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by

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CONTROLLABILITY OF NONLINEAR STOCHASTIC CONTROL SYSTEMS, II.

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Abstract

In the paper sufficient conditions for "deterministic weak controllability" of nonlinear Ito's equations of the form

$$dx(t)=[f(x(t))+\sum_{i=1}^{m}u_i g_i(x(t))]dt+\sum_{j=1}^{\ell}h_j(x(t))dw_j(t),\ t\geq 0$$

are given.

It is a generalization and a more detailed version of the author's paper [5].

1. Introduction

Our purpose in this paper is to consider in more detail the "deterministic weak controllability" studied in [5]. We are looking for sufficient conditions ensuring that the stochastic control system may reach any open set in R^n with positive probability starting from an arbitrary fixed point $p_o \in R^n$. For various types of controllability and more comprehensive literature regarding this problem we refer to [2] and [3].

The basic assumptions in [5] require the global existence of a diffeomorphism $H:R^n \longrightarrow R^n$ which defines globally two submanifolds in R^n one of them being invariant for the control system

$$\frac{dx}{dt}=\sum_{i=1}^{m}u_i g_i(x)$$

This property appears seldom for nonlinear systems and it is our goal in this paper to obtain "deterministic weak controllability" using only the local existence of a diffeomorphism H depending on each $x\in R^n$. The two examples in the final part will point out how those local diffeomorphisms can be constructed for some bilinear control systems.

2. Formulation of the problem and main result

We consider Itô's equation with control of the form

$$1)\quad dx(t)=\Big[f(x(t))+\sum_{i=1}^{m}u_i(t)g_i(x(t))\Big]dt+\sum_{j=1}^{\ell}h_j(x(t))dw_j(t),\quad x(0)=p_0 ,$$

where $w(t)$, $t\geq 0$, is a ℓ-dimensional Wiener process over the fixed probability space $\{\Omega,\mathcal{F},P\}$, and the control $u:[0,\infty)\to R^m$ is deterministic in the set $\mathcal{U}_0$ consisting of all piecewise constant functions with a finite number of discontinuity points.

Denote W the set of all ℓ-dimensional Wiener processes over $\{\Omega,\mathcal{F},P\}$. The functions $f, g_i, h_j: R^n\to R^n$ are supposed to be of class C^1 and satisfying a linear growth condition; in addition g_i is assumed to be of class C^∞.

For each $u(.)\in\mathcal{U}_0$ and $w(.)\in W$, from (1) there is $x^u(.)$, an unique Itô's solution. By solution of (1) we take the pair $(x^u(.), w(.))$ which is unique in probability law, (uniqueness holds even pathwise), see [4]. Denote $\mathcal{V}(y)$ the set consisting of all open sets in R^n containing $y\in R^n$.

Definition 1

The system (1) is deterministic weakly controllable if for any $p_0, p_1\in R^n$, $T>0$, $V\in\mathcal{V}(p_1)$ there exist $u(.)\in\mathcal{U}_0$ and $w(.)\in W$ such that $P\{x^u(T,\omega)\in V\}>0$, where $(x^u(.),w(.))$ is the solution in (1) with $x^u(0)=p_0$.

For each $u(.)\in\mathcal{U}_0$ and $w(.)\in W$, the Markov process $x^u(.)$ (Itô's solution in (1), corresponding to $u(.)\in\mathcal{U}_0$ and $w(.)\in W$) generates a finite family of homogeneous diffusion processes. Let $u(t)=$ $=u^i\in R^m$, $t\in[t_i,t_{i+1})\hat{=}I_i$, $i=0,1,\dots,N$, $t_0=0$, $t_{N+1}=\infty$

Denote $P_i(t,x,A)$ the transition probabilities of the diffusion in (1) corresponding to $u(t)=u^i\in R^m$ and $w(.)\in W$ fixed; $P_i(t,x,A)$ is not depending on $w(.)\in W$. For $t\in I_i$ define

$$P^u(t,x,A)=\int_{R^n} P_1(t_1,x,dy_1)\int_{R^n} P_2(t_2-t_1,y_1,dy_2)\ldots\int_{R^n} P_{i-1}(t_i-t_{i-1},y_{i-1},dy_i)P_i(t-t_i,y_i,A)$$

The system (1) is deterministic weakly controllable iff for any $p_0,p_1\in R^n$, $T>0$, $V\in\mathcal{V}(p_1)$ there exists $u(.)\in\mathcal{U}_0$ such that $P^u(T,p_0,V)>0$.

The system (1) will be studied via the following two subsystems

$$2)\quad dx(t)=f(x(t))dt+\sum_{j=1}^{l} h_j(x(t))dw_j(t)$$

$$3)\quad \frac{dx}{dt}=\sum_{i=1}^{m} u_i(t)g_i(x(t)),\quad u(.)\in\mathcal{U}_0$$

On (2) and (3) we make the following assumptions.

For each $x\in R^n$ there exist a local diffeomorphism of class C^∞, $H:V\to C$, $V\in\mathcal{V}(x)$, C a cube in R^n, with $\frac{\partial H}{\partial y}(y)$ nonsingular $(\forall)y\in V$, and integers $0\leq k<n$, $0\leq m'\leq m$ such that

a) $P^0(T,y,D)>0$ for any $T>0$, $y\in V$, D open in V, if $D\cap M_1(H_1(y),\ldots,H_k(y))\neq\emptyset$, where $M_1(C_1,\ldots,C_k)\triangleq\{y\in V: H_i(y)=C_i,\ i\leq k\}$

b) i) the manifold $M_2(C_{k+1},\ldots,C_n)\triangleq\{y\in V: H_j(y)=C_j,\ j\geq k+1\}$ is invariant for the fields g_{i_l}, $l=1,\ldots,m'$ $(\langle\frac{\partial H_j}{\partial y}(y),g_{i_l}(y)\rangle=0$, $y\in V$, $j\geq k+1$, $l=1,\ldots,m'$, $i_l\in\{1,\ldots,m\})$;

ii) $\dim\mathcal{L}(g_0,g_{i_1},\ldots,g_{i_{m'}})(y)=k$, $(\forall)y\in V$, where $g_0=0$.

By $(g_0,\ldots,g_{m'})$ is denoted the Lie algebra over R generated by $g_0,\ldots,g_{m'}$ using the Lie bracket $[h_1,h_2](x)=\frac{\partial h_1}{\partial x}(x)h_2(x)-\frac{\partial h_2}{\partial x}(x)h_1(x)$, and $\mathcal{L}(g_0,\ldots,g_{m'})(x)$ is the linear subspace in R^n consisting of all vectors $v=h(x)$, $h\in\mathcal{L}(g_0,\ldots,g_{m'})$.

Theorem 1

Let the conditions (a) and (b) be fulfilled. Then (1) is deterministic weakly controllable.

Before giving the proofs some remarks are necessary. If the system (1) fulfils $P^0(t,x,D)>0$ for any $t>0$, $x\in R^n$ and D open in R^n then the conditions (a) and (b) are trivially satisfied with $m'=k=0$ and $H(x)=x$. In this case there is nothing to be proved since the control $u=0$ gives the answer for the problem.

If the system (1) is linear, $f(x)=Ax$, $g_i(x)=b_i$, $h_j(x)=c_j$, $B=(b_1\ldots b_m)$, $C=(c_1\ldots c_\ell)$ and $\mathrm{Rank}(B,C,AC,\ldots,A^{n-1}C)=n$ then the conditions (a) and (b) are almost trivially verified. Indeed, if $\mathrm{Rank}(C,AC,\ldots,A^{n-1}C)=n$ then the solution of the uncontrolled system ($u=0$) is a nondegenerate Gaussian process which gives $P^0(t,x,D)>0$ for any $t>0$, $x\in R^n$ and D open in R^n; therefore (a) and (b) are fulfilled with $m'=k=0$, $H(x)=x$.

In the case $\mathrm{Rank}(C,AC,\ldots,A^{n-1}C)=n-k$, $k\geq 1$, there will exist $v_1,\ldots,v_k\in\{b_1,\ldots,b_m\}$ and $v_{k+1},\ldots,v_n\in(C,AC,\ldots,A^{n-1}C)$ such that the matrix, $V=(v_1,\ldots,v_n)$ is a nonsingular one. Define $H(x)=V^{-1}x$ and since

$$\sum_{j=1}^{n} H_j(x)\,v_j = x\,,\quad \text{where } H(x)=(H_1(x),\ldots,H_n(x))\,,$$

it follows $\left\langle \frac{\partial H_j}{\partial x}(x), v_i\right\rangle = 0$, $j=k+1,\ldots,n$, $i=1,\ldots,k$.

Therefore (b) is fulfilled with $(g_{i_1},\ldots,g_{i_{m'}})=(v_1,\ldots,v_k)$. In addition, the linear variety $S=\{x\in R^n: H_j(x)=C_j,\ j=1,\ldots,k\}$ is invariant for (2) and when it is restricted to S the solution is a Gaussian nondegenerate process, i.e. $P^o(t,x,D)>0$ for any $t>0$, $x\in S$, D open in S.

The situation with (a) and (b) is not trivial when considering biliniar systems especially in verifying condition (a).

3. Auxiliary results and proofs

The following lemma is an obvious consequence of Chow's theorem (see [1]) for deterministic systems.

Lemma 1

Let the condition (b) be fulfilled, $x\in R^n$ fixed and $H:V\longrightarrow C$ the associated local diffeomorphism. Then for any $p_o,p\in V$ and $T>0$ there exists $u(.)\in\mathcal{U}_0$ such that the solution $x^u(.)$ in (3) with $x^u(0)=p_o$ verifies $x^u(T)\in M_1(H_1(p_1),\dots,H_k(p_1))$.

Proof

By hypothesis, the system (3) invariates the manifold $M_2(H_{k+1}(p_o),\dots,H_n(p_o))$ which has the dimension k and by (b), (ii) the Chow theorem is applicable to the system (5).
It follows that p_o can be joined with any $x_2\in M_2(H_{k+1}(p_o),\dots,H_n(p_o))$ in finite time by a trajectory of (3) corresponding to a control $u(.)\in\mathcal{U}_0$ taking its values in $\{\pm e_1,\dots,\pm e_m\}$, where $e_1,\dots,e_m$ is a canonical base in R^m. In particular, let $x_2\in M_2(H_{k+1}(p_o),\dots,H_n(p_o))$ be such that $H_j(x_2)=H_j(p_j)$ $j=1,\dots,k$, i.e. $x_2\in M_1(H_1(p_1),\dots H_k(p_1))$. Then there exists $t_2<\infty$ and $u_2(.)\in\mathcal{U}_0$, $u_2(t)\in\{\pm e_1,\dots,\pm e_m\}$ such that $x^{u_2}(t_2)=x_2$.
By substitution $\tau=\frac{T}{t_2}\,t$ and denoting $\overline{u_2}(\tau)=\frac{t_2}{T}u_2(\frac{t_2}{T}\tau)$, computation gives that $x^{\overline{u_2}}(\tau)=x^{u_2}(\frac{t_2}{T}\tau)$ fulfils (3) with $x^{\overline{u_2}}(0)=p_o$, $x^{\overline{u_2}}(T)=x_2$.
The following is an obvious consequence of the Lemma 1 and condi-

dition (a).

Corollary

Let the conditions (a) and (b) be fulfilled and $x^u(.)$ given by Lemma 1 such that $x^u(0)=p_o$, $x^u(T)\in M_1(H_1(p_1),\dots,H_k(p_1))$, where $T>0$. Then $P^o(t,x^u(T),D)>0$ for any $t>0$, if $D\in\mathcal{V}(p_1)$.

Generally, the solution $x^u(.)$ in (3) is not obtained as the first component of a solution in (1), but it can be approximated by a sequence defined as follows.

Lemma 2

Let $u(.)\in\mathcal{U}_0$ and $x^u(.)$ be the solution in (3) with $x^u(0)=p_o$. Then there exist $\{u_n(.)\}_{n\geq 1}\subseteq\mathcal{U}_0$ and $\{x_n(.),w_n(.)\}_{n\geq 1}$ solutions in (1) with $x_n(0)=p_o$ corresponding to $\{u_n(.)\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} P\{|x_n(\tfrac{T}{n},\omega)-x^u(T)|\leq\varepsilon\}=1, \text{ for any } T>0 \text{ and } \varepsilon>0.$$

Proof

Let $x^u(.)$ be a solution in (3) corresponding to $u(.)\in\mathcal{U}_0$ with $x^u(0)=p_o$. Fix $T>0$ and define $y_n(.)$, Itô's solution in (1), corresponding to $f_n=\frac{1}{n}f$, $h_{jn}=\frac{1}{\sqrt{n}}h_j$, $u(.)\in\mathcal{U}_0$ and $w(.)\in W$, i.e.

$$4)\ y_n(t,\omega)=p_o+\int_0^t f_n(y_n(s,\omega))ds+\sum_{i=1}^m\int_0^t u_i(s)g_i(y_n(s,\omega))ds+\sum_{j=1}^l\int_0^t h_{jn}(y_n(s,\omega))dw_j(s)$$

Substituting $t=n\tau$ in (4) we get that $x_n(\tau,\omega)=y_n(n\tau,\omega)$ is the Itô solution in (1) corresponding to a new Wiener process w^n on the same probability space $\{\Omega,\mathcal{F},P\}$.

Namely,

$$5)\ x_n(\tau,\omega)=p_o+\int_0^\tau f(x_n(s,\omega))ds+\sum_{i=1}^m\int_0^\tau nu_i(ns)g_i(x_n(s,\omega))ds+$$

$$\sum_{j=1}^{l}\int_0^{\tau} h_j(x_n(s,\omega))\,dw_j^n(s)$$

where $\quad w_j^n(s)=\frac{1}{\sqrt{n}}\,w_j(ns)$

We shall prove that $\{y_n(T,\omega)\}_{n\geq 1}$ converges in probability to $x^u(T)$ uniformly with respect to the initial condition p_o in a bounded set $B\subseteq R^n$.

Fix $\delta>0$. Since $E\sup_{0\leq t\leq T}|y_n(t)|^2\leq C$ (see [4] p.105) where the constant C is independent of $p_o\in B$ and $n\geq 1$, it follows that there exists $N_\delta>0$ such that

$$P\{\omega\in\Omega:\sup_{0\leq t\leq T}|y_n(t,\omega)|>N_\delta\}\leq\delta,\ \text{for any } p_o\in B \text{ and } n\geq 1.$$

Denote $A_\delta=\{\omega:\sup_{0\leq t\leq T}|y_n(t)|>N_\delta\}$ and from (4) we get

$$|y_n(t,\omega)-x^u(t)|^2\leq C_\delta\int_0^t|y_n(s,\omega)-x^u(s)|^2ds+|\eta_n(t,\omega)|^2$$

for $\omega\in\Omega-A_\delta,\ t\in[0,T]$, where $\lim_{n\to\infty}E\sup_{0\leq t\leq T}|\eta_n(t,\omega)|^2=0$.

Therefore

$$|y_n(T,\omega)-x^u(T)|^2\leq\sup_{0\leq t\leq T}|\eta_n(t,\omega)|^2\exp.C_\delta T,\ \text{for } \omega\in\Omega-A_\delta$$

and it follows

$$\lim_{n\to\infty}E\,\alpha_\delta(\omega)|y_n(T,\omega)-x^u(T)|=0,\ \text{where } \alpha_\delta(\omega)=\begin{cases}1 & \omega\in\Omega-A_\delta\\ 0 & \omega\in A_\delta\end{cases}$$

For any $\varepsilon>0$ we have

$$\lim_{n\to\infty}P\{\omega\in\Omega\setminus A_\delta:|y_n(T,\omega)-x^u(T)|\geq\varepsilon\}=0$$

and

$$\overline{\lim_{n\to\infty}}\,P\{\omega:|y_n(T,\omega)-x^u(T)|\geq\varepsilon\}\leq\overline{\lim}\,P\{\omega\in\Omega\setminus A_\delta:|y_n(T,\omega)-x^u(T)|\geq\varepsilon\}$$

$$+\overline{\lim}\,P\{\omega\in A_\delta:|y_n(T,\omega)-x^u(T)|\geq\varepsilon\}\leq\delta$$

where $\delta > 0$ is arbitrarily fixed and the left hand side is not depending on δ.

Therefore $y_n(T,\cdot)$ converges in probability to $x^u(T)$ uniformly in $p_o \in B$ and since $x_n(\frac{T}{n},\ )=y_n(T,\omega)$ the proof is complete.

Now we are in position to prove the Theorem.

<u>Proof of Theorem</u>

Let $p_o, p_1 \in R^n$, $T>0$, $V \in \mathcal{V}(p_1)$ be arbitrarily fixed. By hypothesis for any $x \in [p_o, p_1]$ (the line connecting p_o, p_1) there exist $V \in \mathcal{V}(x)$, C a cube in R^n, a diffeomorphism $H: V \longrightarrow C$, and integers $0 \leq k < n$, $0 \leq m' \leq m$ such that (a) and (b) are satisfied. We obtain a covering of the compact set $[p_o, p_1]$ and substracting a finite subcovering $V_1, \ldots, V_N$, without any loss of generality assume $p_o \in V_1$, $p_1 \in V_N$ and $V_i \cap V_{i+1} \neq \emptyset$ for any $i=1,\ldots,N-1$. We choose an open set $\mathcal{O}_i \subseteq \dot{V}_i \cap V_{i+1}$ and $z_i \in \mathcal{O}_i$ a fixed point. Let $T_i > 0$, $\sum_{i=1}^{N} T_i = T$. In each V_i we apply Lemma 1. In V_1 there exist $u^1(\cdot) \in \mathcal{U}_o$ and $x^1(\cdot)$ in (3) with $x^1(0)=p_o$ fulfiling $x^1(T_1) \in M_1(H_1(z_1),\ldots,H_k(z_1))$. Using the Corollary we have $P^o(t, x^1(T_1), \mathcal{O}_1) > 0$ for any $t > 0$. The condition (a) guarantees $P^o(t,y,\mathcal{O}_1) > 0$ $(\forall) t > 0$, if $\mathcal{O}_1 \cap M_1(H_1(y),\ldots,H_k(y)) \neq \emptyset$.

By hypothesis the equations $H_i(z)=c_i$, $i=1,\ldots,k$, can be solved in a neighbourhood of $z=z_1$, $\tilde{c}=\tilde{H}(x^1(T_1))$, where $\tilde{H}=(H_1,\ldots,H_k)$, $\tilde{c}=(c_1,\ldots,c_k)$. We get a continuous function $z=h(\tilde{c})$ such that $z_1=h(\tilde{H}(x^1(T_1))$ and $z(y)=h(\tilde{H}(y))$ verifies $z(y) \in \mathcal{O}_1$ if $|y-x^1(T_1)| \leq \varepsilon_o$ and $\varepsilon_o > 0$ is sufficiently small. Therefore,

7) $\mathcal{O}_1 \cap M_1(H_1(y),\ldots,H_k(y)) \neq \emptyset$ if $|y-x^1(T_1)| \leq \varepsilon_o$

On the other hand, using Lemma 2, for N_1 sufficiently large we get

8) $P^{u_n^1}(\frac{1}{n}T_1, p_o, K) \geq \frac{1}{2}$ if $n \geq N_1$

where $K \triangleq \{y \in V_1 : |y - x^1(T_1)| \leq \varepsilon_0\}$ and $P^{u_n^1}(t, p_0, dy)$ is the probability measure generated by $x_n^1(t,.)$ in Lemma 2. Define

$$\tilde{u}_n^1(t) = \begin{cases} u_n^1(t) & t \in [0, \frac{1}{n} T_1] \\ 0 & t \in [\frac{1}{n} T_1, T_1] \end{cases} \quad \text{and} \quad w_n^1(t) = \frac{1}{\sqrt{n}} w(nt)$$

where $w \in W$ is arbitrarily fixed.

It follows $\tilde{u}_n^1(.) \in \mathcal{U}_0$ and

$$9) \; P^{\tilde{u}_n^1}(T' + \tfrac{1}{n} T_1, p_0, \mathcal{O}_1) = \int_{R^n} P^0(T', y, \mathcal{O}_1) P^{u_n^1}(\tfrac{1}{n} T_1, p_0, dy)$$

Using (7) and (8) in (9) we get $P^{\tilde{u}_n}(T' + \frac{1}{n} T_1, p_0, \mathcal{O}_1) > 0$ for any $T' > 0$ if $n \geq N_1$ and choosing $T_n' = T_1 - \frac{T_1}{n}$ we obtain that for each $n \geq N_1$ there exist $\tilde{u}_n^1(.) \in \mathcal{U}_0$ and $w_n(.) \in W$, $w_n(t) = \frac{1}{\sqrt{n}} w(nt)$, such that

$$10) \; P_n^1(T_1, p_0, \mathcal{O}_1) > 0$$

where $P_n^1(t, p_0, dy)$ is the probability measure generated by $\tilde{x}_n^1(t,.)$ and $(\tilde{x}_n^1(\ ), w_n(.))$ is the solution in (1) corresponding to the control $\tilde{u}_n^1(.)$ which fulfils $\tilde{x}^1(0) = p_0$.

Actually (10) holds in a stronger form

$$10') \; P_n^1(T_1, y, \mathcal{O}_1) > 0 \qquad \text{for each } n \geq N_1$$

and for any y in a sufficiently small neighbourhood of p_0.

Using the same control $u^1(.) \in \mathcal{U}_0$ and the continuity dependence of the solution in (3) with respect to the initial condition we get a neighbourhood B of p_0 such that for any $z \in B$ the solution $x^1(.,z)$ in (3) corresponding to $u^1(.)$ and z will verify

$$|x^1(T_1, z) - x^1(T_1, p_0)| \leq \frac{\varepsilon_0}{2}$$

On the other hand using the uniform convergence with respect to initial condition in Lemma 2 we get

11) $P^{u_n^1}(\frac{1}{n}T_1, z, K_z) \geq \frac{1}{2}$ for any $z \in B$, if $n \geq N_1$,

where $K_z \triangleq \{y \in V_1 : |y - x^1(T_1, z)| \leq \frac{\varepsilon_0}{2}\}$ and N_1 is sufficiently large not depending on z.

Define $\tilde{u}_n^1(\cdot)$ and $w_n(\cdot)$ as above and it follows

$$P^{\tilde{u}_n^1}(T' + \frac{1}{n}T_1, z, \mathcal{O}_1) = \int_{R^n} P^0(T', y, \mathcal{O}_1) P^{u_n^1}(\frac{1}{n}T_1, z, dy)$$

Since $K_z \subseteq \{y \in V_1 : |y - x^1(T_1, p_0)| \leq \varepsilon_0\}$ for any $z \in B$, using (7) and Corollary we obtain $P^0(T', y, \mathcal{O}_1) > 0$ for any $T' > 0$ if $y \in K_z$, $z \in B$.

Therefore for $T' = T_1 - \frac{T_1}{n}$ we have $P^{\tilde{u}_n^1}(T_1, z, \mathcal{O}_1) > 0$

for any $z \in B$ and $n \geq N_1$ which represent (10').

What we obtained in V_1 can be repeated in each neighbourhood V_i. In V_N we obtain $N_N > 0$, $w_n(\cdot) \in W$, $\tilde{u}_n^N(\cdot) \in \mathcal{U}_0$, $\tilde{x}_n^N(\cdot)$ and the neighbourhood $\tilde{\mathcal{O}}_{N-1} \subseteq \mathcal{O}_{N-1}$ of $z_{N-1} \in \mathcal{O}_{N-1}$ such that $\tilde{P}_n^N(T - T'_{N-1}, T'_{N-1}; y, \mathcal{O}_N) > 0$ for any $y \in \tilde{\mathcal{O}}_{N-1}$ and $n \geq N_N$, where $T'_{N-1} = \sum_{i=1}^{N-1} T_i$, $\tilde{P}_n^N(t, s; y, dz)$ is the probability measure generated by $\tilde{x}_n^N(t, \cdot)$ with $\tilde{x}_n^N(s) = y$ and $(\tilde{x}_n^N(\cdot), w_n(\cdot))$ is the solution in (1) corresponding to $\tilde{u}_n^N(\cdot)$.

In each V_i, $i = 2, \ldots, N$ we obtain that there exist $w_n(\cdot) \in W$, $\tilde{u}_n^i(\cdot) \in \mathcal{U}_0$, $\tilde{\mathcal{O}}_i \subseteq \mathcal{O}_i$ and N_i such that

12) $\tilde{P}_n^i(T'_i, T'_{i-1}; y, \tilde{\mathcal{O}}_i) > 0$ for any $y \in \tilde{\mathcal{O}}_{i-1}$ and $n \geq N_i$,

where $T'_i = \sum_{j=1}^{i} T_j$, $\tilde{P}_n^i(t, s; y, dz)$ is the probability measure generated by $\tilde{x}_n^i(t, \cdot)$ with $\tilde{x}_n^i(s) = y$, and $(\tilde{x}_n^i(\cdot), w_n(\cdot))$ is the solution in (1) for $u = \tilde{u}_n^i(\cdot)$.

Define $M = \max\{N_1, \ldots, N_N\}$, $\tilde{u}(\cdot) \in \mathcal{U}_0$ by $\tilde{u}(t) = \tilde{u}_M^i(t)$ for $t \in (T'_{i-1}, T'_i]$ and $\tilde{P}^i(t, s; y, dz) = \tilde{P}_M^i(t, s; y, dz)$, $i = 1, \ldots, N$, where $T'_0 = 0$

Denote $\tilde{P}(t, s; y, dz)$ the probability generated by $\tilde{x}(t, \cdot)$ with $\tilde{x}(s, \cdot) = y$, where $(\tilde{x}(.),\tilde{w}(.))$ is the solution in (1) corresponding to the control $\tilde{u}(.)$ with $\tilde{x}(0)=p_0$; it fulfils

$$\tilde{P}(t,s,y,dz)=\tilde{P}^i(t,s,y,dz) \text{ if } t,s\in[T'_{i-1},T'_i].$$

Therefore, using (10) and (12) for n=M we obtain

13) $\tilde{P}(T'_i,T'_{i-1},y,\tilde{O}_i)>0$ for any $y\in\tilde{O}_{i-1}$, $i=2,\dots,N$

and

14) $\tilde{P}(T_1,0,p_0,\tilde{O}_1)>0$

Finally, since $V\supseteq\tilde{O}_N-O_N$ it follows

15) $\tilde{P}(T,0,p_0,V)\geq\int_{R^n}\tilde{P}(T_1,0,p_0,dy_1)\dots\int_{R^n}\tilde{P}(T'_{N-1},T'_{N-2},y_{N-2},dy_{N-1})\tilde{P}(T,T'_{N-1},y_{N-1},\tilde{O}_N)$

Using (13) and (14) in (15) we obtain

16) $\tilde{P}(T,0,p_0,V)\geq\int_{\tilde{O}_1}\tilde{P}(T_1,0,p_0,dy_1)\dots\int_{\tilde{O}_{N-1}}\tilde{P}(T'_{N-1},T'_{N-2},y_{N-2},dy_{N-1})\tilde{P}(T,T'_{N-1},y_{N-1},\tilde{O}_N)>0$

and the proof is complete.

Definition 2

The system (1) is deterministic weakly controllable from p_0 to p_1 if for any $T>0$ and $V\in \mathcal{V}(p_1)$ there is $w\in W$ and a control $u(\cdot)\in \mathcal{U}_0$ such that $P\{x^u(T,\cdot)\in V\}>0$, where $x^u(\cdot)$ is the Ito solution in (1) corresponding to $w(\cdot)$ and $u(\cdot)$.

The conclusion in Theorem 1 requires that the conditions (a) and (b) hold everywhere in R^n.

In the case we want to pay a particular attention to some pairs p_0, $p_1\in R^n$ we need to replace conditions (a) and (b) with the following.

(*) For p_0, $p_1\in R^n$ there is a continuous curve $x(t)$, $t\in[0,1]$ connecting them ($x(0)=p_0$, $x(1)=p_1$) such that either $x(t)$, $t\in[0,\varepsilon]$, $\varepsilon\in[0,1)$, is an admissible solution for (3), or $P^0(t,p_0,V)>0$ for any $t\in[0,\varepsilon]$ and $V\in\mathcal{V}(x(\varepsilon))$.

In addition, for any $t\in[\varepsilon,1)$ there is a diffeomorphism H as was stipulated in conditions (a) and (b).

Theorem 2

The system (1) is deterministic weakly controllable from p_0 to $p_1\in R^n$ for any pair (p_0,p_1) verifying the condition (*)

The proof of this theorem follows exactly the same way as in Theorem 1 replacing initial point p_0 by $x(\varepsilon)$, and the neighbourhood for p_1 required in (a) and (b) by the arbitrarily fixed $V\in\mathcal{V}(p_1)$

Remark

From the proof of Theorem 1 and Theorem 2 it follows that if we are not interested in the controllability within arbitrary time $T>0$ (see definitions 1 and 2) then replacing "for any $T>0$" in condition (a) and definition 2 by "for some $T>0$" we get another version of Theorem 2 adequate to this case.

Examples

1) Consider a two dimensional system with a scalas Wiener process

1') $dx(t)=(Ax(t)+Bx(t))dt+Cdw(t), \quad t\geq 0, \quad u\in R, \; x=(x_1,x_2)$

where $A=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$, $B=\begin{pmatrix}0 & r\\ -r & 0\end{pmatrix}$, $r>0$, $C=\begin{pmatrix}0\\ 1\end{pmatrix}$

The corresponding subsystems (2) and (3) are

2') $dx_2(t)=x_2(t)dt+dw_t$

3') $\dfrac{dx_1}{dt}=rux_2(t) \qquad \dfrac{dx_2}{dt}=-rux_1(t)$

and they invariates manifolds $x_1=$const., $x_1^2+x_2^2=$const. correspondingly. In this example the line $L=\{x\in R^2;\; x_2=0\}$ plays a special role since the two manifolds have the same tangent space along L.

For $p_o,p_1\in R^2$, $p_i\neq 0$, there are positive constants C_o, C_1 such that

$$p_o\in S_o=\{x\in R^2: x_1^2+x_2^2=C_o\} \quad \text{and } p_1\in \; S_1=\{x\in R^2:\; x_1^2+x_2^2=C_1\}$$

In the case p_o and p_1 are in the same half space $x_2>0$ ($x_2<0$) then they are joined by a line along which the application $h_1(x_1,x_2)=x_1$, $h_2(x_1,x_2)=x_1^2+x_2^2$ is the required diffeomorphism in conditions (a) and (b) of Theorem 1.

If p_o and p_1 are on both sides of the line L then moving p_o on the circle S_o by an admissible trajectory of (3') $\tilde{x}(t)$, $t\in[0,\varepsilon]$, $\tilde{x}(0)=p_o$, $\tilde{x}(\varepsilon)=p_o'$, we can arrange that p_o' and p_1 to be on the same side of L and $p_o'\in L$. In this case the condition

(*) in Theorem 2 is fulfilled choosing as the curve $x(\cdot)$

$$x(t)=\begin{cases}\tilde{x}(t), & t\in[0,\varepsilon]\\ \frac{1-t}{1-\varepsilon}p_o'+\frac{t-\varepsilon}{1-\varepsilon}p_1, & t\in[\varepsilon,1]\end{cases}$$

Finally, if $p_o=0$ and $p_1\in S_1$ then the condition (*) in Theorem 2 holds if we take

$$x(t)=\begin{cases}\frac{t}{\varepsilon}p_o' & t\in[0,\varepsilon]\\ \left(\frac{1-t}{1-\varepsilon}\right)p_o'+\left(\frac{t-\varepsilon}{1-\varepsilon}\right)p_1 & t\in[\varepsilon,1]\end{cases}$$

where $p_o'\in S_1$ and p_1, p_o' are on the same side of L and p_o' is on the line $x_1=0$.

2) Consider a two dimensional bilinear system

1") $dx(t)=(Ax(t)+uBx(t))dt+Cx(t)dw(t)$, $t\geq 0$, $u\in R$, $w(t)\in R$.

where $A=-\begin{pmatrix} d_1 & 0\\ 0 & d_2\end{pmatrix}$, $d_i>0$, $B=\begin{pmatrix} 0 & r\\ -r & 0\end{pmatrix}$, $r>0$, $C=\begin{pmatrix} 0 & m_1\\ m_2 & 0\end{pmatrix}$,

$m_i>0$ and for some positive constants $c_1\neq c_2$ we have

$$d_1=\frac{1}{2}\frac{c_2m_2^2}{c_1}, \quad d_2=\frac{1}{2}\frac{c_1m_1^2}{c_2}, \quad c_1m_1=c_2m_2 .$$

The corresponding subsystems (2) and (3) are

2") $dx_1(t)=-d_1x_1(t)dt+m_1x_2(t)dw(t)$

$dx_2(t)=-d_2x_2(t)dt-m_2x_1(t)dw(t)$

3") $\frac{dx_1}{dt}=urx_2$, $\quad \frac{dx_2}{dt}=-urx_1$

and they invariates the manifolds $c_1x_1^2+c_2x_2^2=\text{const.}$ and $x_1^2+x_2^2=\text{const.}$ correspondingly.

For any pair p_o, $p_1\in R^2$, $p_i\neq 0$ the condition (*) in Theorem 2

The diffeomorfism required in conditions (a) and (b) is defined by $H=(h_1,h_2)$, $h_1(x_1,x_2)=c_1x_1^2+c_2x_2^2$, $h_2(x_1,x_2)=x_1^2+x_2^2$. The matrix $\frac{\partial H}{\partial x}(p)$ is singular for $p=0$, and on the lines

$L_1=\{x\in R^2 : x_2=0\}$, $\quad L_2=\{x\in R^2 : x_1=0\}$.

In order to avoid L_1 and L_2 we consider the circles S_o and S_1 centered at origine such that $p_o\in S_o$, $p_1\in S_1$ and using an admissible control we can arrange that the corresponding trajectory in (3"), will verify $\tilde{x}(0)=p_o$, $\tilde{x}(\varepsilon)=\tilde{p}_o$ for some $\varepsilon\in[0,1)$, where $\tilde{p}_o$ is on the same side of both lines L_1 and L_2 as it is p_1 and $\tilde{p}_o\notin L_1$, $\tilde{p}_o\notin L_2$. Define $x(\cdot)$ in (*) by

$$x(t)=\begin{cases}\tilde{x}(t) & t\in[0,\varepsilon]\\[2ex] \frac{1-t}{1-\varepsilon}p_o+\frac{t-\varepsilon}{1-\varepsilon}p_1 & t\in[\varepsilon,1]\end{cases}$$

It is obvious that the rank condition for (3") required in (b) is fulfilled

It remains to be proved that (2") fulfils condition (a). Since (2") satisfies the conditions in [6] and [7] it follows that the measures generated by a solution $x(\cdot)$ of (2") $x(0)=x_o$, has the property supp $\mu_t(\cdot)=\{x\in R^2 : c_1x_1^2+c_2x_2^2=c_1x_{10}^2+c_2x_{20}^2\}$ for $t>0$, where $\mu_t(dx)$ is the measure generated by $x(t,\cdot)$.

Therefore $P^o(t,x_o,D)>0$ for any $t>0$ if $D\cap \text{supp}\,\mu_t(\cdot)\neq\emptyset$, and D open, where $P^o(t,x_o,D)=P\{\omega\in\Omega : x(t,\omega)\in D,\ x(0,\cdot)=x_o$ $x(\cdot)$ solution in (2")$\}$.

In conclusion the system (1") fulfils the conditions in Theorem 2 for any $p_o,p_1\in R^2$ with $p_o\neq 0$.

R E F E R E N C E S

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