{x}]$ is still a subalgebra of an algebra of finite type, we find a maximal ideal $\underline{m}' \subset A[\frac{1}{x}]$ including $\underline{p}\, A[\frac{1}{x}]$. If $\underline{m} = \underline{m}' \cap A$, we have $A/_{\underline{m}} \subseteq A[\frac{1}{x}]/_{\underline{m}'} = \mathbb{C}$ and so, $\underline{m}$ is a maximal ideal of A containing $\underline{p}$ and $x \notin \underline{m}$, which is a contradiction.

Q.E.D.

Remark 1. With the same proofs as for Proposition 1 and Corollary 1, one establishes that for any subalgebra A of an algebra of finite type over an uncountable field k, A is a Jacobson ring and for any maximal ideal $\underline{m} \subset A$, $A/_{\underline{m}}$ is an algebraic extension of k. This improves a result from [15]. □

COROLLARY 2. *Let $i: X \hookrightarrow X^*$ be an open dense immersion of schemes over $\mathbb{C}$, where X is of finite type over $\mathbb{C}$ and $x \in X^*$ a closed point. Then the residue field $k(x)$ of x is $\mathbb{C}$ and X^* is a Jacobson scheme.*

Indeed, if U is an affine neighbourhood of x in X^* and $V \subset U \cap X$ an affine subset, then $A = \Gamma(U, \mathcal{O}_{X^*_{red}}) \subseteq A' = \Gamma(V, \mathcal{O}_{X_{red}})$ and A' is finitely generated over $\mathbb{C}$. If $\underline{m} \subset A$ is the maximal ideal corresponding to $x \in U$, then $k(x) = A/_{\underline{m}} = \mathbb{C}$. $A = \Gamma(U, \mathcal{O}_{X^*})_{red}$ and $\Gamma(U, \mathcal{O}_{X^*})$ are Jacobson rings.

Q.E.D.

In the situation given in Corollary 2, since X^* is a Jacobson scheme, the map $U \rightsquigarrow U \cap X^*_{cl}$, where X^*_{cl} is the set of all closed points of X^*, establishes an one to one correspondence between the open (resp.closed), subsets of X^* and the open (resp.closed) subsets of X^*_{cl} (cf. [9], ch.IV, §10).

For any open subset $U \subseteq X^*$ and every $f \in \Gamma(U, \mathcal{O}_{X^*})$ we can associate the map $\tilde{f}: U \cap X^*_{cl} \longrightarrow \mathbb{C}$ given by $\tilde{f}(x) =$ (residue of f in x) $\in k(x) = \mathbb{C}$. Since X^* is Jacobson, $f \rightsquigarrow \tilde{f}$ is a ring homomorphism, whose kernel is the ideal of all nilpotents of $\Gamma(U, \mathcal{O}_{X^*})$.

Hence if X^* is reduced, $f \rightsquigarrow \tilde{f}$ is an injective map.

We can consider the weakest topology on the set X^*_{cl} which contains all Zariski open subsets of X^*_{cl} and making continuous all above maps $\tilde{f}$, where $\mathbb{C}$ is considered with the natural Hausdorff topology. We call this topology the fine topology of X^*.

We have the following elementary properties of the fine topology, which will be used in the following:

1) If $f: X \longrightarrow Y$ is a morphism of $\mathbb{C}$-schemes which are generically of finite type over $\mathbb{C}$, then f establishes a continuous map between X_{cl} and Y_{cl} with respect to the fine topologies.

2) If X^* is affine, then the fine topology on X^* is the weakest topology on X^*_{cl} making continuous all functions $\tilde{f}$ associated as above to a set of generators $\{f\}$ of the $\mathbb{C}$-algebra $\Gamma(X^*, \mathcal{O}_{X^*})$.

3) If X^* is affine, $x \in X^*_{cl}$ and U is a neighbourhood of x, then there exist $\varepsilon > 0$ and $f_1, \dots, f_n \in \Gamma(X^*, \mathcal{O}_{X^*})$ such that $U \supseteq \{ \xi \mid \xi \in X^*_{cl}, \ |f_i(\xi) - f_i(x)| < \varepsilon, \text{ for all } i, 1 \le i \le n \}$.

4) Every closed point of X^* has a fundamental system of closed neighbourhoods in X^*_{cl} with respect to the fine topology.

5) If Y^* is a subscheme of X^* such that $Y = Y^* \cap X \neq \phi$, then the fine topology of Y^* is the restriction to Y^* of the fine topology of X^*.

In particular, the restriction of the fine topology of X^* to X is the usual fine topology of the complex algebraic prevariety X_{cl}.

6) If $i_\alpha : X_\alpha \hookrightarrow X^*_\alpha$, $\alpha = 1, \dots, n$ are open dense immersions of $\mathbb{C}$-schemes and X_α are of finite type over $\mathbb{C}$, then $i_1 \times \dots \times i_n : X_1 \times \dots \times X_n \hookrightarrow X^*_1 \times \dots \times X^*_n$ is an open dense immersion, $(X^*_1 \times \dots \times X^*_n)_{cl} = (X^*_1)_{cl} \times \dots \times (X^*_n)_{cl}$ and the fine topology on $X^*_1 \times \dots \times X^*_n$ is the product of the fine topologies on X_α, $1 \le \alpha \le n$.

7) If X^* is a separated scheme, then the fine topology on X^* is Haussdorf.

§2. ON THE FINE TOPOLOGY OF COMPLEX SCHEMES WITH CLOSED 1-CODIMENSIONAL POINTS

The aim of this chapter is to prove an important particular case of the main result of this paper:

Proposition 2 - Let $i:X \hookrightarrow X^*$ be an open immersion of integral schemes over $\mathbb{C}$, where X is of finite type over $\mathbb{C}$, $X^* - X$ is a closed 1-codimensional point x of X^* and $\mathcal{O}_{X^*,x}$ is noetherian. Then the following assertions are equivalent:

i) X^*_{cl} is locally compact in the fine topology

ii) x has a compact neighbourhood in X^*_{cl} in the fine topology

iii) $\dim X^* = 1$

iv) X^* is of finite type over $\mathbb{C}$.

Remark 2. In Proposition 1, as well as in Lemma 1 and 3 which follow, the fact that $X^* - X$ is a closed 1-codimensional point x of X^* and $\mathcal{O}_{X^*,x}$ is noetherian, implies that X^* is noetherian (see [6], Lemma 3), but we shall not use this assertion in the paper. $\square$

To prove Proposition 2, we need some preparatory facts.

Let $i:X \hookrightarrow X^*$ be an open immersion of integral schemes over $\mathbb{C}$, where X is of finite type over $\mathbb{C}$, X^*-X is a closed 1-codimensional point x of X^* and $\mathcal{O}_{X^*,x}$ is a noetherian normal ring.

Then $\mathcal{O}_{X^*,x}$ is a discrete valuation ring. Let $t \in \mathcal{O}_{X^*,x}$ be a local parameter. By replacing X^* with a Zariski open neighbourhood of x, we may assume that X^* is affine, $t \in \Gamma(X^*,\mathcal{O}_{X^*})$ and t is invertible in $\Gamma(X,\mathcal{O}_X)$.

Then $X = \{\xi \in X^*$, such that $t(\xi)\neq 0\}$ and so X is affine. Hence the ring of fractions $\Gamma(X^*,\mathcal{O}_{X^*})[\frac{1}{t}] = \Gamma(X,\mathcal{O}_X)$. We may find $f_1,\dots,f_n \in \Gamma(X^*,\mathcal{O}_{X^*})$ such that $\Gamma(X,\mathcal{O}_X) = \mathbb{C}[\frac{1}{t},f_1,\dots,f_n]$. Indeed, $\Gamma(X^*,\mathcal{O}_{X^*})[\frac{1}{t}]$ is of the form $\mathbb{C}[\frac{f_1}{t^{\alpha_1}},\dots,\frac{f_n}{t^{\alpha_n}}]$ where

$f_1,\dots,f_n\in\Gamma(X^*,\mathcal{O}_{X^*})$ and then $\Gamma(X^*,\mathcal{O}_{X^*})[\frac{1}{t}]\supseteq\mathbb{C}[\frac{1}{t},f_1,\dots,f_n]\supseteq$ $\supseteq\mathbb{C}[\frac{f_1}{t^{\alpha_1}},\dots,\frac{f_n}{t^{\alpha_n}}]=\Gamma(X^*,\mathcal{O}_{X^*})[\frac{1}{t}]$.

We may assume that $f_1(x)=f_2(x)=\dots=f_n(x)=0$.

In fact, by Corollary 2, $k(x)=\mathbb{C}$, and we can replace f_i by $f_i-f_i(x)$ for every $1\le i\le n$.

Let us denote $A=\Gamma(X^*,\mathcal{O}_{X^*})$ and $\underline{m}\subset A$ the maximal ideal corresponding to the closed point $x\in X^*$. The completion in $\underline{m}$ - adic topology $\hat{A}_{\underline{m}}$ of $A_{\underline{m}}=\mathcal{O}_{X^*,x}$ is $\mathbb{C}$-isomorphic with the ring of formal power series $\mathbb{C}[[T]]$, since $\hat{A}_{\underline{m}}$ is a discrete valuation ring and $k(x)=A/_{\underline{m}}=\mathbb{C}$. We have a natural inclusion of $\mathbb{C}$-algebras $A\subseteq\mathbb{C}[[T]]=\hat{A}_{\underline{m}}$, and so all elements of A have expansions in power series in $\mathbb{C}[[T]]$. Replacing T by the power series corresponding to t, we may suppose that in $\mathbb{C}[[T]]$ we have $t=T$. If $f\in\underline{m}$ with $f=\sum_{k=1}^{\infty}c_kT^k$ in $\mathbb{C}[[T]]$, then $(\mathrm{Rest}\,f)_\ell=\sum_{k=\ell+1}^{\infty}c_kT^{k-\ell}$ is equal with $\dfrac{f-c_1t-\dots-c_\ell t^\ell}{t^\ell}$ in $C((T))$ and so $(\mathrm{Rest}\,f)_\ell\in\mathbb{C}[[T]]\cap Q(A)=\hat{A}_{\underline{m}}\cap Q(A_{\underline{m}})$ where by $Q(A)$ (resp. $Q(A_{\underline{m}})$) we denoted the field of fractions of A (resp. $A_{\underline{m}}$). By [3], Ch. III, §3.5, Cor. 1, we have $\hat{A}_{\underline{m}}\cap Q(A_{\underline{m}})=A_{\underline{m}}$ and so $(\mathrm{Rest}\,f)_\ell\in$ $\in A_{\underline{m}}\cap A[\frac{1}{t}]=A$ for all $l\ge0$. Since $(\mathrm{Rest}\,f)_\ell=((\mathrm{Rest})_{\ell+1}+c_{\ell+1})t$, it follows $(\mathrm{Rest}\,f)_\ell\in\underline{m}$ for all $l\ge0$.

Let us denote $B=\mathbb{C}[t,f_1,\dots,f_n,\dots,(\mathrm{Rest}\,f_k)_\ell,\dots]_{1\le i,k\le n,\ \ell\ge1}$ the subalgebra of A generated by $\{t,f_i,(\mathrm{Rest}\,f_k)_\ell\}_{1\le i,k\le n,\ \ell\ge1}$ and $\underline{n}=(t)$ the ideal generated in B by t. We have $B[\frac{1}{t}]=\mathbb{C}[\frac{1}{t},t,f_1,\dots,f_n]=A[\frac{1}{t}]$ (since $B[\frac{1}{t}]\supseteq\mathbb{C}[\frac{1}{t},f_1,\dots,f_n]=A[\frac{1}{t}]$) and $t,f_i,(\mathrm{Rest}\,f_k)_\ell\in\underline{n}$ for all $1\le i,k\le n$, and $l\ge1$. Hence $B/_{\underline{n}}=\mathbb{C}$, and $\underline{n}=\underline{m}\cap B$. It follows that $B_{\underline{n}}\subseteq A_{\underline{m}}$ is an inclusion of local rings, $\bigcap_{k=1}^{\infty}\underline{n}^kB_{\underline{n}}\subseteq\bigcap_{k=1}^{\infty}\underline{m}^kA_{\underline{m}}=0$ and so $B_{\underline{n}}$ is a discrete valuation ring. It is easy to see that $\hat{A}_{\underline{m}}=\hat{B}_{\underline{n}}$ and so in the field $Q(\hat{A}_{\underline{m}})=Q(\hat{B}_{\underline{n}})$ we have $A_{\underline{m}}=Q(A)\cap\hat{A}_{\underline{m}}=Q(B)\cap\hat{B}_{\underline{n}}=B_{\underline{n}}$ (cf. [3], Ch. III, §3, Cor. 4). Then in the field $Q(A)=Q(B)$ we have $A=A[\frac{1}{t}]\cap A_{\underline{m}}=B[\frac{1}{t}]\cap B_{\underline{n}}=B$.

Therefore we established the following

LEMMA 1. Let $i: X \hookrightarrow X^*$ be an open immersion of integral schemes over $\mathbb{C}$, where X is of finite type over $\mathbb{C}$ and $X^* - X$ is a closed 1-codimensional point x of X^* with $\mathcal{O}_{X^*,x}$ normal and noetherian. Then there exists a Zariski open affine neighbourhood U of x in X^*, such that $\Gamma(U, \mathcal{O}_{X^*}) = \mathbb{C}[T, f_1, \ldots, f_n, \ldots, (\operatorname{Rest} f_k)_\ell, \ldots]$ where $f_k = \sum_{i=1}^{\infty} c_i^{(k)} T^i$ is a formal power series in $\mathbb{C}[[T]]$, $(\operatorname{Rest} f_k)_\ell = \sum_{i=\ell+1}^{\infty} c_i^{(k)} T^{i-\ell}$ for all $1 \le k \le n$ and $\ell \ge 1$, and $\mathbb{C}[T, f_1, \ldots, f_n, \ldots, (\operatorname{Rest} f_k)_\ell, \ldots]$ is the $\mathbb{C}$-subalgebra of $\mathbb{C}[[T]]$ generated by $\{T, \ldots, f_k, \ldots, (\operatorname{Rest} f_k)_\ell, \ldots\}_{1 \le k \le n, \ell \ge 1}$

Remark 3 . Conversely, in a ring of the type $A = \mathbb{C}[T, f_1, \ldots, f_n, \ldots (\operatorname{Rest} f_k)_\ell \ldots]$ where f_i, $(\operatorname{Rest} f_k)_\ell$ are as in Lemma 1, the ideal $\underline{m} = (T)$ is maximal, $A_{\underline{m}}$ is dominated by $\mathbb{C}[[T]]$ and so $A_{\underline{m}}$ is a discrete valuation ring and $A[\frac{1}{T}] = \mathbb{C}[T, \frac{1}{T}, f_1, \ldots, f_n]$ is finitely generated over $\mathbb{C}$. Then the open immersion $i: X = \operatorname{Spec} A[\frac{1}{T}] \hookrightarrow X^* = \operatorname{Spec} A$ have all properties given in the hypothesis of Lemma 1.

We may point out that the proof of Lemma 1 shows that we can replace $\mathbb{C}$ in Lemma 1 by an algebraically closed uncountable field.□

In the situation and with the notations given in Lemma 1, if $A = \mathbb{C}[T, f_1, \ldots, f_n, \ldots, (\operatorname{Rest} f_k)_\ell, \ldots]_{1 \le k \le n, \ell \ge 1}$, the field of fractions $Q(A)$ is generated by $T, f_1, \ldots, f_n$. Let us suppose that

$T, f_1, \ldots, f_m$ are algebraically independent over $\mathbb{C}$ and

$f_{m+1}, \ldots, f_n$ are algebraic over the field $\mathbb{C}(T, f_1, \ldots, f_m)$

Let us denote $B = \mathbb{C}[T, f_1, \ldots, f_m, \ldots, (\operatorname{Rest} f_k)_\ell, \ldots]_{1 \le k \le m, \ell \ge 1}$ the subalgebra of A generated by $f_1, \ldots, f_m, \ldots, (\operatorname{Rest} f_k)_\ell, \ldots$, for $1 \le k \le m$, $\ell \ge 1$.

For $1 \le k \le m$, f_k satisfies an equation of the type:

$$T^{q_k} P_{k,p_k} f_k^{p_k} + \ldots + P_{k,1} f_k + P_{k,0} = 0$$

where $q_k \ge 0$, $p_k > 0$, $P_{k,i}$ are polynomials in $T, f_1, \ldots, f_m$ and $P_{k,p_k} \notin \underline{n} = TB$. If $P = P_{1,p_1} P_{2,p_2} \cdots P_{m,p_m}$ then $A[\frac{1}{T}, \frac{1}{P}] = \mathbb{C}[\frac{1}{P}, \frac{1}{T}, T, f_1, \ldots, f_n]$ is finite over $B[\frac{1}{T}, \frac{1}{P}] = \mathbb{C}[\frac{1}{P}, \frac{1}{T}, T, f_1, \ldots, f_m]$

We can change $P \in B$, such that $P \notin \underline{n}$ and $A[\frac{1}{T},\frac{1}{P}]$ is normal and finite over $B[\frac{1}{T},\frac{1}{P}]$. Indeed, with P as before, there exists $Q \in A[\frac{1}{T},\frac{1}{P}]$ such that $A[\frac{1}{T},\frac{1}{P},\frac{1}{Q}]$ is normal. If Q satisfies the equation $Q^q + P_{q-1}Q^{q-1} + \dots + P_0 = 0$ with $P_0,\dots,P_{q-1} \in B[\frac{1}{T},\frac{1}{P}]$ and $P_0 \neq 0$, then $A[\frac{1}{T},\frac{1}{P},\frac{1}{P_0}]$ is finite over $B[\frac{1}{T},\frac{1}{P},\frac{1}{P_0}]$ and $A[\frac{1}{T},\frac{1}{P},\frac{1}{P_0}] \supseteq A[\frac{1}{T},\frac{1}{P},\frac{1}{Q}]$ since Q is invertible in $A[\frac{1}{T},\frac{1}{P},\frac{1}{P_0}]$. Hence $A[\frac{1}{T},\frac{1}{P},\frac{1}{P_0}]$, being a ring of fractions of $A[\frac{1}{T},\frac{1}{P},\frac{1}{Q}]$, is normal and it is finite over $B[\frac{1}{T},\frac{1}{P},\frac{1}{P_0}]$. But P_0 can be written as $T^\alpha P^\beta P_0'$, where $\alpha,\beta \in \mathbb{Z}$ and $P_0' \in B\setminus\underline{n}$. Then $A[\frac{1}{T},\frac{1}{PP_0'}]$ is normal and finite over $B[\frac{1}{T},\frac{1}{PP_0'}]$ and we can replace P by PP_0'.

Let us denote by B' the integral closure of $B[\frac{1}{P}]$ in Q(A). The integral closure of $B_{\underline{n}} = B[\frac{1}{P}]_{\underline{n}B[\frac{1}{P}]}$ in Q(A) is the ring offractions B'_S of B' with respect to $S = B\setminus\underline{n}$. $B_{\underline{n}}$ being a discrete valuation ring, it follows that B'_S is a free B -module of finite type (see [3], Ch.VI, §8, Cor.1). Let $\{\frac{\alpha_1}{s},\dots,\frac{\alpha_p}{s}\}$ be a basis of the $B_{\underline{n}}$-module B'_S, with $\alpha_1,\dots,\alpha_p \in B'$ and $s \in S$. The integral closure of $B[\frac{1}{T},\frac{1}{P}]$ in Q(A) is $B'[\frac{1}{T}]$ and it is a $B[\frac{1}{T},\frac{1}{P}]$ - module of finite type. Let $\{\frac{\alpha_{p+1}}{T^{n_1}},\dots,\frac{\alpha_q}{T^{n_q}}\}$ be a set of generators of this module with $\alpha_{p+1},\dots,\alpha_q \in B'$. Then $\alpha_1,\dots,\alpha_p,\alpha_{p+1},\dots,\alpha_q$ generate the $B[\frac{1}{P}]$ - module B'. Indeed, if $b' \in B'$ then $b' = \sum_{i=1}^{p} \frac{a_i}{s_i}\frac{\alpha_i}{s} = \sum_{i=1}^{q-p} \frac{a_i'}{T^{m_i}}\frac{\alpha_{p+i}}{T^{n_i}}$ with a_i, $a_i' \in B[\frac{1}{P}]$ and $s_i \in S$; therefore we have $s'b' = \sum_r \alpha_i' \alpha_i$ and $T^r b' = \sum \beta_i' \beta_i$ with $s' \in S$ and $\alpha_i', \beta_i' \in B[\frac{1}{P}]$; since the ideal generated in $B[\frac{1}{P}]$ by s' and T^r is $B[\frac{1}{P}]$, it follows that b' can be written as a linear combination of α_i with coefficients in $B[\frac{1}{P}]$.

Therefore B' is a finite $B[\frac{1}{P}]$- module.

Since $A[\frac{1}{T},\frac{1}{P}]$ is normal and finite over $B[\frac{1}{P},\frac{1}{T}]$ it follows that $B'[\frac{1}{T}] = A[\frac{1}{T},\frac{1}{P}]$.

Let us denote $\underline{m}=TA$. $A_{\underline{m}}$ is a discrete valuation ring dominating $B_{\underline{n}}$. Hence $A_{\underline{m}} \supseteq B'_S$ and if $\underline{n}' = \underline{m}A_{\underline{m}} \cap B'$, then $A_{\underline{m}} = (B')_{\underline{n}'}$, since $(B')_{\underline{n}'}$ is a valuation ring. Clearly $\underline{m}A[\frac{1}{P}] \cap B' = \underline{n}'$

Thus in the field Q(A) we have the following inclusion of rings: $B' \subset B'[\frac{1}{T}] \cap B'_{\underline{n}'} = A[\frac{1}{T}, \frac{1}{P}] \cap A_{\underline{m}} = A[\frac{1}{P}]$.

Let $V = \mathrm{Spec}\, A[\frac{1}{P}]$, $W = \mathrm{Spec}\, B[\frac{1}{P}]$, $W' = \mathrm{Spec}\, B'$ and $j : V \longrightarrow W'$ the morphism of schemes associated to the inclusion $B' \subset A[\frac{1}{P}]$. Then j is an open immersion. Indeed, $T \in \Gamma(W', \mathcal{O}_{W'}) = B'$ vanishes only in the points of W' lying over $\underline{n}B[\frac{1}{P}] \in W$; hence the zerous of T on W' are (finitely many) closed 1-codimensional points of W'. We have that $j|_{V_T} : V_T \longrightarrow W'_T$ (V_T and W'_T are the sets of all non-zerous of T) is an isomophism, because $B'[\frac{1}{T}] = A[\frac{1}{T}, \frac{1}{P}]$. If $x \in V$ is the point corresponding to $\underline{m}\, A[\frac{1}{P}]$ then j(x) is the point corresponding to $\underline{n}'$ and so $\mathcal{O}_{W', j(x)} = \mathcal{O}_{V, x}$. Hence $j(V) = j(V_T) \cup \{j(x)\} = W'_T \cup \{j(x)\}$ is an open subset of W' and in Q(A) we have $\Gamma(j(V), \mathcal{O}_{W'}) = \Gamma(j(V_T), \mathcal{O}_{W'}) \cap \mathcal{O}_{W', j(x)} = \Gamma(V_T, \mathcal{O}_V) \cap \mathcal{O}_{V,x} = \Gamma(V, \mathcal{O}_V)$. Since j(V) is affine (cf. [7], Lemma 2), it follows that j is an open immersion. In the following we shall identify V with j(V) and x with j(x). If $p : W' \longrightarrow W$ is the morphism of schemes induced by the inclusion $B[\frac{1}{P}] \subseteq B'$, p is unramified in x. We shall denote y=p(x).

It follows that V is a scheme of finite type over W. Therefore $A[\frac{1}{P}]$ is a $B[\frac{1}{P}]$-algebra of finite type. Since $A[\frac{1}{T}]$ is also a $B[\frac{1}{T}]$- algebra of finite type and P and T generate in B the ideal B, it is easy to see that A is a B-algebra of finite type.

Since $Q(A) = \mathbb{C}(T, f_1, \ldots, f_n)$ and $\{T, f_1, \ldots, f_m\}$ is a transcendental basis of Q(A) over $\mathbb{C}$, we can find $g \in \mathbb{C}(T, f_1, \ldots, f_m)$ such that $Q(A) = \mathbb{C}(T, f_1, \ldots, f_m, g)$. In Q(A) we can write $g = \frac{g'}{R}$, where $g', R \in \underline{m} \subset A$. Let $R^p + r_1 R^{p-1} + \ldots + r_p = 0$ be an equation satisfied by $R \in A[\frac{1}{P}]$ with $r_i \in B[\frac{1}{P}]$ and $r_p \neq 0$. If we write $r_p = \frac{r'}{P^{p_1}} T^{p_2}$ in $B[\frac{1}{P}]$, where $r' \in B \setminus \underline{n}$ and $p_1, p_2 \geq 0$, then in $A[\frac{1}{Pr'}, \frac{1}{T}] = A[\frac{1}{P}, \frac{1}{r'}, \frac{1}{T}]$, r_p and R are invertible.

By changing P with Pr', we may suppose that R is invertible in $A[\frac{1}{T}, \frac{1}{P}]$.

Then $g = \frac{f}{(TP)^l}$ in $A[\frac{1}{T},\frac{1}{P}] = A[\frac{1}{TP}]$, where $f \in \underline{m} \subset A$.

Let us denote $A' = \mathbb{C}[T, f_1, ---, f_m, f, ---, (\mathrm{Rest}\, f_k)_l, ---, (\mathrm{Rest}\, f)_l, ---]_{1\le k\le m, l\ge 1}$ $\subset \mathbb{C}[[T]]$. Since in $C((T))$ $(\mathrm{Rest}\, f)_l \in Q(A)\cap \mathbb{C}[[T]] = Q(A_{\underline{m}})\cap \hat{A}_{\underline{m}} = A_{\underline{m}}$ and $(\mathrm{Rest}\, f)_k \in A[\frac{1}{T}]$, it follows that for all $l \geq 1$, $(\mathrm{Rest}\, f)_l \in A_{\underline{m}} \cap A[\frac{1}{T}] = A$. Therefore $A \supseteq A' \supseteq B$. Since $Q(A) = \mathbb{C}(T, f_1, ---, f_m, g)$ and $A[\frac{1}{T},\frac{1}{P}] \supseteq A'[\frac{1}{T},\frac{1}{P}] = \mathbb{C}[\frac{1}{T},\frac{1}{P}, T, f_1, ---, f_m, f] \supseteq \mathbb{C}[\frac{1}{T}, T, P, f_1, ---, f_m, g]$, the rings $A[\frac{1}{T},\frac{1}{P}]$ and $A'[\frac{1}{T},\frac{1}{P}]$ have the same field of fractions.

Thus we can find $Q \in A'$, $Q \neq 0$, such that $A[\frac{1}{T},\frac{1}{P},\frac{1}{Q}] = A'[\frac{1}{T},\frac{1}{P},\frac{1}{Q}]$ If $Q^r + \delta_1 Q^{r-1} + \dots + \delta_r = 0$ is an equation of $Q \in A'[\frac{1}{P}]$ over $B[\frac{1}{P}]$ with $\delta_r \neq 0$ and if in $B[\frac{1}{P}]$ we have $\delta_r = \frac{\delta'}{P^{r_1}} T^{r_2}$ with $\delta' \in B \setminus \underline{n}$ and $r_1, r_2 \geq 0$, then in $A'[\frac{1}{P\delta'}, \frac{1}{T}] = A'[\frac{1}{P},\frac{1}{\delta'},\frac{1}{T}]$, δ_r and Q are invertible.

By changing P with $P\delta'$ we may suppose that Q is invertible in $A'[\frac{1}{T},\frac{1}{P}]$. Then $A[\frac{1}{T},\frac{1}{P}] = A'[\frac{1}{T},\frac{1}{P}]$.

Let us denote $\underline{m}' = TA'$, the ideal generated by T in A'. It results $\hat{A}_{\underline{m}} = \hat{A}'_{\underline{m}'} = \mathbb{C}[[T]]$, and, next, $A_{\underline{m}} = A'_{\underline{m}'}$. Then in $Q(A) = Q(A')$ we have $A[\frac{1}{P}] = A[\frac{1}{T},\frac{1}{P}] \cap A_{\underline{m}} = A'[\frac{1}{T},\frac{1}{P}] \cap A'_{\underline{m}'} = A'[\frac{1}{P}]$.

The ring $A[\frac{1}{P}] = A'[\frac{1}{P}]$ is a finitely generated $B[\frac{1}{P}]$ - algebra. There are finitely many elements from $\{f, ---, (\mathrm{Rest}\, f)_l, ---\}_{l\ge 1}$ generating $A[\frac{1}{P}]$ over $B[\frac{1}{P}]$. Since in A, for all $l \geq 0$ we have the relations $(\mathrm{Rest}\, f)_l = [(\mathrm{Rest}\, f)_{l+1} + \alpha_{l+1}]T$ with $\alpha_{l+1} \in \mathbb{C}$ and $(\mathrm{Rest}\, f)_0 = f$, it follows that $A[\frac{1}{P}]$ has a generator of the form $(\mathrm{Rest}\, f)_l \in A$ over $B[\frac{1}{P}]$, for l sufficiently big.

Therefore we proved the following

LEMMA 2. <u>Let</u> $f_1, ---, f_m, f_{m+1}, ---, f_n \in \mathbb{C}[[T]]$. <u>Suppose that</u> $T, f_1, ---, f_m$ <u>are algebraically independent over</u> $\mathbb{C}$ <u>and</u> $f_{m+1}, ---, f_n$ <u>are algebraic over the field</u> $\mathbb{C}(T, f_1, ---, f_m)$. <u>Denote</u> $A = \mathbb{C}[T, f_1, ---, f_n, ---, (\mathrm{Rest}\, f_k)_l, ---]_{1\le k\le n, l\ge 1}$ <u>and</u>

$B = \mathbb{C}[T, f_1, \ldots, f_m, \ldots, (\mathrm{Rest}\, f_k)_\ell \ldots]_{1 \le k \le m, \ell \ge 1}$ the $\mathbb{C}$-subalgebras of $\mathbb{C}[[T]]$ generated by the indicated elements. Then A is finitely generated over B and there exists $P \in B$, $P \notin TB$, such that $A[\frac{1}{P}]$ is generated as $B[\frac{1}{P}]$-algebra by an element of A. The morphism $V = \mathrm{Spec}\, A[\frac{1}{P}] \xrightarrow{\pi} W = \mathrm{Spec}\, B[\frac{1}{P}]$ is unramified at the point x corresponding to the maximal ideal $TA[\frac{1}{P}]$.

Remark 4. As we have shown in the proof, P can be choosen such that π is quasifinite (more precisely, such that $V - \{x\} \longrightarrow W - \{\pi(x)\}$ is finite). Moreover, in Lemma 2 we may choose P such that π is étale, but we shall not use this fact in the following. □

Concerning the above morphism $V \xrightarrow{\pi} W$ we may prove

LEMMA 3. In the situation and with notations given in Lemma 2, the morphism $V \xrightarrow{\pi} W$ induces a homeomorphism with respect to the fine topologies between a fine open neighbourhood of x in V and a fine open neighbourhood of $\pi(x)$ in W.

Proof. - Firstly, we shall show that the ring B in Lemma 2 is factorial. Indeed, $B[\frac{1}{T}] = \mathbb{C}[\frac{1}{T}, T, f_1, \ldots, f_m]$ is a ring of fractions of the polynomial ring $\mathbb{C}[T, f_1, \ldots, f_m]$ and so it is factorial. Let $\underline{p} \subset B$ be a prime ideal of height one. If $\underline{p} \neq \underline{n} = TB$, then $\underline{p}B[\frac{1}{T}]$ is a prime ideal of height one in $B[\frac{1}{T}]$ and so there exists $P_1 \in B$ such that $\underline{p}B[\frac{1}{T}]$ is generated by P_1. We may suppose that P_1 is not divisible by T in $\mathbb{C}[[T]]$. In fact, if $P_1 = T^\alpha R$, where $\alpha \geq 0$ and $R \in \mathbb{C}[[T]]$ is not divisible by T, then $R = \frac{P_1}{T^\alpha} \in Q(B) \cap \mathbb{C}[[T]] = Q(B_{\underline{n}}) \cap \hat{B}_{\underline{n}} = B_{\underline{n}}$ and so $R \in B[\frac{1}{T}] \cap B_{\underline{n}} = B$. Then R is a generator of $\underline{p}B[\frac{1}{T}]$. Hence suppose that P_1 not divisible by T in $\mathbb{C}[[T]]$. If $P_2 \in \underline{p}$ then there exist $P_3 \in B$ and $\beta \geq 0$ such that $T^\beta P_2 = P_3 P$. In $\mathbb{C}[[T]]$, T^β divides P_3 and so $\frac{P_3}{T^\beta} \in \mathbb{C}[[T]] \cap Q(B) = \hat{B}_{\underline{n}} \cap Q(B_{\underline{n}}) = B_{\underline{n}}$;

then $\frac{P_3}{T^\beta} \in B_{\underline{n}} \cap B[\frac{1}{T}] = B$. Therefore $\underline{p}$ is generated by $P_{\underline{f}}$.

By Lemma 2, $\Gamma(V,\mathcal{O}_V)$ has a generator f as $\Gamma(W,\mathcal{O}_W)$ - algebra and $\Gamma(W,\mathcal{O}_W)$ is factorial, as a ring of fractions of B. The kernel $\underline{p}$ of the canonical homomorphism $h: \Gamma(W,\mathcal{O}_W)[X] \longrightarrow \Gamma(V,\mathcal{O}_V)$, defined by $h(X)=f$, is a prime ideal of height 1, because $\dim \Gamma(W,\mathcal{O}_W) = \dim \Gamma(V,\mathcal{O}_V)$. Since $\Gamma(W,\mathcal{O}_W)[X]$ is factorial, $\underline{p}$ is generated by a polynomial $R \in \Gamma(W,\mathcal{O}_W)[X]$ and so $\Gamma(V,\mathcal{O}_V) = \Gamma(W,\mathcal{O}_W)/(R)$. Therefore $\pi : V \longrightarrow W$ is the morphism associated to a homomorphism of the type $\Gamma(W,\mathcal{O}_W) \longrightarrow \Gamma(W,\mathcal{O}_W)[X]/(R)$. Since π is unramified at x then the $\mathbb{C}$-vector space $\Gamma(W,\mathcal{O}_W)[X]/(R) \otimes_{\Gamma(W,\mathcal{O}_W)} \Gamma(W,\mathcal{O}_W)/\underline{n}\,\Gamma(W,\mathcal{O}_W)$ is of dimension one. If $y=\pi(x)$ and $R(X) = a_k X^k + \dots + a_1 X + a_0$ where $a_0,\dots,a_k \in \Gamma(W,\mathcal{O}_W)$, this means that the equation in X $a_k(y)X^k + \dots + a_1(y)X + a_0(y) = 0$ has $f(x) \in \mathbb{C}$ as unique solution and with multiplicity one (where $a_1,\dots,a_k$ are considered as functions on W_{cl}).

Using the implicit function Theorem or the formula of residues for analytic functions, be an find $\varepsilon, \delta > 0$ and a complex continuous function ξ of $k+1$ variables, defined in an open neighbourhood in $\mathbb{C}^{k+1}$ of the point $(a_0(y),\dots,a_k(y)) \in \mathbb{C}^{k+1}$, such that for $w \in W$, with $|a_i(w) - a_i(y)| < \delta$ for all $0 \le i \le k$, we have $|\xi(a_0(w),\dots,a_k(w)) - f(x)| < \varepsilon$ and $a_k(w)\xi^k(a_0(w),\dots,a_k(w)) + \dots$
$\dots + a_1(w)\xi(a_0(w),\dots,a_k(w)) + a_0(w) = 0$

In particular, $\xi(a_0(y),\dots,a_k(y)) = f(x)$. Via the above homomorphism h, V is the closed subscheme of the scheme W x Spec $\mathbb{C}[X]$ defined by the equation $R(X)=0$. Then $V_{cl} - \{x\}$ is the closed algebraic subvariety of the variety $(W_{cl} - \{y\}) \times \mathbb{C}$, defined also by $R(X)=0$, because x is the unique point of V lying over y, and so $\pi|_{V_{cl}-\{x\}} : V_{cl} - \{x\} \longrightarrow W_{cl} - \{y\}$ is the projection on $W_{cl} - \{y\}$. Of course, the natural inclusion of sets $V_{cl} \subseteq W_{cl} \times \mathbb{C}$ is defined by $v \mapsto (\pi(v), f(v))$.

Let us denote $W_\delta = \{w \in W_{ce} \mid |a_i(w) - a_i(y)| < \delta$, for all $0 \le i \le k$ and $V_{\delta,\varepsilon} = \{v \in V_{ce} \mid |a_i(\pi(v)) - a_i(y)| < \delta, |f(v) - f(x)| < \varepsilon$, for all $0 \le i \le k\}$. Then W_δ and $V_{\delta,\varepsilon}$ are fine open neighbourhoods of y and x and $\pi(V_{\delta,\varepsilon}) \subseteq W_\delta$. we shall define the following map $\rho: W_\delta \longrightarrow V_{\delta,\varepsilon}$ in the following manner: if $w \in W_\delta - \{y\}$, then $\rho(w)$ is the point of $V_{ce} - \{x\} \subseteq W_{ce} \times \mathbb{C}$ corresponding to $(w, \xi(a_0(w), \dots, a_k(w)) \in W_{ce} \times \mathbb{C}$ (this point satisfies the equation $R(X)=0$) and $\rho(y)=x$. It is clear that $\pi\rho = 1_{W_\delta}$ and we claim that ρ is continuous. In fact, it suffices to prove that all functions corresponding to a set of generators of the $\mathbb{C}$-algebra $\Gamma(V, \mathcal{O}_V)$ composed with ρ are continuous $\mathbb{C}$ - valued functions on W_δ . Since f is a generater of the $\Gamma(W, \mathcal{O}_W)$--algebra $\Gamma(V, \mathcal{O}_V)$ and all functions corresponding to $\Gamma(W, \mathcal{O}_W) \subseteq \Gamma(V, \mathcal{O}_V)$ are continuous, we must prove that $f\rho$ is continuous. It is clear because $f\rho(w) = \xi(a_0(w), \dots, a_k(w))$ for $w \in D_\delta - \{y\}$, $(f\rho)(y) = f(x) = \xi(a_0(y), \dots, a_k(y))$ and $\xi, a_0, \dots, a_k$ are continuous.

If we take above $\varepsilon, \delta > 0$ with the supplementary property that for any $w \in W$, such that $|a_i(w) - a_i(y)| < \delta$, there exists a unique solution ξ of the equation $a_k(w)\xi^k + \dots + a_1(w)\xi + a_0(w) = 0$, with $|\xi - f(x)| < \varepsilon$, then we have $\rho\pi = 1_{V_{\delta,\varepsilon}}$

Q.E.D.

Remark 5 - Using the structure Theorem for étale morphisms, one can prove that an étale morphism of finite type of noetherian $\mathbb{C}$-schemes, generically of finite type over $\mathbb{C}$, is locally homeomorphism with respect to the fine topologies. □

The last preparatory fact for the proof of proposition 2 is the following:

LEMMA 4. Let $i: X \hookrightarrow X^*$ be an open immersion of integral schemes over $\mathbb{C}$, where X is of finite type over $\mathbb{C}$ and $X^* - X$ is a closed 1-codimensional point x of X^*, such that $\mathcal{O}_{X^*,x}$ is noetherian. Then the normalization morphism $p: X^{*N} \longrightarrow X^*$ is a proper continuous map between X^{*N}_{cl} and X^*_{cl} with respect to the fine topologies, if X^*_{cl} is locally compact.

Proof. If suffices to prove that for a Zariski open affine neighbourhood U of $x \in X^*$, $p|_{p^{-1}(U)} : p^{-1}(U) = U^N \longrightarrow U$ is a proper map between U^N_{cl} and U_{cl} with respect to the fine topologies (cf. [4], Ch.I, §10, Prop.3), since $X^{*N}_{cl} - p^{-1}(x) \longrightarrow X^*_{cl} - \{x\}$ is proper.

Hence we may assume that X^* is affine and let us denote $A = \Gamma(X^*, \mathcal{O}_{X^*})$, $A^N = \Gamma(X^{*N}, \mathcal{O}_{X^{*N}})$ and $X^N = p^{-1}(X)$.

The fibers of p are finite since $\mathcal{O}_{X^*,x}$ is noetherian (of dimension one , cf. [3], Ch.VII, §2, Cor .1). It suffices to prove that if $Z \subseteq X^{*N}_{cl}$ is a closed subset in the fine topology, then $p(Z) \subseteq X^*_{cl}$ is closed in the fine topology (cf. [4], Ch.I, §10,2,Th.1).

If $Z \cap p^{-1}(x) \neq \phi$, then $x \in p(Z)$ and $p(Z) - \{x\} = p(Z - p^{-1}(x))$ is closed in X_{cl}, since $Z - p^{-1}(x)$ is closed in X^N_{cl} and $p|_{X^N_{cl}} : X^N_{cl} \longrightarrow X_{cl}$ is closed with respect to the fine topologies, as finite morphism of complex algebraic varieties. Then p(Z) is closed in $X^*_{cl} = X_{cl} \cup \{x\}$ in the fine topology.

Suppose that $Z \cap p^{-1}(x) = \phi$. Let $x_1, \ldots, x_n$ be the points of X^{*N} lying over x. For every i, $1 \leq i \leq n$, there exists a finite subset $\{f_{i1}, \ldots, f_{in_i}\}$ of $\Gamma(X^{*N}, \mathcal{O}_{X^{*N}}) = A^N$ and $\varepsilon_i > 0$ such that $|f_{i1}(x_i)| < \varepsilon_i, \ldots, |f_{in_i}(x_i)| < \varepsilon_i$ and for every $z \in Z$, there exists j, $1 \leq j \leq n_i$ with $|f_{ij}(z)| \geq \varepsilon_i$ (where f_{ij} are considered as functions on X^{*N}_{cl}) , because $x_i \notin Z$. Replacing f_{ij} by $\frac{1}{\varepsilon_i} f_{ij}$ we may assume that $\varepsilon_i = 1$ for all i, $1 \leq i \leq n$.

Let $A' = A[\ldots, f_{ij}, \ldots]_{1 \leq i \leq n,\, 1 \leq j \leq n_i}$ be the A-subalgebra of A^N generated by $\{f_{ij}\}_{1 \leq i \leq n,\, 1 \leq j \leq n_i}$. The inclusion

$A^N \supseteq A' \supseteq A$ give a factorization of p, $X^{*N} \xrightarrow{q} X'^* = \text{Spec } A' \xrightarrow{p'} X^*$, with q surjective and p' finite.

Then q(Z) is closed in $X'^*_{\mathbb{C}}$. Indeed, for all i,j, $1 \leq i \leq n$, $1 \leq j \leq n_i$, f_{ij} are functions on $X'^*_{\mathbb{C}}$ and $|f_{ij}(z')| < 1$, for all i,j, and all $z' \in q(p^{-1}(x))$; for any $1 \leq i \leq n$ and $z' \in q(Z)$, there exists j, $1 \leq j \leq n_i$, such that $|f_{ij}(z')| \geq 1$. Therefore $q(p^{-1}(x)) = p'^{-1}(x)$ does not meet the closure of q(Z) in $X'^*_{\mathbb{C}}$ in the fine topology. We have $q(Z) \subset q(X^N) = p'^{-1}(X) = X'$ and q(Z) is closed in $X'_{\mathbb{C}}$ in the fine topology, because $q|_{X^N_{\mathbb{C}}} : X^N_{\mathbb{C}} \longrightarrow X'_{\mathbb{C}}$ is a finite morphism of complex algebraic varieties. From these two properties of q(Z), it results that q(Z) is closed in $X'^*_{\mathbb{C}}$ in the fine topology.

To prove that p(Z) is closed $X^*_{\mathbb{C}}$, we shall show that p' is a proper continuous map between $X'^*_{\mathbb{C}}$ and $X^*_{\mathbb{C}}$ with respect to the fine topologies. Let $K \subseteq X^*_{\mathbb{C}}$ be a compact subset. If $m = \sum_{i=1}^{n} n_i$, the generators $\{f_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq n_i}$ of A' give a closed immersion of X'^* in $X^* \times \text{Spec } \mathbb{C}[T_1, \dots, T_m]$ and so the fine topology on $X'^*_{\mathbb{C}}$ is induced by the product of the fine topologies on $(X^* \times \text{Spec } \mathbb{C}[T_1, \dots, T_m])_{\mathbb{C}} = X^*_{\mathbb{C}} \times \mathbb{C}^m$. Of course, the natural inclusion of sets $X'^*_{\mathbb{C}} \subseteq X^*_{\mathbb{C}} \times \mathbb{C}^n$ is given by the map $\xi \rightsquigarrow (p'(\xi), \dots, f_{ij}(\xi), \dots)$. For any i,j, $1 \leq i \leq n$, $1 \leq j \leq n_i$, let $f_{ij}^{m_{ij}} + a^{(ij)}_{m_{ij}-1} f_{ij}^{m_{ij}-1} + \dots + a^{(ij)}_1 f_{ij} + a^{(ij)}_0 = 0$ be an equation with $a^{(ij)}_k \in A$, satisfied by f_{ij}. For $\xi \in p'^{-1}(K)$ such that $f_{ij}(\xi) \neq 0$ we have the equality:

$$1 = -\frac{a^{(ij)}_{m_{ij}-1}(p'(\xi))}{f_{ij}(\xi)} - \dots - \frac{a^{(ij)}_0(p'(\xi))}{f_{ij}^{m_{ij}}(\xi)}$$

Since all functions $a^{(ij)}_k$ are bounded on K, it follows from here that f_{ij} must be bounded on $p^{-1}(K)$. Hence there exists $M > 0$ such that for all i,j, $1 \leq i \leq n$, $1 \leq j \leq n_i$ and all $\xi \in p'^{-1}(K)$, $|f_{ij}(\xi)| \leq M$ Denote $\overline{D}_M = \{z \mid z \in \mathbb{C}, |z| \leq M\}$. Then the subset $f^{-1}(K) \subseteq X'^*_{\mathbb{C}} =$

$= X^*_{\mathbb{C}} \times \mathbb{C}^m$ is contained in $\overbrace{}^{K\times(\bar{D}_N)^m \text{ and so it}}$ is compact in the fine topology. Since $X^*_{\mathbb{C}}$ is locally compact and $X'^*_{\mathbb{C}}$, being affine, is Hausdorff with respect to the fine topology, it follows that p' is a proper map between $X'^*_{\mathbb{C}}$ and $X^*_{\mathbb{C}}$, by [4], Ch.I, §10,3, Prop.7.

Q.E.D.

Proof of Proposition 2. It is clear that i) is equivalent with ii). The assertion iii) is equivalent with iv), by [5], Theorem 3,1).

Since iv) $\Rightarrow$ i), we shall prove ii) $\Rightarrow$ iii).

If $p: X^{*N} \longrightarrow X^*$ is the normalization morphism, for all points $x^N \in X^{*N}$ lying over x, $\mathcal{O}_{X^{*N}, x^{*N}}$ is noetherian ring, by Krull-Akizuki Theorem. By Lemma 4, p is proper with respect to the fine topologies on $X^{*N}_{\mathbb{C}}$, $X^*_{\mathbb{C}}$. Then by [4], Ch.I, §10,2, Prop.6, all $x^N \in X^{*N}_{\mathbb{C}}$ lying over x have compact neighbourhoods in the fine topology. By restricting X^{*N} to a Zariski open neighbourhood V^* of a point x^N lying over x, if we prove the assertion ii) $\Rightarrow$ iii) for $j: V^* \cap p^{-1}(x) \hookrightarrow V^*$ and $x^N \in V^*$, it follows dim $V^*=1$ and so dim $X^*=1$ (cf. [5], Lemma 1 a)).

Therefore we can suppose that X^* is normal.

By Lemma 1, we may replace X^* by a Zariski open neighbourhood of x in X^* and suppose that $X^* = \operatorname{Spec} \mathbb{C}[T, f_1, \ldots, f_n, \ldots, (\operatorname{Rest} f_k)_\ell, \ldots]_{1\le k\le n,\, \ell\ge 1}$ where $f_k = \sum_{i=1}^{\infty} c_i^{(k)} T^i \in \mathbb{C}[[T]]$
$X = X^*_T = \operatorname{Spec} \mathbb{C}[\frac{1}{T}, T, f_1, \ldots, f_n]$ and x is the closed point of X^* corresponding to the maximal ideal generated by T in $\mathbb{C}[T, f_1, \ldots, f_n, \ldots, (\operatorname{Rest} f_k)_\ell, \ldots]_{1\le k\le n,\, \ell\ge 1}$.

By hypothesis x has a compact neighbourhood $U \subset X^*_{\mathbb{C}}$ in the fine topology. The field of fractions of $\mathbb{C}[T, \ldots, f_i, \ldots, (\operatorname{Rest} f_k)_\ell, \ldots]$ is generated over $\mathbb{C}$ by $T, f_1, \ldots, f_n$ and we may assume that $T, f_1, \ldots, f_m$ are algebraically independent over $\mathbb{C}$, and $f_{m+1}, \ldots, f_n$ are algebraic over the field $\mathbb{C}(T, f_1, \ldots, f_m)$.
Let us denote $Y^* = \operatorname{Spec} \mathbb{C}[T, f_1, \ldots, f_m, \ldots, (\operatorname{Rest} f_k)_\ell, \ldots]_{1\le k\le m,\, \ell\ge 1}$

$Y = Y^*_T = \mathrm{Spec}\,\mathbb{C}[\frac{1}{T}, T, f_1, \dots, f_m]$, $y = Y^* - Y$ the point corresponding to the maximal ideal generated by T in $\mathbb{C}[T, f_1, \dots, f_m, \dots, (\mathrm{Rest}\, f_k)_\ell, \dots]_{1\le k\le m,\ \ell\ge 1}$ and $\varphi : X^* \longrightarrow Y^*$ the natural mop hism of schemes given by the inclusion of rings. It is clear that $y=\varphi(x)$. Then, by Lemma 3, there exists a fine open neighbourhood of x in X^* which is homeomorphic with a fine open neighbourhood of y in Y^*. Since $x \in X^*_{cl}$ has a compact neighbourhood, then $y \in Y^*_{cl}$ has also a compact neighbourhood.

We shall prove that this fact is not possible if $m = \dim Y^* - 1 \geq 1$.

Indeed, the compact neighbourhood of y contains a neighbourhood of the type: y and all $\eta \in Y^*_{cl}, \eta \neq y$, satisfying some inequalities:

$$\begin{cases} |T(\eta)| \le \varepsilon \\ \text{-------} \\ |f_j(\eta)| \le \varepsilon_j \\ \text{-------} \\ |(\mathrm{Rest}\, f_k)_\ell(\eta)| = |(\sum_{i=\ell+1}^{\infty} c_i^{(k)} T^{i-\ell})(\eta)| = \left| \dfrac{f_k(\eta) - \sum_{i=1}^{\ell} c_i^{(k)} T(\eta)^i}{T(\eta)^\ell} \right| \le \varepsilon_{k\ell} \\ \text{--------} \end{cases}$$

where $\varepsilon, \varepsilon_j, \varepsilon_{k\ell} > 0$ and j, k, l are finitely many indexes such that $1 \le j, k \le m$, $l \ge 1$. (Of course, we considered T, f_j, $(\mathrm{Rest}\, f_k)_\ell \in \Gamma(X^*, \mathcal{O}_{X^*})$ as functions on X^*_{cl}). If we put $(\mathrm{Rest}\, f_k)_0 = f_k$, then for all $l \ge 0$, we have $(\mathrm{Rest}\, f_k)_\ell(\eta) = [(\mathrm{Rest}\, f_k)_{\ell+1}(\eta) + c^{(k)}_{\ell+1}]\, T(\eta)$ Hence, by changing ε, we may suppose that the compact neighbourhood of y contains a neighbourhood K of the type: y and all $\eta \in Y^*_{cl}$, $\eta \neq y$, satisfying some inequalities:

$$\begin{cases} |T(\eta)| \le \varepsilon \\ \text{-------} \\ |(\mathrm{Rest}\, f_k)_{\ell_k}(\eta)| \le \varepsilon_k \qquad 1 \le k \le m,\ \ell_k \ge 1 \\ \text{--------} \end{cases}$$

where $\varepsilon, \varepsilon_k > 0$ and where for any k, $1 \le k \le m$, we have a unique inequality $|(\mathrm{Rest}\, f_k)_{\ell_k}(\eta)| \leq \varepsilon_k$ (with $\ell_k \geq 1$). Since K is closed in the fine topology, it follows that K is compact.

Let $C \subset Y^*$ be the Zariski closed subset given by the equations $f_k - \sum_{i=1}^{\ell_k} c_i^{(k)} T^i = 0$, $1 \leq k \leq m$. Since $y \in C$ and y is a closed 1-codimensional point in Y^*, $\{y\}$ is an irreducible component of C. Then $C' = C - \{y\}$ is a Zarkiski closed subset of Y^* and so $C'_{\mathbb{C}} = \{\eta \in Y^*_{\mathbb{C}} \mid \eta \neq y,\ f_k(\eta) - \sum_{i=1}^{\ell_k} c_i^{(k)} T(\eta)^i = 0$, for all k, $1 \leq k \leq m\}$ is closed in $Y^*_{\mathbb{C}}$ in the fine topology. Since $C'_{\mathbb{C}} \subset Y^*_{\mathbb{C}} - \{0\} =$ $= (\mathrm{Spec}\, \mathbb{C}[\frac{1}{T}, T, f_1, \dots, f_m])_{\mathbb{C}} \subseteq \mathbb{C}^{m+1}$ is a complex algebraic variety, the map $\pi : \overline{D}_\varepsilon - \{0\} = \{z \in \mathbb{C} \mid z \neq 0, |z| \leq \varepsilon\} \longrightarrow C'_{\mathbb{C}} \cap K$, defined by $\pi(t) = (t, \sum_{i=1}^{\ell_1} c_i^{(1)} t^i, \dots, \sum_{i=1}^{\ell_m} c_i^{(m)} t^i) \in \mathbb{C}^{m+1}$, establishes a homeomorphism between $\overline{D}_\varepsilon - \{0\}$ and $C'_{\mathbb{C}} \cap K$. Since $C'_{\mathbb{C}} \cap K$ is compact, it follows $\overline{D}_\varepsilon - \{0\}$ is compact, which is not true.

Therefore m=0 and so dim X^*=dim X=1.

Q.E.D.

§3. THE MAIN RESULT

In this section we shall establish the following

Theorem. Let $i: X \hookrightarrow X^*$ be an open immersion of integral schemes over $\mathbb{C}$, where X is of finite type over $\mathbb{C}$ and $x \in X^*$ a closed point such that $\mathcal{O}_{X^*,x}$, is noetherian. Then x has a Zariski open neighbourhood of finite type over $\mathbb{C}$ iff x has a compact neighbourhood in the fine topology.

To prove Theorem we need the following

Lemma 5. Let $i: X \hookrightarrow X^*$ be an open immersion of integral schemes over $\mathbb{C}$, where X is of finite type over $\mathbb{C}$ and $x \in X^*_{cl}$ such that $\mathcal{O}_{X^*,x}$ is noetherian. Then the following assertions are equivalent:

i) x has a Zariski open neighbourhood of finite type over $\mathbb{C}$

ii) if $p: Y \to X^*$ is a finite morphism of schemes such that Y is integral, generically of finite type over $\mathbb{C}$ and contains closed 1-codimensional points lying over x, then $\dim Y=1$.

Proof. It is clear that i) $\Rightarrow$ ii), by [5], Lemma 1. We will prove ii) $\Rightarrow$ i) by induction on $\dim X^*$. If $\dim X^*=1$, then X^* is of finite type over $\mathbb{C}$ (cf.[5], Th.3.1)).

Suppose $\dim X^* > 1$ and let $\{x\} = Z_0 \subset Z_1 \subset \dots \subset Z_n = X^*$ be a saturated chain of closed irreducible subsets. We have $n \geq 2$, since, otherwise, $n=1$ and x is a closed 1-codimensional point of X ; hence, by ii), $\dim X^*=1$, which is a contradiction. By changing Z_{n-1}, we may suppose that $Z_{n-1} \cap X \neq \phi$. In fact, in the local noetherian ring $\mathcal{O}_{X^*, Z_{n-2}}$ of the integral subscheme Z_{n-2} in X^*, we have a maximal chain of prime ideals $m \supset p' \supset 0$ corresponding to $Z_{n-2} \subset Z_{n-1} \subset X^*$. By a theorem of McAdam (see [12], Prop.1), we may find a maximal chain of prime ideals $\underline{m} \supset \underline{p}' \supset 0$ such that $\underline{p}$ does not include the prime ideals of $\mathcal{O}_{X^*, Z_{n-2}}$ corresponding to (finitely many) irreductible components of X^*-X containing Z_{n-2}. Then we may change Z_{n-1} and replace it by the closed irreducible subscheme of X^* corresponding to $\underline{p}'$.

Therefore we can apply the induction hypothesis to Z_{n-1} (which contains the open $\mathbb{C}$-subscheme of finite type $Z_{n-1} \cap X \neq \phi$), since $\dim Z_{n-1} < \dim X^*$ and Z_{n-1} has still the property ii) of X^*. Then x has a Zariski open neighbourhood V in Z_{n-1}, of finite type over $\mathbb{C}$. Then $\{x\} = Z_0 \cap V \subset Z_1 \cap V \subset \ldots \subset Z_{n-1} \cap V = V$ is a saturated chain of closed irreducible subsets of V and so $n-1 = \dim V = \dim Z_{n-1}$ (cf. [5], Lemma 1). Since $\dim Z_{n-1} = \dim (Z_{n-1} \cap X) = \dim X - 1 = \dim X^* - 1$, it follows that $n = \dim X^*$. Therefore all maximal chains of closed irreducible subsets of X^* passing through x have the same length.

If $Z \subset X^*$ is any closed integral subscheme, passing through x, we can find a maximal chain $\{x\} = Z_0 \subset Z_1 \subset \ldots \subset Z_n = X^*$ of closed irreducible subsets of X^* such that $Z = Z_k$ for certain k. Since $0 = \dim.al._{\mathbb{C}} K(Z_0) < \dim.al._{\mathbb{C}} K(Z_1) < \ldots$ $\ldots < \dim.al._{\mathbb{C}} K(Z_n) = \dim X^*$, where $K(Z_i)$ is the field of rational functions on the integral subscheme Z_i, (see [5], proof of Lemma 1), we have $\dim Z_i = \dim.al._{\mathbb{C}} K(Z_i) = i$, since $\dim \mathcal{O}_{Z_{i+1}, Z_i} = \operatorname{codim}_{Z_{i+1}} Z_i = 1$. Therefore the generic point z_i of Z_i has the following properties: $\dim \mathcal{O}_{Z_{i+1}, z_i} = 1$ and $\dim.al._{k} k(z_i) = i$, where $k(z_i) = K(Z_i)$ is the residue field of z_i. By [5], Remark 1, applied for $i = n-1, n-2, \ldots, k$ to Z_i, it results that $Z = Z_k$ is generically algebraic over $\mathbb{C}$. Hence by the induction hypothesis, x has a zariski open neighbourhood in Z, which is of finite type over $\mathbb{C}$. Thus $\mathcal{O}_{Z,x}$ is a ring essentially of finite type over $\mathbb{C}$.

It follows that for every prime ideal $\underline{p}$, $0 \neq \underline{p} \subset \mathcal{O}_{X^*, x}$

the ring $\mathcal{O}_{X^*,x}/\underline{p}$ is essentially of finite type over $\mathbb{C}$, and so universally japanese. By a Lemma of Marot (see [11], Lemma 2), the integral closure of $\mathcal{O}_{X^*,x}$ in its field of quotients is noetherian.

Therefore if $p: X^{*N} \longrightarrow X^*$ is the normalization morphism of X^*, then for all points $x^N \in X^{*N}$ lying over x, $\mathcal{O}_{X^{*N},x}$ is a noetherian ring. All points $x^N \in X^{*N}$ lying over x are closed in X^{*N} but not of codimension one, since, otherwise, using the fact that there exist finitely many $x^N \in X^{*N}$ lying over x, by a standard procedure one constructs an integral scheme Y, which is finite over X^* and dominating it, having closed 1-codimensional points lying over x; by ii), it follows that $1=\dim Y=\dim X^*$, which contradicts the assumption $\dim X^*>1$.

Let $x^N \in X^{*N}$ be a point lying over x and $\underline{m}=\underline{p}_0 \supset \underline{p}_1 \supset \dots \supset \underline{p}_k = 0$ a maximal chain of prime ideals of $\mathcal{O}_{X^{*N},x^N}$. Then $k\geq 2$. Let us denote by $\{x\} = Z_0 \subset Z_1 \subset \dots \subset Z_k = X^{*N}$ the corresponding maximal chain of closed irreducible subsets. Since $p|_{Z_{k-1}}: Z_{k-1} \longrightarrow p(Z_{k-1})$ is integral, $p(Z_{k-1}) \neq X^*$ and $K(Z_{k-1}) \supseteq K(p(Z_{k-1}))$ is a finite extension (by [14], Theorem 33.10), it follows that x^N has a Zariski open neighbourhood W in Z_{k-1}, which is of finite type over $\mathbb{C}$, because $x\in p(Z_{k-1})$ has this property. Then $\{x\} = Z_0 \cap W \subset Z_1 \cap W \subset \dots \subset Z_{k-1} \cap W = W$ is a maximal chain of closed irreducible subsets of W, and so $k-1=\dim W=\dim Z_{k-1}$. Since $\mathcal{O}_{X^{*N},Z_{k-1}}$ is noetherian, by changing Z_{k-1} we may assume that $Z_{k-1} \cap p^{-1}(x) \neq \phi$ (in the same manner as above). Then $\dim Z_{k-1} = \dim (Z_{k-1} \cap p^{-1}(x)) =$
$= \dim p^{-1}(x) - 1 = \dim Z_k - 1$, since $p^{-1}(x)$ is a $\mathbb{C}$-scheme of finite type and $\operatorname{codim}_{X^{*N}} Z_{k-1} = 1$. Therefore $k=\dim X^{*N}$, and so $\dim \mathcal{O}_{X^{*N},x^N} = \dim X^{*N}$. Then, by [5], Lemma 4 , it follows that x^N has a Zariski open neighbourhood W_{x^N} of finite type over $\mathbb{C}$. Then $\widetilde{W} = X^* - p(X^{*N} - \bigcup_{x^N \to x} W_{x^N})$ is a Zariski open neighbourhood of x, of finite type over $\mathbb{C}$, since $p^{-1}(\widetilde{W}) \subseteq \bigcup_{x^N \to x} W_{x^N}$ and $p|_{p^{-1}(W)}: p^{-1}(\widetilde{W}) \to \widetilde{W}$ is finite.

Q.E.D.

Remark 6. Lemma 5 is true for an arbitrary base field. $\square$

Proof of Theorem Suppose that x has a compact neighbourhood K in the fine topology. By restriction X^* to a Zariski open neighbourhood of x, we may assume that X^* is affine. Let $p: Y^* \longrightarrow X^*$ be a finite morphism of schemes such that Y^* is integral, contains an open subscheme Y of finite type over $\mathbb{C}$ and a closed 1-codimensional point y lying over x. Then Y^* is a closed subscheme of a scheme of the type $X^* \times_{\mathbb{C}} \mathrm{Spec}\, \mathbb{C}[T_1, \ldots, T_n]$ ans so $Y^*_{\mathbb{C}}$ is a closed topological subspace of $X^*_{\mathbb{C}} \times \mathbb{C}^n$, with respect to the fine topologies. Then $(K \times \mathbb{C}^n) \cap Y^*_{\mathbb{C}}$ is a neighbourhood of y in $Y^*_{\mathbb{C}}$, and so there exists $\varepsilon > 0$, such that $(K \times \overline{D}_\varepsilon) \cap Y^*_{\mathbb{C}}$ is still a neighbourhood of y in $Y^*_{\mathbb{C}}$ in the fine topology, where $\overline{D}_\varepsilon = \{ z \in \mathbb{C} \mid |z| \leq \varepsilon \}$; moreover, it is compact. It is clear that $\mathcal{O}_{Y^*,y}$ is noetherian, since $\mathcal{O}_{X^*,x}$ is noetherian and Y^* is of finite type over X^*. Since y is a closed 1-codimensional point of Y^*, then $\{y\}$ is an irreducible component of $Y^* - Y$. If Z is the union of all components of $Y^* - Y$ different from $\{y\}$, and $Y'^* = Y^* - Z$, then Y is an open subscheme of Y'^* of finite type over $\mathbb{C}$ and $Y'^* - Y = \{y\}$, where y is still a closed point of codimensional one in Y'^*. Applying Proposition 2 to the open immersion $Y \hookrightarrow Y'^*$, it follows $\dim Y'^* = 1$. Since Y'^* is an open subscheme of Y^*, we have $\dim Y = \dim Y'^* = 1$ by [5], Lemma 1. From Lemma 5 it results that x has a Zariski open neighbourhood of finite type over $\mathbb{C}$.

Corollary 3 - Let $f: X \longrightarrow Y$ be a dominant morphism of schemes over $\mathbb{C}$, where X is of finite type over $\mathbb{C}$, Y is reduced and $y \in Y$ is a closed point such that $\mathcal{O}_{Y,y}$ is noetherian. Then y has a Zariski open neighbourhood of finite type over $\mathbb{C}$ ifff y has a compact neighbourhood in the fine topology.

COROLLARY 4. Let $f: X \longrightarrow Y$ be a dominant morphism of schemes over $\mathbb{C}$, where X is of finite type over $\mathbb{C}$ and Y is noetherian. Then Y is of finite type over $\mathbb{C}$ iff the fine topology of Y is locally compact.

We left to the reader to establish these consequences of the Theorem , using the fact that a $\mathbb{C}$-scheme dominated by a $\mathbb{C}$ -scheme of finite type is generically of finite type (cf. [5], Lemma 5). From Corollary 4 results easy the following

Corollary 5. A noetherian subalgebra A of a $\mathbb{C}$ - algebra of finite type is of finite type iff the Gel'fand topology on the set of all maximal ideals of A is locally compact.

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