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Foundind the classical mechanics and the special theory of relativity on the principle of inertia

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To the memory of Iulian Popovici

1. Introduction

In this paper we try to show that the principle of inertia when explicitly formulaţed from a mathematical point of view assure us that the universe of events has a structure of analytical manifold generated by an atlas, the coordinate transformations between the charts of this atlas being homographies. Under stronger hypothesis (for collineations of class C^3), the coordinate transformation between two inertial reference systems was proved to be a homography (for instance, see [6] or the original proof in [8]). In our treatment we drop out any continuity hypothesis, as we first made in [2]. Moreover in this paper we characterize homographies whose domains are not open in $\mathbb{R}^m$ (see 1.1). This is essentially used (in § 3, especially 3.13) in order to obtain the differentiable structure of the universe of events. Thus on the universe of events M, the principle of inertia leads to a differentiable structure; from which either the Galilei structure of classical mechanics or the Weyl structure of the special theory of relativity were derived (see § 3). (The problem of using more "physical" postulates in order to deduce that M has a differentiable manifold structure was treated, in the case of the general theory of relativity, for example in [5], [9]).

This paper is dedicated to the memory of our late friend and teacher in geometry, dr. Iulian Popovici.

Let U be a subset in $\mathbb{R}^m$. A function $F: U \longrightarrow \mathbb{R}^m$ is called a lineation (or a collineation) if F maps any three collinear points of U in collinear points.

For any point $x \in \mathbb{R}^m$ we consider the set $C_x = \{y; Q(x-y) > 0\}$ where

(1.0) $Q(z)=(z^1)^2-(z^2)^2- \dots -(z^m)^2,$

for $z=(z^1,\dots,z^m)$ in the standard coordinate system of $\mathbb{R}^m$. A straight line (line, for short) l is named a <u>time line</u> if there exists $x \in l$ s.t. $l \subset C_x \cup \{x\}$. The line determined by two points $y, z \in \mathbb{R}^m$ will be denoted by yz. The segment determined by two points $x, y \in \mathbb{R}^m$ is denoted by $[x,y]$. An affine 2-plane (plane, for short), resp. k-plane with $k > 2$, which contains a time line is named a <u>time plane</u>, resp. <u>time k-plane</u>.

A function $F: U \longrightarrow \mathbb{R}^m$ is called a <u>partial lineation</u> if for any time line d, F maps any three points from a connected subset of $d \cap U$ in collinear points.

On $\mathbb{R}^m$ we consider the euclidean topology.

In order to introduce a differentiable structure on the universe of events and to derive the Galilei and Weyl groups we prove the following

1.1. Theorem. <u>Let</u> $m \geqslant 2$, <u>let</u> U <u>be a subset of</u> $\mathbb{R}^m$ <u>s.t. the intersection of</u> U <u>with any time plane</u> V <u>is open in</u> V <u>and let</u> $F: U \longrightarrow \mathbb{R}^m$ <u>be an injective partial lineation. Suppose that:</u> U <u>is connected by segments; there exist a time plane</u> T <u>and three non-collinear points</u> $v_0, v_1, v_2 \in T \cap U$ <u>s.t.:</u> $F(v_0), F(v_1), F(v_2)$ <u>are not</u>

collinear; v_0, v_1, v_2 are contained in a connected component T^o of $T \cap U$.

Then F is the restriction to U of a homography.

(1.2 A function $H: \mathbb{R}^m \setminus E \longrightarrow \mathbb{R}^m$ is called a homography if

$$H(x)^i = \left(\sum_{j=1}^{m} H_{i,j} x^j + H_{i,m+1}\right) / \left(\sum_{j=1}^{m} H_{m+1,j} x^j + H_{m+1,m+1}\right),\ i=1,\ldots,m,$$

where: $H_{i,j} \in \mathbb{R}, i,j=1,\ldots,m+1$ and $x=(x^1,\ldots,x^m) \in \mathbb{R}^m \setminus E$; E is the empty set or the hyperplane of $\mathbb{R}^m$ having the equation $\sum_{j=1}^{m} H_{m+1,j} x^j + H_{m+1,m+1} = 0$; the rank of the matrix $H_{i,j}$ is m+1).

1.3. Remark. There exist subsets U as in 1.1 but which are not open in the euclidean topology. For instance take a ball B of centre x and take a sequence $Z=\{z_n; n \in \mathbb{N}\}$ which converges to x s.t. $xz_n, n \in \mathbb{N}$ are distinct time lines and any time plane contains only a finite number of them. Then $B \setminus Z$ is not open but has the properties of the set U from 1.1.

Theorem 1.1 is proved in § 2. An axiomatic presentation of the principle of inertia is given in § 3; it leads to a structure of analytic manifold for the universe of events. Weaker conditions which imply Galilei or Weyl group are given. A generalization of theorem 1.1 is proved in § 4. Some physical implication of our treatment of the principle of inertia are sketched in § 3. They are made by D.C. Rădulescu.

2. The proof of Theorem 1.1.

2.1. For U as in 1.1 and for any line d of $\mathbb{R}^m$, the set $d \cap U$ is open in d because there is a time plane including d.

To prove 1.1 we need some preliminaries.

2.2. Lemma. Let N be a subset of $\mathbb{R}^m$ and let $F: N \longrightarrow \mathbb{R}^m$ be an injective function. Let $[x,w] \subset N$. Suppose that in the vertical (i.e. parallel with the x^1 axis) time plane V through x and w the set $V \cap N$ contains an open set Z s.t. $[x,w] \subset Z$. If F is a partial lineation on Z, then F is a lineation on $[x,w]$.

Particularly if $F: U \longrightarrow \mathbb{R}^m$ is as in 1.1, then F is a lineation on any segment $[x,w] \subset U$ and on any convex subset W of U.

Proof. Put d=xw. Let s be an open segment of $d \cap Z$ s.t. $[x,w] \subset s$.

For any $v \in s$ we prove that there exist $a,c \in s$ s.t. $v \in (a,c)$ and $F([a,c])$ is contained in a line. Indeed if d is a time line this is obvious. Let d be not a time line. Let e be the parallel to the x^1 axis through v. Then V is generated by d and e. Let f be the line of V orthogonal to e in v and let e_1, resp. f_1, be a half-line of e, resp. f, with v as origin. In V we consider the oriented axes f_1, e_1. Let $r > 0$ s.t. the open disk $B(v,4r) \subset Z$.

Suppose that the slope of d is positive. Take c on d (resp. a on d) s.t. the f_1 coordinate of c (resp. a) is r (resp. -r). Take y to be the point (r,3r). Then the disk $B(y,r/2)$ is included in $B(v,4r)$ and in the part of C_a which meets e_1. Let $a_1, a_2 \in (y,c)$ s.t. $a_2 \in (y,a_1)$ and $a_1 \in B(y,r/2)$. Choose a time line $g \subset V$ s.t. the slope of g be negative and $y \in g$. Let $b \in (a,c)$. Put $\{b_i\} = ba_i \cap g$ and $\{c_i\} = aa_i \cap cb_i$ with i=1,2. It results that $b_i, c_i \in B(y,r/2)$ for i=1,2. By Desargues' theorem it follows that $y \in c_1c_2$. It is easy to check that a_1a_2, b_1b_2, c_1c_2 and aa_i, bb_i, cc_i with i=1,2, are time lines. If $z \in N$, put $z' = F(z)$. Since F is an injective partial lineation on Z it follows that any three collinear points from $a,b,c,a_1,b_1,c_1,a_2,b_2,c_2,y$, excepting a,b,c have collinear images. If the sets given by the intersection of Z with the distinct lines aa_i, bb_i, cc_i, with i=1,2, and a_1a_2, b_1b_2, c_1c_2 have the images pm distinct nine lines, then

Desargues' theorem shows that a',b',c' are collinear. If any two of the above nine sets have the images on the same line, we see that a',b',c' are also collinear by a straight forward investigation of the following possibilities: a'_i,b'_i,c'_i, are collinear for a i; two of the above lines through y' coincide; two lines through a' or b' or c' coincide; a line through a' and a line through y' coincide.

If the slope of d is negative, change f_1 with its opposite half-line.

Now cover $[x,w]$ with a finite number of open segments $s_i \subset s$, s.t. $F(s_i)$ is contained in a line; hence $F([x,w])$ is contained in aline.

Let U,F be as in 1.1. Let $[x,w] \subset U$. The intersection of a vertical time plane through x and w with U is open. Hence F is a lineation on $[x,w]$, by the first part of 2.2. If W is a convex subset of U, then F is a lineation on W since F is a lineation on any segment of W.

<u>Remark</u>. The first conclusion of 2.2 holds also if the considered time plane V is not vertical (by a slight modification of the proof).

2.3. Proposition. <u>Let</u> W <u>be an open set in</u> $\mathbb{R}^2$ <u>and let</u> $f: W \longrightarrow \mathbb{R}^2$ <u>be an injective lineation. If</u> f(W) <u>is not contained in a line, then</u> f <u>is the restriction to</u> W <u>of a homography</u>.

A direct proof of 2.3 was given in [2]. A generalization of 2.3 to open subsets of planes over ordered fields is given in [3]; its proof is an adaptation to open subsets of the proof given in [4], Theorem 3.3.

2.4. Let N be a subset of $\mathbb{R}^m$ and A_r be an affine r-plane in

$\mathbb{R}^m$ s.t $A_r \cap N \neq \emptyset$; we say that a function $F: N \longrightarrow \mathbb{R}^m$ is a t-lineation on $Z := A_r \cap N$ if F is a lineation on Z and F(Z) generates an affine t-plane. Let F be an r-lineation on $Z := A_r \cap N$; we say that F is an r-homography on Z if Z generates A_r and F acts on Z as the restriction to Z of a homography of $\mathbb{R}^m$.

In 2.5– 2.8, N is a subset of $\mathbb{R}^m$ and $F: N \longrightarrow \mathbb{R}^m$ is an injective function.

2.5. If for a plane A the set $A \cap N$ contains a non-empty set W which is open in A and the function F is a 2-lineation on $A \cap N$, then F is a 2-homography H on $A \cap N$.

Indeed F is a 2-homography on W, by 2.3. Since F is an injective lineation, the image of any point z of $A \cap N$ is determined by the action of F on W (by taking two distinct lines through z which meet W). Moreover from $z \in A \cap N$ it follows $z \in \mathrm{dom}\, H$ (this can be seen by extending H to the projective envelope P^m of $\mathbb{R}^m$; see also the proof of 2.11). It follows that: for any $C \subset A \cap N$ which is open in A, then F(C) is an open set in the affine 2-plane generated by $F(A \cap N)$; for any line l in A with $l \cap N \neq \emptyset$, F is an 1-homography on $l \cap N$.

2.6. Let A be a plane and l_1, l_2 two distinct lines of A which meet in $x \in N$ s.t. $l_i \cap N$ contains an open segment s_i of l_i and $x \in s_i$ for i=1,2. If F is a lineation on $A \cap N$ then F is an 1-or 2-lineation on $A \cap N$.

Indeed, by hypothesis $F(l_i \cap N)$ generates a line l'_i for i=1,2, Let A' be the affine plane determined by l'_1, l'_2. For any point $y \in A \cap N$ there exists a line l s.t. $l \cap l_i = \{y_i\} \neq \emptyset$, $y_i \in s_i$, i=1,2 and $y_1 \neq y_2$. It follows $F(y) \in A'$. If l'_1, l'_2 coincide (respectively are distinct) then A' is an one (resp.two) dimensional affine plane.

2.7. Let A, l_1, l_2, s_1, s_2 be as in 2.6. If F is an 1-homography on s_i and a lineation on $l_i \cap N$ for $i=1,2$, then $F(l_1 \cap N)$ and $F(l_2 \cap N)$ generates an affine plane A'.

Indeed, suppose A' is a line. Hence $F(s_1) \cap F(s_2)$ is an open set of the line generated by $F(l_1 \cap N)$ and $F(l_2 \cap N)$. But $F(s_1) \cap F(s_2) = F(s_1 \cap s_2) = \{F(x)\}$ since F is injective; contradiction. Hence A' is a plane.

2.8. Let $x \in N$ and H_1, H_2 be two homographies of $\mathbb{R}^m$. Suppose that for any line l through x there is a subset l^x of $l \cap N$, s.t.: l^x contains x and at least two other distinct points; H_1 and H_2 coincide on l^x. Then $H_1 = H_2$.

Indeed, let $v_o = x$ and $v_1, \ldots, v_m$ be affinely independent with $v_i \in (v_o v_i)^x$. Let $w_1 = (v_o + v_1 + \ldots + v_{m+1})/(m+1)$. Take $w \in (v_o w_1)^x$ with $w \neq x$ and w is not in the hyperplane given by $v_1, \ldots, v_m$. Then $v_o, \ldots, v_m, w$ are in general position; since H_1 and H_2 coincide on them, it results that $H_1 = H_2$.

2.9. Lemma. <u>Let V be an open connected set in a plane and let $F: V \longrightarrow \mathbb{R}^m$ be an injective function. Suppose that for any $x \in V$, there is a convex open subset C of V with $x \in C$ s.t. F is a lineation on C. Then F is an 1-lineation or a 2-lineation on V.</u>

<u>Proof</u>. Let V_1, V_2 be convex open subsets of V s.t. $V_3 := V_1 \cap V_2 \neq \emptyset$ and F is a lineation on V_1 and V_2. Hence V_3 is convex. We prove that F is a lineation on $V_1 \cup V_2 =: W$. Take $z \in V_3$ and the segments $s_i \subset V_3$ with $z \in s_i$, for $i=1,2,3$. Denote by F_j the restriction of F to V_j, where $j=1,2,3$; F_1, F_2, F_3 are lineations. We distinguish two cases:

a) $F_3(s_i)$ are contained in the same line d for $i=1,2,3$. Since F_1, F_2, F_3 are injective lineations it is clear that for any $y \in W$ we have $F(y) \in d$.(Indeed take a line $l \ni y$, $l \not\ni z$ s.t. $l \cap s_i \neq \emptyset$ for at least two indices $i=1,2,3$.It results $F(l \cap V) \subseteq d$.Thus F is an 1-lineation on W.

b) The sets $F_3(s_i)$, $i=1,2,3$,are not contained in the same line.Then they are contained in a plane by 2.6.Then, by 2.3,F_1,F_2,F_3 are restrictions of 2-homographies to their respective domains.Since then 2-homographies coincide on V_3, they are equal.

Let $V_1,\dots,V_n$ be open convex subsets of V s.t. F is a lineation on $V_1,\dots,V_n$.If $W:=V_1 \cup \dots \cup V_n$ is connected, then it follows by induction on n that F is a lineation on W.

Finally let x,y,z be any distinct collinear points of V.Take the continuous path $g \subset V$,resp. $h \subset V$,connecting x and y,resp. y and z.There exists a finite number of convex open sets $V_1,\dots,V_n$ in V which cover g and h and F is a lineation on $V_1,\dots,V_n$.By above,F is a lineation on $V_1 \cup \dots \cup V_n$.Thus F(x),F(y),F(z) are collinear.

2.1o.Remark.Let U,F be as in 1.1 and A be a (time) plane. Then F is a 1-or 2-lineation on any connected open subset of $A \cap U$.This results from 2.2, 2.6 and 2.9.

For any $x \in \mathbb{R}^m$ we define a time-x-connected set U_x to be a set constructed in the following way: for any time plane A containing x,take a connected open set $A^x \subset A$ with $x \in A^x$;U_x is the union of these sets A^x.

2.11.Theorem.Let U_x be a time-x-connected set.Let $F: U_x \longrightarrow \mathbb{R}^m$ be an injective function s.t.: for any time plane A through x,F is a lineation on A^x;there exists a time plane V through x for which F is a 2-lineation on V^x.Then F is an m-homography on U_x.

In order to prove 2.11, firstly we prove two Lemmas.

2.12. Lemma. *Let U_x and F be as in 2.11. Then for any time plane A through x the function F is a 2-homography on $A \cap U_x$. Thus for any line l through x, F is an 1-homography on $l \cap U_x$.*

Proof. From 2.3 it follows that F is a 2-homography on V^x. Now for any A^x we distinguish two cases:

(i) $A \cap V$ is a line l (which contains x). It results that F is an 1-homography on an open segment s of l which contains x. Suppose that F is an 1-lineation on A^x. Then choose the time lines $l_1 \subset V, l_2 \subset A$, s.t. $x \in l_1, l_2$. Let W be the time plane generated by l_1, l_2. It results that F is a 2-lineation on W^x. Indeed since F is a 1-lineation on A^x it follows that $F(l_2 \cap A^x)$ is included in the line l' determined by $F(l \cap A^x)$. But l' is distinct from the line determined by $F(l_1 \cap V^x)$ since F is a 2-homography on V^x. Thus F is a 2-lineation on W^x; hence, by 2.3, F is a 2-homography on W^x. Consequently F is an 1-homography on an open segment s_2 of $l_2 \cap A^x$ which contains x. By 2.7 for F, s and s_2, it follows that $F(A^x)$ is not included in a line; contradiction. Hence F is a 2-lineation on A^x. Then F is a 2-homography on A^x, by 2.3.

(ii) $A \cap V = \{x\}$. Let l be a line of A through x. Let l_1 be a time line of V through x and B the plane generated by l and l_1. Case (i) for B and V shows that F is a 2-homography on B^x. By case (i) for A and B it results that F is a 2-homography on A^x (for any time plane through x).

Now for any line l containing x, F is an 1-homography on $l \cap U_x$. Namely, let $y \in l \cap U_x$ and let A be any time plane s.t. $y \in A^x$. By above, F is an 1-homography on $l \cap A^x$. The actions of these homographies on l do not depend on A and y since they coincide on an open segment of l containing x.

Finally for any time plane A containing x, F is a

2-homography on $A \cap U_x$. Indeed, let h be a homography of $\mathbb{R}^m$ which acts as F on A^x. Let $y \in A \cap U_x$. Let g be a homography of $\mathbb{R}^m$ which acts as F on $xy \cap U_x$. Since g,h coincide on $xy \cap A^x$ they coincide on xy. Hence $h(y)=g(y)$. Since $g(y)=F(y)$ it follows that h acts as F on $A \cap U$.

2.13. Lemma. *Let F and U_x be as in 2.11. For $v \in U_x$, put $v'=F(v)$. Then for any $r \in \{1,\dots,m\}$ and for any time r-plane A_r through x it follows:*

(i,r) *$F(A_r \cap U_x)$ generates and affine r-plane A'_r*

(ii,r) *For any line l of A'_r through x', the set $l \cap F(A_r \cap U_x)$ contains at least two distinct points, hence, by 2.12 it contains a non-empty open subset of l.*

Proof. Induction on r. For r=1 and r=2 apply 2.12. Let $r \geqslant 3$ and suppose that 2.13 is true for r-1. Let A_r be a time r-plane through x. Since A_r is a time r-plane there is v_1 in A_r s.t. xv_1 is a time line. Let $a_2,\dots,a_r$ in A_r s.t. $x,v_1,a_2,\dots,a_r$ are affinely independent. By hypothesis, in the time plane generated by x,v_1,a_i, there is a point v_i in U_x s.t. xv_i is a time line distinct from xv_1 and $[x,v_i] \subset U_x$, for $i=2,\dots,r$. Then $x,v_1,v_2,\dots,v_r$ are affinely independent. Hence $x,v_1,\dots,v_{r-1}$ generate a time (r-1)-plane A_{r-1}. By (i,r-1) the set $F(A_{r-1} \cap U_x)$ generates an affine (r-1)-plane A'_{r-1} and satisfies (ii,r-1).

Now, we prove (i,r). Namely, if $v'_r \in A'_{r-1}$, then the line $x'v'_r$ is generated by x' and $z' \in F(A_{r-1} \cap U_x)$ with $z \in A_{r-1} \cap U_x$. But the points x,v_r,z are not collinear. From 2.12 it follows that x',v'_r,z' are not collinear; contradiction. Hence $v'_r \notin A'_{r-1}$. Let A'_r be the affine r-plane generated by A'_{r-1} and v'_r. Let $u \in A_r \cap U_x$. If $u \in A_{r-1}$ then $u' \in A'_{r-1}$.

by 2.12. If $u \notin A_{r-1} \cup xv_r$ let W be the plane given by x, v_r, u. Since F is a 2-homography on $W \cap U_x$ it follows that u' is contained in the plane generated by x', v_r' and $F(W \cap A_{r-1} \cap U_x)$. Hence $u' \in A_r'$ and (i,r) is proved.

To prove (ii,r), observe first that (ii,r) is verified for $x'v_r'$, by 2.12.

Now let l be a line of A_r' with $x' \in l, v_r' \notin l$. The plane B generated by l and v_r' is contained in A_r', hence its intersection with A_{r-1}' is a line k. By (ii,r-1) there is v in A_{r-1} with $v \neq x$ and v' in k. Let A be the plane given by x, v_r and v. By 2.12 F is a 2-homography on $A \cap U_x$. Hence $F(A \cap U_x)$ includes a set Z which contains x' and is open in B. Thus $l \cap Z$ is a non-empty open set of l.

<u>Proof of 2.11.</u> For any $r \in \{1,\dots,m\}$ and for any $v_0,\dots,v_r$ affinely independent in $\mathbb{R}^m$, recall that the set

$$S_r = \{y; y = \sum_{i=0}^{r} t_i v_i, \sum_{i=0}^{r} t_i = 1, 0 < t_j < 1, v_j \in \mathbb{R}^m, j = 0,\dots,r\}$$

is named the r-simplex of vertices $v_0,\dots,v_r$ and is denoted also by $(v_0,\dots,v_r)$.

We prove by induction on $r \in \{1,\dots,m\}$ the following property

(a,r) For any time r-plane A_r containing x there exist the points $v_0 = x, v_1,\dots,v_r \in A_r$ and a homography H_r of $\mathbb{R}^m$ s.t.: $[v_0,v_i] \subset A_r \cap U_x$ and v_0v_i are time lines for $i=1,\dots,r$; the restrictions of F and H_r to $A_r \cap U_x$ coincide. (<u>Remark.</u> From (a,r) it follows that dom $H_r \supset (v_0,\dots,v_r)$. Indeed if H_r has a hyperplane E (in $\mathbb{R}^m$) of singularities, then E does not meet $[v_0,v_i]$ for $i=1,2,\dots,r$ since H_r and F coincide on $A_r \cap U_x$. Hence E does not meet $(v_0,\dots,v_r)$.)

The assertions (a,1) and (a,2) were proved in 2.12.

For any $z \in U_x$, put $z'=F(z)$. For $r \geqslant 2$ suppose (a,r) is true. Then we show that $(a,r+1)$ is also true. Thus for $r=m$, (a,m) proves 2.11.

As in the proof of 2.13 we choose $v_o=x, v_1, \ldots, v_{r+1} \in A_{r+1} \cap U_x$ s.t.: $[v_o, v_i] \subset A_r \cap U_x$ and $v_o v_i$ are time lines for $i=1,\ldots,r+1$. Take $w_r \in (v_o,\ldots,v_r) \cap U_x$ s.t. $v_o,\ldots,v_r,w_r$ are in general position (This is possible since for any time line $l \ni x$ we have that $l \cap U_x$ is non-empty and open in l, by 2.13 (ii,r)). In the affine 2-plane B generated by v_o, v_{r+1}, w_r, change eventually the point v_{r+1} on (v_o, v_{r+1}) and w_r on (v_o, w_r) s.t. $[v_{r+1}, w_r]$ be included in B^x and take $w \in (v_{r+1}, w_r)$. It follows that $v_o,\ldots,v_{r+1},w$ are in general position. Moreover $w' \in (v'_{r+1}, w'_r)$ by $(a,2)$ applied to B. Now take a homography H_{r+1} with: $H_{r+1}(v_i)=v'_i$, $i=,\ldots,r+1$, $H(w)=w'$.

If H_{r+1} is an affine transformation then $w_r \in \text{dom } H_{r+1}$. If H_{r+1} has a hyperplane E of singularities (see 1.2) then $w_r \notin E$. Indeed observe that $H_{r+1}(A_r \setminus E) \subseteq A'_r$ (see the definition of H_{r+1}). If $w_r \in E$ then the line $w'v'_{r+1}$ which contains $H_{r+1}(wv_{r+1} \setminus \{w_r\})$ must be parallel with A'_r (this can be seen extending H_{r+1} to the projective envelope of $\mathbb{R}^m$; this extension maps E on the hyperplane to infinity in its codomain). But F is a lineation on A_r and wv_{r+1} by (a,r) $(a,1)$. Hence $w'_r \in A'_r$ and so $w'v'_{r+1}$ is not parallel with A'_r. Thus $w_r \in \text{dom } H_{r+1}$ and $H_{r+1}(w_r)=w'_r$.

From (a,r) we have $w_r \in \text{dom } H_r$ and $w'_r=H_r(w_r)$. Therefore, F, H_{r+1} and H_r coincide on $A_r \cap U_x$. Hence F and H_{r+1} coincide also on $B \cap A_r \cap U_x \neq \emptyset$. Moreover H_{r+1} coincides with F also on $v_{r+1}w \cap U_x$ since F and H_{r+1} are 1-homographies on $v_{r+1}w \cap U_x$ and their actions coincide in the three distinct points v_{r+1}, w and w_r. Since F and H_{r+1} are 2-homographies on $B \cap U_x$ and they coincide on $v_{r+1}w \cap U_x$ and $B \cap A_r \cap U_x$ it results that F and H_{r+1} coincide on $B \cap U_x$, particularly on $v_{r+1}v_o \cap U_x$.

Now take any point z from $(A_{r+1} \cap U_x) \setminus (B \cup A_r)$. Denote by A the plane generated by v_o, v_{r+1} and z. By 2.13 it follows that $A \cap A_r$ contains an open interval s on a line $l \subset A_r$. Since F is a 2-homography on $A \cap U_x$ by (a,2), and F and H_{r+1} coincide on $v_o v_{r+1} \cap U_x$ and s, it results $H_{r+1}(z)=z'$.

<u>Remark</u>. In 2.11 we can replace: "for any time plane A through x, F is a lineation on A^x", by "F is a partial lineation on U_x". Indeed using the remark after 2.2 and 2.9, if F is a partial lineation it results that F is a lineation on any A^x.

<u>Proof of theorem 1.1</u>. Let $x \in U$. If A is a time plane through x let A^x be the connected component of $A \cap U$ which contains x. Then F is a lineation on A^x, by 2.1o. Put $U_x = \bigcup A^x$, where A runs over the time planes through x. Then U_x is a time-x-connected set. Let $x \in T^o$; since F is a 2-lineation on $T^x=T^o$, from 2.11 it follows that F acts as a homography H on U_x. Let $y \in U$ s.t. $[x,y]$ is a temporal segment included in U. By above F is a 2-homography on any plane containing x,y; then F acts as a homography G on U_y, by 2.11. For any time plane A through x and y, H and G coincide with F on $A^x=A^y$. Then $H=G$ by 2.1 and 2.8. Let $z \in U$. Since U is connected by segment, let $z_o=x, z_1, \dots, z_n=z$ s.t. $[z_i, z_{i+1}] \subset U$, for $i=0,\dots,n-1$. Taking the vertical plane A_i which contains $[z_i, z_{i+1}]$ we can join z_i and z_{i+1} by a finite number of temporal segments included in $A_i \cap U$, since this set is open in A_i. Hence renumbering the points we can suppose that $[z_i, z_{i+1}]$ are temporal segments, for $i=0,\dots,n-1$. By above F acts as H on U_{z_1}; by induction it

results that F acts as H on U_{z_n}. Hence $F(z)=H(z)$. Thus F coincides with H on U.

3. Deducing the differentiable structure of the universe of events, the Galilei and Weyl groups.

3.1. Let M be a set; its elements are called events. Let P be a family of subsets of M; the elements of P are named particles. The family P satisfies the following axioms:

A1. For any $x \in M$, there is $p \in P$ with $x \in p$. For any $p \in P$, the set p has at least two elements.

A2. If $x,y \in M, x \neq y$, and $p,q \in P$ s.t. $x,y \in p,q$, then $p=q$.

A3. For any $p \in P$ and for any $x,y \in p$ there is defined a subset $[x,y]$ of p, called segment, s.t.:

$[x,y] = [y,x]$; $x,y \in [x,y]$; $[x,x] = \{x\}$

if $a,b \in [x,y]$, then $[a,b] \subseteq [x,y]$

let $a,b,c,d \in p$, s.t. $[b,c] \cap [a,d] \neq \emptyset$: if $b,c \notin [a,d]$ then $[a,d] \subset [b,c]$; if $b \in [a,d]$ and $c \notin [a,d]$, then either $[b,d] \subseteq [a,c]$ and $[b,c] \cap [a,d] = [b,d]$, or $[a,b] \subset [c,d]$ and $[b,c] \cap [a,d] = [a,b]$.

The set $[x,y] \setminus \{x,y\}$ is denoted (x,y).

A4. If $p \in P$ and $x,y,z,t \in p$ s.t $(x,y) \cap (z,t) \neq \emptyset$, then $[x,y] \cup [z,t]$ is a segment.

3.2. Definition. A subset N of M is called quasi-open if:

QO1. For any x in N and any $p \in P$ with $x \in p$, there are $a,b \in p \cap N$ s.t. $x \in (a,b) \subseteq N$.

QO2. For any $x \in N$ and $p,q \in P$ with $x \in p,q$ and $p \neq q$, there exists $[a,b] \subseteq p \cap N$ s.t. $x \in (a,b)$ and for any $[c,d] \subseteq [a,b]$ with $x \in (c,d)$, there is $[e,f] \subseteq q \cap N$ with

$x\in(e,f)$ s.t. for any $z\in[e,f]$, it follows that c and z, resp. d and z, belong to a particle and $[c,z],[d,z]\subseteq N$.

Note that the empty set is quasi-open.

3.3. Remark. If N_1, N_2 are quasi-open, then $N_1\cap N_2$ is quasi-open.

Indeed, suppose that $N:=N_1\cap N_2$ is not empty. Let $x\in N$ and $p\in P$ with $x\in p$. By QO1, there are $a_i, b_i\in p\cap N_i$ s.t. $x\in(a_i,b_i)\subseteq N_i$, for $i=1,2$. By A3 it follows that $[a_1,b_1]\cap[a_2,b_2]=[a,b]$ and $x\in(a,b)$; thus QO1 is satisfied for N,x,p. Let $x\in N$ and $p,q\in P$ with $x\in p,q$ and $p\neq q$. Let $[a_i,b_i]\subset p\cap N_i$, given by QO2 for x,p,q,N_i, for $i=1,2$. Then $[a_1,b_1]\cap[a_2,b_2]=[a,b]$, by A3. It is easy to see that $[a,b]$ is the interval on p necessary in QO2 for x,p,q,N.

3.4. Remark. Let N be quasi-open, let $x\in N$ and let N_x be the set of points $y\in N$, s.t. there exist $z_0=x, z_1,\dots,z_n=y$, with $[z_1,z_{i+1}]$ a segment contained in N, for $i=0,\dots,n-1$. Then N_x is quasi-open. (The proof is straight forward).

The set N_x is called the segment-connected component of x in N.

3.5. Definitions. Let N be a subset of M.

(i) An injective function $h: N\longrightarrow\mathbb{R}^4$ is called a linear chart if:

C1. For any $p\in P$ and for any $x,y,z\in p\cap N$ the points hx, hy, hz are collinear.

(ii) A linear chart $h: N\longrightarrow\mathbb{R}^4$ is called preinertial if:

C2. N is quasi-open.

C3. For any segment $[x,y]\subset N$, it follows $h([x,y])=[hx,hy]\subset\mathbb{R}^4$. (When no confusion can appear we denote $h(x)$ by hx.)

C4. For any $x \in N$ and for any temporal line d through hx, there is a particle p s.t. $x \in p$ and $h(p \cap N) \subseteq d$.

3.6. Lemma. Let $h: N \longrightarrow \mathbb{R}^4$ be a preinertial chart. Let $x,y \in N$, $x \neq y$ and put $a=hx$, $b=hy$. If $[a,b]$ is included in $h(N)$ and ab is a time line, then there is a particle p with $x,y \in p$ and $[x,y] \subset N$. If d is a time line in $\mathbb{R}^4$, then $d \cap h(N)$ is open in d.

Indeed, let $c \in [a,b]$ and $c' \in N$ with $hc'=c$. By C4, there exists a particle p_c through c' s.t. $h(p_c \cap N) \subseteq ab$. By QO1, there are distinct $x_c, y_c \in p_c \cap N$ with $c' \in (x_c,y_c) \subset N$. By C3, $h([x_c,y_c]) = [hx_c,hy_c] \subset ab$. The set $[a,b]$ is compact and (hx_c,hy_c) with $c \in [a,b]$ is an open covering of it; hence there are $c_0=a, c_1, \ldots, c_n=b$, s.t. (hx_{c_i}, hy_{c_i}) with $i=0,\ldots,n$ cover $[a,b]$. By A2 it follows that $p_{c_0}, p_{c_1}, \ldots, p_{c_n}$ are the same particle p and $[x,y] \subset p$. By A4 it follows that $[x,y] \subset N$. The last assertion results from C4, QO1 and C3.

Remark. In 3.7, 3.8 we use C4 in the weaker form.

C4'. For any $x \in N$ and for any time plane A through hx, there are two time lines d_1, d_2 of A through hx and two particles p_1, p_2 through x s.t. $h(p_i \cap N) \subseteq d_i$.

3.7. Remarks. Let $h: N \longrightarrow \mathbb{R}^4$ be a preinertial chart

(i) Let $N' \subset N$ be quasi-open. Then $h': N' \longrightarrow \mathbb{R}^4$ with $h'x=hx$ for any $x \in N'$ is (obviously) a preinertial chart.

(ii) The intersection of h(N) with any time plane A of $\mathbb{R}^4$ is open in A, hence with any line d is open in d. Indeed, let $y \in h(N) \cap A$. Let $x \in N$ with $hx=y$. Let d, g be distinct time lines of A through y. By C4 let p, q be particles through x with $h(p \cap N) \subseteq d$ and $h(q \cap N) \subseteq g$. Let $[a,b] \subset p \cap N$ be the segment given by QO2 for N, x, p, q. Let $[e,f] \subset q \cap N$ be the segment associated to $[a,b]$ in QO2. Using QO2 and C3 it results that the convex domain with vertices ha, he, hb, hf is included in $h(N) \cap A$ and its interior contains y.

3.8. Lemma. Let $h_i : N_i \longrightarrow \mathbb{R}^4$ be linear charts for $i=1,2$, s.t.: h_1 is a preinertial chart; h_2 verifies C2 and C3; $N' := N_1 \cap N_2$ is not empty. Let N be the segment-connected component of $x \in N'$. Then the function $F : h_1(N) \longrightarrow \mathbb{R}^4$ defined by $F(y) = h_2 h_1^{-1}(y)$ for any $y \in h_1(N)$ is the restriction of a homography to $h_1(N)$.

Indeed, N is quasi-open by 3.3 and 3.4. By 3.7, the intersection of $h_1(N)$ with any time plane A is open in A. By C3, $h_1(N)$ is connected by segments. The function F is injective. Let d be a time line and let a_1, a_2, a_3 be three points of a connected subset $[a,b]$ of $d \cap h_1(N)$. By 3.6, there is a particle p s.t. $h_1^{-1}(a_i) \in p \cap N$, for $i=1,2,3$. Then $F(a_1), F(a_2), F(a_3)$ are collinear by C1 for h_2. Hence F is a partial lineation. Let T be a time plane with $T' := T \cap h_1(N) \neq \emptyset$. Let $a \in T'$ and d_1, d_2 two time lines of T through a. Let $[a_i, b_i] \subset T' \cap d_i$ with $a \in (a_i, b_i)$, for $i=1,2$. Let $x \in N$ with $h_1(x) = a$. By 3.6, there is a particle p_i through x and $[x_i, y_i] \subset p_i \cap N$ with $x \in (x_i, y_i)$ and $h_1(x_i) = a_i$, $h_1(y_i) = b_i$, for $i=1,2$. By C3 it follows that $h_2([x_i, y_i]) = [h_2(x_i), h_2(y_i)]$, for $i=1,2$. Since h_2 is injective, it follows that $h_2(x_1)h_2(y_1) \neq h_2(x_2)h_2(y_2)$. Hence $h_1(N)$, F verify the hypotheses of 1.1.

3.9. Definition. A preinertial chart $h : N \longrightarrow \mathbb{R}^4$ is called open if $h(N)$ is open in $\mathbb{R}^4$.

3.10. Lemma. Let $h_i : N_i \longrightarrow \mathbb{R}^4$ be open preinertial charts for $i=1,2$. If $h_1(N_1 \cap N_2)$ is open in $\mathbb{R}^4$, then $h_2(N_1 \cap N_2)$ is also open.

Proof. Let $b \in h_2(N_1 \cap N_2)$. Let $x \in N_1 \cap N_2$ with $h_2 x = b$. Put $a = h_1 x$. Let N be the segment-connected component of x in $N_1 \cap N_2$ and $F : h_1(N) \longrightarrow \mathbb{R}^4$ defined by $F(c) = h_2 h_1^{-1}(c)$.

Then F is the restriction of a homography H, by 3.8. Moreover $h_1(N)$ is open in a. Indeed, we can suppose that $a=(0,0,0,0)$ and let $B(a,r) \subseteq h_1(N_1 \cap N_2)$ be a closed ball of centre a. Let C be the intersection of $B(a,r)$ with the cylinder $(x^2)^2+(x^3)^2+$ $+(x^4)^2=s^2$ for an $s \in (0,r/\sqrt{2})$. Let $c \in C$. If ac is a time line then by 3.6 there is $y \in N_1 \cap N_2$ s.t. $[y,x] \subset N_1 \cap N_2$. Hence $y \in N$. If ac is not a time line, the line through c parallel to the x^1 axis meets the boundary of $B(a,r)$ in two points e,f. Then ef, ac, af are time lines. Let $y,z \in N_1 \cap N_2$ s.t. $h_1y=e$, $h_2z=f$. By 3.6, $[x,y]$, $[y,z] \subset N_1 \cap N_2$ and $t=h^{-1}(c) \in [y,z]$. Hence $t \in N$. It follows that $C \subset h_1(N)$. Hence $F(C)$ contains an open neighbhourhood of b in $h_2(N)$.

3.11. Definition. The open preinertial charts $h_i : N_i \longrightarrow \mathbb{R}^4$, with $i=1,2$, are named compatible if $h_1(N_1 \cap N_2)$ is open (equivalently $h_2(N_1 \cap N_2)$ is open, by 3.1o).

3.12. Definition. A preinertial chart $h: N \longrightarrow \mathbb{R}^4$ is called adequate if it satisfies also the axioms:

C5. For any particle p and for any $x,y \in p \cap N$, if $[hx,hy] \subset h(N)$, then $[x,y] \subset N$.

C6. For any $p \in P$ and $x,y \in p$ with $x \neq y$ and $[x,y] \subset p \cap N$ and for any segment s of $h(x)h(y) \cap h(N)$ which includes hx, hy, it follows $h^{-1}(s) \subset p$.

Observe that C5 and C6 are verified for the particles p s.t. $h(p \cap N)$ is included in a time line, cf. 3.6.

3.13. Proposition. Let $h_i : N_i \longrightarrow \mathbb{R}^4$ with $i=1,2$, be open, preinertial charts. If one of then is adequate, then h_1 and h_2 are compatible.

Proof. Suppose h_1 is adequate. Let $x \in N_1 \cap N_2$ and let N be the segment-connected component of x in $N_1 \cap N_2$. Then the function $F: h_1(N) \longrightarrow \mathbb{R}^4$ defined by $h_2h_1^{-1}$ is the restriction of a homography H, cf. 3.8. Put $b=h_1x$ and $a=h_2x$. We shall prove that $h_2(N)$ is open in a. Take the closed balls $B(b,r_1) \subset h_1(N_1) \cap \text{dom } H$ and $B(a,r_2) \subset h_2(N_2)$, with $r_1, r_2 > 0$. Then there is $r > 0$ s.t. $B(a,r) \subset B(a,r_2) \cap H(B(b,r_1))$. Let d_2 be a time line through a and put $d_2 \cap B(a,r) = \{e_2, f_2\}$. By 3.6, there is a particle p and $u_2, v_2 \in N_2 \cap p$ s.t.: $h_2(u_2)=e_2$, $h_2(v_2)=f_2$ and $[u_2,v_2] \subset N_2$. By QO1, there are $u', v' \in p \cap N_1$ with $[u',v'] \subset N_1$. Put $h_1(u')h_1(v') \cap B(b,r_1) = [e_1,f_1]$. By C6, there are $u_1, v_1 \in p \cap N_1$ s.t. $h_1(u_1)=e_1$, $h_1(v_1)=f_1$. Then $[u_1,v_1] \subset N_1$, cf. C5. Put $[u,v] := [u_1,v_1] \cap [u_2,v_2] \ni x$. The points $H(e_1), e_2, f_2, H(f_1)$ are collinear with a. The segment $[h_2u, h_2v]$ contains a, hence, by A3, it includes $[e_2,f_2]$. It follows that $[e_2,f_2] \subset h_2(N)$. Let C be the intersection of $B(a,r)$ with the cylinder $(x^2)^2 + (x^3)^2 + (x^4)^2 = s^2$, for an $s \in (0, r/\sqrt{2})$. Let $c \in C$. If ac is a time line, then there is $y \in N$ s.t. $h_2y=c$, cf. above. If ac is not a time line, then the line through c parallel to the x^1 axis meets the boundary of $B(a,r)$ in two points e, f s.t. ef, ae, af are time lines. By above let $y, z \in N$ s.t. $h_2y=e$, $h_2z=f$; also $[x,y], [x,z] \subset N$. By 3.6, $[y,z] \subset N_2$. Since $h_1y=H^{-1}(e)$ and $h_1z=H^{-1}(f)$ it follows that $[h_1y, h_1z] \subset B(b,r_1) \subset h_1(N_1)$. Then $[y,z] \subset N_1$, cf. C5; hence $[y,z] \subset N$. By C3, there is $t \in [y,z]$ with $h_2t=c$. It follows that $C \subset h_2(N)$. Hence $h_2(N_1 \cap N_2)$ is open.

<u>Remark</u>. The assertions proved in 3.6- 3.8, 3.10, 3.13 remain true even if C1 and C4 are replaced by the weaker axioms:

C1". For any $p \in P$ and for any x, y, z contained in a segment of $p \cap N$, the points hx, hy, hz are collinear.

C4". For any $x \in N$ and for any temporal line d through hx, there is a particle p s.t.: $x \in p$; $h(p \cap N)$ contains a segment s with $s \subset d$, and hx is in the interior of s.

3.14. <u>Definition</u>. A preinertial chart $h: N \longrightarrow \mathbb{R}$ is called inertial if it satisfies also the axiom:

C7. For any particle p which meets N, the set $h(p \cap N)$ lies on a time line.

Observe that any inertial chart is adequate, by 3.6.

3.15. Theorem (W). Let $h_i : N_i \longrightarrow \mathbb{R}^4$ be open inertial charts with $i=1,2$ and $N := N_1 \cap N_2 \neq \emptyset$. Then $F : h_1(N) \longrightarrow \mathbb{R}^4$, defined by $F(y) = h_2 h_1^{-1}(y)$ for any $y \in h_1(N)$ is a Weyl transformation.

Proof. The sets $h_1(N)$ and $h_2(N)$ are open, by 3.13 and 3.1o. By 3.8, F is the restriction to $h_1(N)$ of a homography H of $\mathbb{R}^4$. Let $y \in h_1(N)$; then F maps any time line through y in a time line through F(y), by C4 for h_1 and C7 for h_2. Hence $F(h_1(N) \cap C_y) \subseteq C_{F(y)}$. Moreover $F^{-1}(C_{F(y)}) \subseteq C_y$, by C4 for h_2 and C7 for h_1. Hence

(3.15.1) $$F(h_1(N) \cap C_y) = h_2(N) \cap C_{F(y)}$$

By 3.8, F is the restriction to $h_1(N)$ of a homography H of $\mathbb{R}^4$; hence F is a homomorphism on its image. From 3.15.1 it results that

(3.15.2) $$F(h_1(N) \cap C^L_y) = h_2(N) \cap C^L_{F(y)} ,$$

where $C^L_1 = \{ z \in \mathbb{R}^4 ; Q(x-z) = 0 \}$. By a standard argument it follows that F maps any light ray in a light ray. Hence F is a Weyl transformation (see for instance 3.2o bellow).

3.16. Let a, b be real non-negative numbers. On $\mathbb{R}^4$ consider the quadratic form

(3.16.1) $$Q(x) = (ax^1)^2 - (bx^2)^2 - (bx^3)^2 - (bx^4)^2$$

for any $x = (x^1, \ldots, x^4) \in \mathbb{R}^4$.

(3.16.2) If $a, b \neq 0$, using the dilatation $(x^1, \ldots, x^4) \longmapsto (x^1/a, x^2/b, x^3/b, x^4/b)$ we can suppose that $a=b=1$.

(3.16.3). If $a \neq 0$ and $b=0$, then the "light" cone $C^L_x := \{ y ; Q(x-y) = 0 \}$ becomes the hyperplane $E_x : x^1 - y^1 = 0$ and any line through x not contained in E_x is a "time" line.

In 3.5 - 3.15 we worked with C1-C7 enounced for the quadratic form (3.16.1) with a=b=1.

(3.16.4). In 3.17 let C1-C7 be enounced for a quadratic form (3.16.1) with $a \neq 0$.

For $x,y \in \mathbb{R}^4$ with $x^1=y^1$ we define the spatial distance between x and y as $d(x,y):=(x^2-y^2)^2+(x^3-y^3)^2+(x^4-y^4)^2$.

3.17. Theorem. Let $h_i: N_i \longrightarrow \mathbb{R}^4$ be open inertial charts (in the sense of 3.16.4) with i=1,2 and $N:=N_1 \cap N_2 \neq \emptyset$. Define $F: h_1(N) \longrightarrow \mathbb{R}^4$ by $F(y)=h_2h_1^{-1}(y)$ for any $y \in h_1(N)$.

(G) Let $a \neq 0$ and b=0. Then F is a quasi-galilean transformation (see 3.18.1). If, moreover, F preserves the spatial distances given in 3.18, then F is a Galilei transformation.

(W) Let $a \neq 0$ and $b \neq 0$. Then F is a Weyl transformation.

Proof. The assertion (W) was proved in 3.15 via 3.16.2.

(G) First observe that 3.6-3.14 are true also for (3.16.4) with $a \neq 0$ and $b \neq 0$.

Thus, as in the proof of 3.15, we obtain that: $h_i(N)$ is open, i=1,2; F is the restriction to $h_1(N)$ of a homography H of $\mathbb{R}^4$; for any $y \in h_1(N)$,

$$F(h_1(N) \cap E_y)=h_2(N) \cap E_{F(y)}$$

Now (G) follows from 3.18.

3.18. Proposition. Let $H: \mathbb{R}^4 \setminus A \longrightarrow \mathbb{R}^4$ be a homography. W.l.g. let H(d)=d, where d=(0,0,0,0). Suppose that: $H(E_d \setminus A) \subseteq E_d$; $H(E_y \setminus A) \subseteq E_{H(y)}$ for some y=(t,0,0,0) with $t \neq 0$. Then H is a quasi-Galilei transformation.

Put a=(0,1,0,0), b=(0,0,1,0), c=(0,0,0,1) and z=(t,1,0,0). If moreover, H preserves the spatial distances between d,a,b,c and between y and z, then H is a Galilei transformation (with eventually $H_{11} \neq 1$).

<u>Proof</u>. Write $H(x)^1=(\sum_{j=1}^{4} H_{1j}x^j+H_{15})/(\sum_{j=1}^{4} H_{5j}x^j+H_{55})$. Since $H(d)=d$, it follows that $H_{15}=\ldots=H_{45}=0$ and $H_{55}\neq 0$. Multiplying the matrix of H, we can suppose that $H_{55}=1$. Since $H(E_d\setminus A)\subseteq E_d$, it results that $H_{12}=H_{13}=H_{14}=0$.

From $H(E_y\setminus A)\subseteq E_{H(y)}$ it follows that $t'=H_{11}t/(H_{51}t+H_{52}x^2+H_{53}x^3+H_{54}x^4+1)$. Then $t'(H_{51}t+1)-H_{11}t=0$ and $H_{52}=H_{53}=H_{54}=0$. Hence

(3.18.1) $\qquad H(x)^1=H_{11}x^1/(H_{51}x^1+1)$

A homography which satisfy 3.18.1 is called a <u>quasi-Galilei transformation</u>.

Since H acts as a linear transformation on E_d and preserves the distances between d,a,b,c, it follows that

$$\sum_{i=2}^{4} H_{ij}H_{ik}=\delta_{jk} \text{ for any } j,k=2,3,4$$

(Indeed, $d(H(a),H(d))=d(a,d)$ implies that $\sum_{i=2}^{4} H_{i1}^2=1$ and $d(H(a),H(b))=d(a,b)$ implies that $\sum_{i=2}^{4} H_{i2}H_{i3}=0$ and so on).

Since $d(H(y),H(z))=d(y,z)$ it results that $(H_{51}t+1)^2=\sum_{i=2}^{4} H_{i2}^2=1$. Hence $H_{51}=0$, i.e. H is a Galilei transformation (multiplying the first coordinate with $1/H_{11}$).

3.19. <u>Remark</u>. Usually for a Galilei transformation it is asked that the time be absolute. Hence in order that a homography H, with $H(d)=d$, be a Galilei transformation, we can impose $H(x)^1=x^1$ for any $x=(x^1,x^2,x^3,x^4)$. From this condition it results that:

(3.19.1) $\qquad H_{5i}=0$ for $i=1,2,3,4$

$H_{1i}=0$ for $i=2,3,4$

$H_{11}=1$

In fact (3.19.1) can be deduced from the following weaker conditions:

$H(x)^1=x^1$ for $x=(1,0,0,0),(2,0,0,0),(0,1,0,0),(0,0,1,0),$

(0,0,0,1),(1,1,0,0),(1,0,1,0),(1,0,0,1).

Indeed, $H(x)^1=x^1$ means $H_{51}(x^1)^2+H_{52}x^1x^2+H_{53}x^1x^3+H_{54}x^1x^4-$ $-(H_{11}-1)x^1-H_{12}x^2-H_{13}x^3-H_{14}x^4=0$.

The first two points give $H_{51}-(H_{11}-1)=0$ and $4H_{51}-$ $-2(H_{11}-1)=0$; hence $H_{11}=1$ and $H_{51}=0$. The third gives $H_{12}=0$ and so on. The sixth gives $H_{52}=0$ and so on.

If, moreover, H preserves the spatial distances between d,a,b,c from 3.18, then H is a Galilei transformation.

3.2o. Proposition. Let $H:\mathbb{R}^4 \setminus A \longrightarrow \mathbb{R}^4$ be a homography. W.l.g. let $H(d)=d$, where $d=(0,0,0,0)$. Let Q be the standard quadratic form on the Minkowski space $\mathbb{R}^4$ (1.0). Suppose that $Q(H(y)-H(x))=$ $=0$ for the following pairs of points: $x=(0,0,0,0)$ and y one of $y_2=(1,1,0,0)$, $z_2=(1,-1,0,0)$, $y_3=(1,0,1,0)$, $z_3=(1,0,-1,0)$, $y_4=(1,0,0,1)$, $z_4=(1,0,0,-1)$, $y_{23}=(1,1/\sqrt{2},1/\sqrt{2},0)$, $y_{24}=(1,1/\sqrt{2},0,1/\sqrt{2})$ and $y_{34}=(1,0,1/\sqrt{2},1/\sqrt{2})$; $x=(1,0,0,0)$ and y one of $w_1=(1/2,1/2,0,0)$, $w_2=(1/2,0,1/2,0)$, $w_3=(1/2,0,0,1/2)$ and $w_4=(1/2,-1/2,0,0)$.

Then H is a Weyl transformation.

Proof. From $H(d)=d$ it follows $H_{i5}=0$ for $i=1,2\ldots,4$ and $H_{55}\neq 0$. We can take $H_{55}=1$. Let $x=(x^1,x^2,x^3,x^4)$ and $y=(y^1,y^2,y^3,y^4)$. Put $x^5=1=y^5$. Suppose that $Q(x-y)=0$ and $Q(H(x)-H(y))=0$. The second condition means

$$(3.2o.1)\qquad \Big[\sum_{k,j=1}^{5} x^jy^k(H_{5k}H_{1j}-H_{5j}H_{1k})\Big]^2=$$

$$=\sum_{i=2}^{4}\Big[\sum_{k,j=1}^{5} x^jy^k(H_{5k}H_{ij}-H_{5j}H_{ik})\Big]^2$$

Let $x=(0,0,0,0)$. Then (3.2o.1) becomes

$$(3.2o.2)\qquad \sum_{k,j=1}^{4} y^ky^jH_{1k}H_{1j}=\sum_{i=2}^{4}\sum_{k,j=1}^{4} y^ky^jH_{ik}H_{ij}$$

Put $\varepsilon_1=1$ and $\varepsilon_2=\varepsilon_3=\varepsilon_4=-1$. Then (3.2o.2) becomes

$$(3.2o.3)\quad \sum_{i=1}^{4}\ \sum_{k,j=1}^{4} y^k y^j \varepsilon_i H_{ik} H_{ij}=0$$

Put $A_{kj}=\sum_{i=1}^{4}\varepsilon_i H_{ik}H_{ij}$. Then $A_{kj}=A_{jk}$.

Taking $y=y_2,z_2$,(3.2o.3) becomes respectively $A_{11}+2A_{12}+A_{22}=0$ and $A_{11}-2A_{12}+A_{22}=0$. Hence $A_{12}=0=A_{21}$ and $A_{11}=-A_{22}$. Analogously from $y=y_3,z_3$ and $y=y_4,z_4$ it follows that $A_{13}=0=A_{31}$, $A_{11}=-A_{33}$ and $A_{14}=0=A_{41}$, $A_{11}=-A_{44}$.

Taking $y=y_{23}$ it follows that $A_{23}=0$. Then $y=y_{24},y_{34}$ give $A_{24}=0$, $A_{34}=0$. Hence, for $\lambda:=A_{11}$,

$$(3.2o.4)\quad \sum_{i=1}^{4}\varepsilon_i H_{ik}H_{ij}=\lambda\varepsilon_k\delta_{kj}\ ,\ \text{for } k,j=1,2,3,4$$

Let $x=(1,0,0,0)$. Then (3.2o.1) becomes

$$\Big[\sum_{k=1}^{5} y^k H_{5k}H_{11}-\sum_{k=1}^{4} y^k H_{1k}(H_{51}+1)\Big]^2=$$

$$=\sum_{i=2}^{4}\Big[\sum_{k=1}^{5} y^k H_{5k}H_{i1}-\sum_{k=1}^{4} y^k H_{ik}(H_{51}+1)\Big]^2$$

This gives

$$A_{11}\sum_{k,j=1}^{5} y^k y^j H_{5k}H_{5j}-2(H_{51}+1)\sum_{k=1}^{5}\ \sum_{j=1}^{4} y^k y^j H_{54}A_{1j}+$$

$$+(H_{51}+1)^2\sum_{k,j=1}^{4} y^k y^j A_{kj}=0$$

Using (3.2o.4) it becomes

$$\Big(\sum_{k=1}^{5} H_{5k}y^k\Big)\Big[\sum_{j=1}^{5} H_{5j}y^j-2y^1(H_{51}+1)\Big]=0$$

The first sum is not zero (as denominator in H(y)). Taking y s.t. $y^1=1/2$, it follows

$$(-1/2)H_{51}+y^2H_{52}+y^3H_{53}+y^4H_{54}=0$$

From $y=w_1,w_2,w_3,w_4$, it follows $H_{51}=H_{52}=H_{53}=H_{54}=0$.

Hence H is a Weyl transformation.

We globalize now these considerations.

3.21. Suppose there exists a set A^o of open, adequate, preinertial charts s.t. their domains cover M. By 3.13 the set A^o extends canonically to an atlas A of M, s.t. any element of A is an open, adequate, preinertial chart. Thus A gives a structure of differentiable manifold on M; the coordinate transformations are given by homographies, cf. 3.8.

If M admits a set A^o as above, we say that M satisfy the <u>generalized principle of inertia.</u>

3.22. Let A be as in 3.21. We say that M,A satisfies the <u>Galilei principle of inertia</u> if there is a subset G^o of A s.t.: the domains of the charts from G^o cover M; for any $h_1, h_2 \in G^o$ with $N=\mathrm{dom}\ h_1 \cap \mathrm{dom}\ h_2 \neq \emptyset$, the function $h_2 h_1^{-1}: h_1(N) \longrightarrow \mathbb{R}^4$ is a Galilei transformation. The set G^o extends canonically to an atlas G of M s.t. the coordinate transformations are galilean. Observe that for $h_1, h_2 \in A$ the function $h_2 h_1^{-1}$ is a Galilei transformation if it satisfies conditions similar to that in 3.18 and 3.19.

3.23. Let A be as in 3.21. We say that M,A satisfies the <u>Einstein principle of inertia</u> if there is a subset E^o of A s.t.: the domains of the charts from E^o cover M; for any $h_1, h_2 \in E^o$ with $N=\mathrm{dom}\ h_1 \cap \mathrm{dom}\ h_2 \neq \emptyset$, the function $h_2 h_1^{-1}: h_1(N) \longrightarrow \mathbb{R}^4$ is a Weyl transformation. The set E^o extends canonically to an atlas E of M s.t. the coordinate transformations are Weyl. Observe that for $h_1, h_2 \in A$, the function $h_2 h_1^{-1}$ is a Weyl transformation if it satisfies conditions similar to that of 3.2o.

3.24. Following [7], the phenomenon of light propagation in M can be described by a binary relation $M' \subset M \times M$ s.t.: if $(u,v) \in M'$, then $(v,u) \in M'$ and $u \neq v$.

Let $N \subset M$. A function $f: N \longrightarrow \mathbb{R}^4$ is called a <u>luminal chart</u> if

(3.24.1) for any distinct $u,v \in N$, the pair (u,v) belongs to M' iff $Q(f(u)-f(v))=0$ (with Q given in 1.0).

Let A be as in 3.21.

Remark that if $h_1, h_2 \in A$ are luminal, with $\mathrm{dom}\, h_1 \cap \mathrm{dom}\, h_2 \neq \emptyset$, then $h_2 h_1^{-1}$ is a Weyl transformation.

(3.24.2) We say that M,A satisfies the <u>principle of the constancy of the velocity of light propagation</u> if there is a subset C^o of A s.t.: the domains of the charts from C^o cover M; any chart from C^o is luminal.

By the above remark it results that: C^o has the properties of E^o from 3.23; C^o extends canonically to an atlas C of M, whose charts are luminal.

3.25. Let M,E,A be as in 3.23. In general it does not follow that M,A satisfies the principle pf the constancy of the velocity of light propagation. Suppose, moreover, there is a luminal chart h in E. Then for any $h_1 \in E$, the chart $h_1' : \mathrm{dom}\, h \cap \mathrm{dom}\, h_1 \longrightarrow \mathbb{R}^4$, $h_1'(x)=h_1(x)$ is luminal; but we can not deduce that h_1 is luminal.

The Michelson-Morley experiment (see, for exemple, [1]). suggests that M,A satisfies the Einstein principle of inertia and E has some luminal charts. But this experiment, being local, does not imply that M,A satisfies the principle of the constancy of the velocity of light propagation. The most strange consequence of this would be that the speed of the light emitted by distant stars would arrive at us with a speed c' different from the speed c of the light emitted by our local sources. For example, this would explain the redshift of that light, without the hypothesis of the expansion of the universe.

The independence of the principle of inertia (3.21, 3.22, 3.23) from the constant speed of light propagation can furnish other consequences; we shall develope some of them in a forthcoming paper.

3.26. In § 4 we show that theorem 1.1 remains valid also for cones defined by a quadratic form whose coefficients depend on the vertex of the cone. This is also an argument for the above discussed independence of these principles.

4. Theorem 1.1 for variable cones

Let $B: \mathbb{R}^m \longrightarrow \mathbb{R}$ be a continuous function s.t. $B(x) \geqslant 0$ for any $x \in \mathbb{R}^m$. For any $x \in \mathbb{R}^m$ define:

$Q^x(z)=(z^1)^2-B(x)^2\left[(z^2)^2+\ldots+(z^m)^2\right]$ with $z=(z^1,\ldots,z^m)$ in the standard coordinate system of $\mathbb{R}^m$;

$C_x=\{z \in \mathbb{R}^m;\ Q^x(x-z)>0\}$.

Note that the standard Minkowski structure is obtained if $B=1$. Taking B non-constant we permit, from the physical point of view, that the speed of light propagation be variable.

A line l is named an <u>x-time line</u> if $l \subset C_x \cup \{x\}$ (hence $x \in l$).

Let l be an x-time line. Then there are $a,b \in l$ with $x \in (a,b)$ s.t. for any $z \in (a,b)$. l is also a z-time line (since B is continuous).

Let U be a subset of $\mathbb{R}^m$. A function $F: U \longrightarrow \mathbb{R}^m$ is called a <u>partial lineation</u> if for any $a,b \in U$ s.t. ab is a z-time line for any $z \in (a,b)$, we have that $F((a,b))$ is included in a line.

A plane A is named an <u>x-time plane</u> if A contains an x-time line.

A segment $[a,b]$ is called <u>temporal</u> if ab is a z-time line for any $z \in [a,b]$.

4.1. Theorem. <u>Let</u> $m \geqslant 2$ <u>and let</u> $F: U \longrightarrow \mathbb{R}^m$ <u>be a partial lineation. Suppose that:</u> F <u>is injective;</u> U <u>is connected by segments; for</u>

any $x \in U$ and A an x-time plane, $U \cap A$ includes an open subset of A containing x; there exist a plane T and three non-collinear points $v_0, v_1, v_2 \in T \cap U$ s.t.: $F(v_0), F(v_1), F(v_2)$ are non-collinear; v_0, v_1, v_2 are contained in a connected component T^o of $T \cap U$; T^o contains a point x_o s.t. T is an x_o-time plane.

Then F is the restriction to U of a homography. The proof of 4.1 follows the proof of 1.1.

First we remark that if V is a vertical plane, then V is an x-time plane for any $x \in V \cap U$, hence $V \cap U$ is open in V.

Let $x \in U$. If A is an x-time plane then there exists an open and bounded subset $\bar{A}^x$ of A s.t.: $x \in \bar{A}^x$; A is a z-time plane for any $z \in \bar{A}^x$; B is upper bounded (by an $a(A,x)$) on $\bar{A}^x$. Indeed B is continuous, hence upper bounded on compacts. Taking $a=a(A,x)$ instead of $B(y)$ in Q^y for any $y \in \bar{A}^x$ we obtain $Q'(z)=(z^1)^2-a^2\left[(z^2)^2+\ldots+(z^m)^2\right]$. Put $C'_y = \{z \in \mathbb{R}^m; Q'(y-z) > 0\}$ for any y in $\bar{A}^x$; these C'_y have the same slope. Moreover, $C'_y \subseteq C_y$ for any $y \in \bar{A}^x$. Now since on any $\bar{A}^x$ we have a family of cones with the same slope, we can adapt the proof of 1.1.

The enounce of 2.2 remains the same. In the proof we have to replace the slope "one" of the cones with a constant slope given by the set $\bar{A}^x$.

Thus from 2.2, 2.6 and 2.9 it results that

(4.2) for U, F as in 4.1, F is a 1-or 2-lineation on any connected open subset of $A \cap U$, where A is any plane of $\mathbb{R}^m$.

(Hence F is a 2-lineation on the set T^o from 4.1).

We have to modify the definition of U_x in 2.11 as follows: for any x-time plane A, take a connected open subset A^x of A with $x \in A^x$; U_x is the union of these sets A^x.

In the enounce of 2.12 we take A to be an x-time plane, and the proof of the new 2.12 remains the same.

Remark that in the proof of 2.11 we need 2.13 only for vertical r-planes (i.e. containing the parallel p to the x^1 axis through x).

In the enounce of 2.13 we take A_r to be an x-time r-plane, i.e. containing an x-time line (particularly we can restrict to those A_r which contain the line p). In the proof of the new 2.13 we take xv_1 to be an x-time line; then there are v_i s.t. xv_i be an x-time line. Using the new 2.12 and 2.13, the proof of the new 2.11 remains the same.

In order to prove 4.1, observe also that: if $[x,y] \subset U, x \neq y$, there are $z_0, \ldots, z_n$ with $z_0 = x, z_n = y$ and $[z_i, z_{i+1}] \subset U$ are temporal segments for $i=0,\ldots,n-1$. (Indeed, in the vertical plane V containing x,y, the set $V \cap U$ is open in V; hence we can choose the desired z_i in $V \cap U$, since B is continuous).

The proof of 4.1 is an adaptation of that of 1.1 as follows. Let $x \in U$. If A is an x-time plane take W to be the maximal open subset of A s.t. $x \in W \subset A \cap U$; then take A^x to be the connected open component of x in W. Now U_x is the union of these sets A^x. Start with an $x \in T^o$. Then the restriction of F to U_x is an m-homography.

Let $y \in U$ s. t. $[x,y]$ is a temporal segment. Let A be a plane containing x,y; then $A^x = A^y$. Indeed A is a z-time plane for any $z \in [x,y]$; hence A^x and A^y contain the connected open set $\bigcup_{z \in [x,y]} A^z$, i.e. coincide with it.

The rest of the proof goes on unchanged.

Minor modifications in the axioms and definitions which appear in 3.5. - 3.12 permit the proof of Proposition 3.13 in the case of variable cones also and therefore the statement of the principle of inertia in the form given in 3.21. Hence in order to construct a physical theory based on the principle of inertia it is not necessary that the principle of the constancy of the velocity of light propagation be satisfied. Intuitively, what is necessary is that for any chart, at any event, the possible particle velocities fill a cone whose slope varies continuously in $\mathbb{R}^4$.

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ABOUT THE FOUNDATIONS OF THE SPECIAL THEORY OF RELATIVITY

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To the memory of Iulian Popovici

ABSTRACT. The principle of the constancy of the velocity of light propagation is used to introduce a differentiable structure on the universe of events, E. Namely, using a theorem proved in [2] and some axioms imposed to E it is shown how E can be endowed with an atlas s.t. the coordinate transformations between the charts of this atlas be given by conformal (or Weyl) transformations.

1. INTRODUCTION

In this paper we show that the principle of the constancy of the velocity of light propagation when explicitly formulated from a mathematical point of view, can be used to introduce on the universe of events E a structure of analytical manifold generated by a subatlas, the coordinate transformations between the charts of this subatlas being conformal transformations. To this end we use Theorem 1.1 from [2] and some procedures developped in [1], § 3.

This paper is dedicated to the memory of our late friend and teacher in geometry, dr. Iulian Popovici with whom we started this approach to the foundation of the special theory of relativity.

In the m-dimensional real affine space $\mathbb{R}^m, m \geq 2$, we consider the Minkowski quadratic form Q given by

(1.1) $\quad Q(x)= x_1^2 - x_2^2 - \dots - x_m^2,$

where $(x_1, x_2, \dots, x_m)$ are the canonical coordinates in $\mathbb{R}^m$. We say that the pair $M=(\mathbb{R}^m, Q)$ is the <u>m-dimensional Minkowski space.</u>

For any $x \in \mathbb{R}^m$, the set

(1.2) $\quad C_x = \{y; Q(y-x)=0\}$

is named the <u>light cone</u> of vertex x. A <u>light line</u> of M is a straight line which lies in a light cone. A <u>light segment</u> of M is a closed segment which lies on a light line. Let v be an arbitrary vector in $\mathbb{R}^m$. If $Q(v)=0$ we say that v is a <u>light vector</u>. Note that a light segment [x,y] defines the light vector x-y.

The <u>Lorentz group</u> of the Minkowski space M consists of all linear applications $F:\mathbb{R}^m \longrightarrow \mathbb{R}^m$ s.t. $Q(F(x))=Q(x)$ for any $x \in \mathbb{R}^m$. The <u>Weyl group</u> of M is generated by the Lorentz group and the translations and dilations of $\mathbb{R}^m$.

Let $(G_{\mu\nu}), G_{\mu\nu} \in \mathbb{R}$, $\mu,\nu=1,\dots,m+2$, be a matrix (of rank m+2) which satisfies

(1.3) $$\sum_{\rho,\sigma=1}^{m+2} G_{\mu\rho} \, G_{\nu\sigma} \, \eta_{\rho\sigma} = \eta_{\mu\nu}$$

where $\eta_{\mu\nu}$ are given by

(1.4) $$\eta_{ij} = \varepsilon_i \, \delta_{ij}, \quad i,j=1,\dots,m, \quad \varepsilon_1 = -\varepsilon_2 = \dots = -\varepsilon_m = 1$$

$$\eta_{ri} = \eta_{ir} = 0, \; i=1,\dots,m, \; r=m+1, \; m+2$$

$$\eta_{rs} = \eta_{sr} = -2, \quad \eta_{rr} = \eta_{ss} = 0, \; r=m+1, \; s=m+2$$

The matrix $(G_{\mu\nu})$ defines a function $G:\mathbb{R}^m \setminus P \longrightarrow \mathbb{R}^m$ by

$$(1.5)\quad G(x)_i = \frac{G_{ij}x_j+G_{ir}Q(x)+G_{is}}{G_{sj}x_j+G_{sr}Q(x)+G_{ss}},\ i,j=1,\dots,m, r=m+1, s=m+2,$$

where P is the empty set or the surface given by the equation $G_{sj}x_j+G_{sr}Q(x)+G_{ss}=0$.

This function is called a conformal transformation and it maps any light segment onto a light segment. If we take $G_{sj}=G_{sr}=G_{ir}=0$ in (1.5), we obtain a Weyl transformation, cf. (1.3) and (1.4).

In [2] was proved the following

1.6 THEOREM. In a Minkowski space $M=(\mathbb{R}^m,Q)$ of dimension $m \geqslant 3$, let $F: U \longrightarrow \mathbb{R}^m$ be an injective map defined on a connected open set U in $\mathbb{R}^m$. If F maps any light segment contained in U onto a light segment, then F is the restriction to U of a conformal transformation.

Following closely the method developped in [1] § 3, in the next paragraph we impose some axioms on the universe of events in order to obtain its differentiable structure.

2. DERIVING THE DIFFERENTIABLE STRUCTURE OF THE UNIVERSE OF EVENTS

2.1. Let E be a set; its elements are called events. Let L be a family of subsets of M; the elements of L are named luminal rays. The family L satisfies the following axioms:

A.1. For any $x \in E$, there is $p \in L$ with $x \in p$. For any $p \in L$, the set p has at least two elements.

A.2. If $x,y \in E$, $x \neq y$; and $p,q \in L$ s.t. $x,y \in p,q$, then $p=q$.

A.3. For any $p \in L$ and for any $x,y \in p$ there is defined a subset $[x,y]$ of p, called segment s.t.:

$[x,y]=[y,x]$; $x,y \in [x,y]$; $[x,x] = \{x\}$

$\underline{\text{if}}$ $a,b \in [x,y]$,then $[a,b] \subseteq [x,y]$

$\underline{\text{let}}$ $a,b,c,d \in p$ s.t. $[b,c] \cap [a,d] \neq \emptyset$: if $b,c \notin [a,d]$ $\underline{\text{then}}$ $[a,d] \subset [b,c]$; $\underline{\text{if}}$ $b \in [a,d]$ $\underline{\text{and}}$ $c \notin [a,d]$ $\underline{\text{then either}}$ $[b,d] \subset [a,c]$ $\underline{\text{and}}$ $[b,c] \cap [a,d] = [b,d]$ $\underline{\text{or}}$ $[a,b] \subset [c,d]$ $\underline{\text{and}}$ $[b,c] \cap [a,d] = [a,b]$.

The set $[x,y] \setminus \{x,y\}$ is denoted (x,y).

A.4. $\underline{\text{If}}$ $p \in L$ $\underline{\text{and}}$ $x,y,z,t \in p$ $\underline{\text{s.t.}}$ $(x,y) \cap (z,t) \neq \emptyset$, $\underline{\text{then}}$ $[x,y] \cup [z,t]$ $\underline{\text{is a segment}}$.

In 2.2. - 2.1o, N is a non-empty subset of E.

2.2. Definition. N is called luminally complete if:

Bi. For any $l \in L$ and any $x,y \in l \cap N$ it follows that $[x,y] \subseteq N$.

Bii. For any $l \in L$ and any $x \in N$ with $x \in l$, there are $y,z \in l \cap N$ s.t. $x \in (y,z) \subseteq N$.

2.3. Let $N_1, N_2 \subseteq E$ with $N := N_1 \cap N_2 \neq \emptyset$. If N_1, N_2 are luminally complete, then N is luminally complete.

Indeed N satisfies Bi since from $l \in L$ and $x,y \in l \cap N$ it results $[x,y] \subseteq N_1, N_2$, i.e., $[x,y] \subseteq N$.

Now take $x \in N$ and $l \in L$ with $x \in l$. Bii for N_1 (resp. N_2) assures the existence of $[y_1,z_1] \subseteq l \cap N_1$ with $x \in (y_1,z_1)$ (resp. $[y_2,z_2] \subseteq l \cap N_2$ with $x \in (y_2,z_2)$). Cf. A3, x is contained in the segment $[y,z] := [y_1,z_1] \cap [y_2,z_2] \subseteq l \cap N$, where y,z are two of the points y_1, z_1, y_2, z_2. It follows $x \neq y \neq z$. Thus $x \in (y,z)$.

2.4. Definition. A function $h: N \longrightarrow \mathbb{R}^4$ is named a luminal chart if h is injective and for any $x,y \in N$ with $x \neq y$ the following holds true: there is $l \in L$ s.t. $x,y \in l \cap N$ iff $Q(hx-hy)=0$. (When no confusion can appear we denote $h(x)$ by hx.)

2.5. Let $h: N \longrightarrow \mathbb{R}^4$ be a luminal chart. Then for any $l \in L$ and any $x,y,z \in l \cap N$ it results that hx, hy, hz are collinear on a light line.

Indeed, the assertion is evidently true for $x=y\neq z$ or $x=y=z$. In the case $x\neq y\neq z$, from

(2.5.1) $Q(x-y)=Q(y-z)=Q(z-x)=0$,

it results that $(x_1-y_1)(x_1-z_1)-\sum_{i=2}^{4}(x_i-y_i)(x_i-z_i)=0$. This implies that the light vectors $x-y$ and $x-z$ are linearly dependent (for a proof see for example [2], the proof of Lemma 2.2 in the case $n=2$). Thus the points hx, hy, hz are collinear on a light line.

2.6. <u>Definition</u>. A luminal chart $h: N \longrightarrow \mathbb{R}^4$ is named <u>adequate</u> if:

Ci. <u>For any</u> $[x,y]\subset N$ <u>it results</u> $h([x,y])=[hx,hy]\subset \mathbb{R}^4$.

2.7. <u>Definition</u>. A set $U\subseteq \mathbb{R}^4$ is named <u>light complete</u> if:

Di. <u>For any light line</u> $l\subset \mathbb{R}^4$ <u>and for any</u> $x,y\in l\cap U$, it follows $[x,y]\subset U$.

Dii. <u>For any</u> $v\in U$ <u>and any light line</u> d <u>with</u> $v\in d$, <u>there are</u> $u,t\in d\cap U$ <u>s.t.</u> $v\in(u,t)\subseteq U$.

2.8. LEMMA. <u>Let</u> U <u>be a light complete subset of</u> $\mathbb{R}^4$. <u>Then</u> U <u>is open in</u> $\mathbb{R}^4$.

<u>Proof</u>. Let b be an arbitrary point in $\mathbb{R}^4$ and let $e_1,\dots,e_n$ with $1\leqslant n\leqslant 4$, be light vectors of $\mathbb{R}^4$ which are linearly independent. The prism

$$P_n=\{x\in\mathbb{R}^4; x=\sum_{i=1}^{n} a_ie_i+(1-\sum_{i=1}^{n} a_i)b,\ -1\leqslant a_j\leqslant 1, j=1,\dots,4\}$$

is said to be a <u>light n-prism</u> centered at b and generated by the vectors $e_1,\dots,e_n$. An affine n-variety V of $\mathbb{R}^4$ whose translation space contains n linearly independent light vectors is called a <u>light n-plane</u>.

We prove by induction on $n\in\{1,\dots,4\}$ the following property (a,n) For any $v \in U$ and any light n-plane V_n containing v, there exists a light n-prism P_n s.t.: v is contained in the interior of P_n; $P_n \subset V_n$; $P_n \subset U$. (Thus U is open in v since, by (a,4), an open ball of centre v and included in P_4 can be constructed).

The assertion (a,1) is true by Dii. Suppose (a,n) is true for $n \in \{1,2,3\}$. Then we prove (a,n+1). Let V_{n+1} be a light (n+1)-plane which contains v and is generated by the light vectors $e_1, \ldots, e_{n+1}$. Let $[c,b] \subset U$ with $v \in (c,b)$ and b-c parallel to e_{n+1}. We can put $e_{n+1}=b-c$. Take V_n and T_n two light n-planes s.t. any point x of V_n (resp. T_n) is of the form $x=c+\sum_{i=1}^{n} a_i e_i$ (resp. $x=b+\sum_{i=1}^{n} a_i e_i$), where $a_i \in \mathbb{R}$, for $i=1,\ldots,n$. Then, by (a,n), there is a light n-prism $P_n^* \subset U$ (resp. $P_n' \subset U$) which contains c (resp. b) and is generated by n light vectors $e_1^*, \ldots, e_n^*$ (resp. $e_1', \ldots, e_n'$) s.t. $P_n^* \subset V_n$ (resp. $P_n' \subset T_n$). Also from (a,n) it is clear that we can choose $e_i \parallel e_i' \parallel e_i$, for $i=1,\ldots,n$. Then construct the light (n+1)-prism R_{n+1}^* (resp. R_{n+1}') generated by $e_1^*, \ldots, e_n^*, e_{n+1}$ (resp. $e_1', \ldots, e_n', e_{n+1}$) which contains [c,b] and P_n^* (resp. P_n'). Put $P_{n+1} := R_{n+1}^* \cap R_{n+1}'$. By construction P_{n+1} is a light (n+1)-prism, whose interior contains v. Moreover $P_{n+1} \subset U$ since any $x \in P_{n+1}$ lies on a light segment (parallel to e_{n+1}) whose extremities y,z are on P_n^* and P_n', respectively. Since U satisfies Di, from $y,z \in U$ it results $[y,z] \subset U$.

2.9. Let N satisfying Bi. Let $h: N \longrightarrow \mathbb{R}^4$ be an adequate luminal chart. Then h(N) satisfies Di.

Indeed, for any $hx, hy \subset h(N)$ with $hx \neq hy$ and $Q(hx-hy)=0$, since h is a luminal chart it follows that there is $l \in L$ with $x,y \in l \cap N$. From Bi it results $[x,y] \subseteq N$. Therefore, from Ci, $h([x,y]) = [hx,hy] \subseteq h(N)$.

2.10. <u>Definition.</u> A luminal chart $h: N \longrightarrow \mathbb{R}^4$ is called <u>open</u> if $h(N)$ is open in $\mathbb{R}^4$.

2.11 PROPOSITION. <u>Let $N_1, N_2 \subseteq E$ be luminally complete and let $h_i: N_i \longrightarrow \mathbb{R}^4$, $i=1,2$, be adequate, luminal charts. If one of them is an open chart, then $h_i(N_1 \cap N_2)$, $i=1,2$, are open sets in $\mathbb{R}^4$.</u>

<u>Proof</u>. Suppose h_1 is open. Put $N := N_1 \cap N_2$. If $N \neq \emptyset$, then N is luminally complete, by 2.3. Particularly, by 2.9, $h_1(N)$ satisfies Di. Now we prove that $h_1(N)$ satisfies also Dii. (Hence $h_1(N)$ is light complete, and, by 2.8, it is an open set). Take $b \in h_1(N)$ and a light line d with $b \in d$. Since $h_1(N_1)$ is open we can find $[a,c] \subset d \cap h_1(N_1)$ s.t. $b \in (a,c)$. Put $x = h_1^{-1}(b)$, $y_1 = h_1^{-1}(a)$, $z_1 = h_1^{-1}(c)$. Because h_1 is luminal and N_1 satisfies Bi it results that there is $l \in L$ s.t.: $x \in l$, $[y_1, z_1] \subset l \cap N_1$; $x \in (y_1, z_1)$. By Bii for N_2, it follows that there exists $y_2, z_2 \in l \cap N_2$ with $x \in (y_2, z_2) \subseteq l \cap N_2$. From A3 it results that there exists a segment $[y,z]$ s.t.: $x \in (y,z)$; $[y,z] = [y_1, z_1] \cap [y_2, z_2] \subseteq l \cap N$; $[y,z]$ contains at least two distinct events from $[y_1, z_1]$. Since h_1 is adequate it follows that $h_1([y,z]) = [h_1 y, h_1 z] \subseteq d \cap N$ and $b \in (h_1 y, h_1 z)$. This proves Dii for $h_1(N)$. Now take any connected component U of the open set $h_1(N)$. Since h_2 is an adequate luminal chart it follows that the function $F := h_2 h_1^{-1}: h_1(N) \longrightarrow \mathbb{R}^4$ is injective and maps any light segment in $h_1(N)$ onto a light segment. Hence by 1.6, the restriction of F to U is a conformal transformation. Therefore $F(h_1(N)) = h_2(N)$ is an open set.

2.12. Suppose there exist: a set A^o of open, adequate luminal charts $h_a: N_a \longrightarrow \mathbb{R}^4$ s.t.: their domains cover E; each set $N_a \subseteq E$ is luminally complete. By 2.11 the set A^o is a subatlas of E. It extends canonically to an adequate luminal atlas A of E. Indeed let us define the adequate luminal atlas A generated by A^o as being the maximal set of adequate, luminal charts $h: N \longrightarrow \mathbb{R}^4$ which satisfy: for any chart $h_a: N_a \longrightarrow \mathbb{R}^4$ of A^o, it follows that $h_a(N \cap N_a)$ is an open set of $\mathbb{R}^4$. Then, if $U_a := h_a(N \cap N_a) \neq \emptyset$, the function $hh_a^{-1}: U_a \longrightarrow \mathbb{R}^4$ maps any light segment $[c,b]$ of U_a onto a light segment. Indeed put $x = h_a^{-1}c$, $y = h_a^{-1}b$; since N_a is luminally complete it results that $[x,y] \subset N_a$. Thus from $h_a([x,y]) = [c,b]$ and the injectivity of h_a it follows that $[x,y] \subset N_a \cap N$. Then, since h is an adequate luminal chart, $h([x,y]) = hh_a^{-1}([c,b])$, is a light segment.

Thus, by 1.6, it results that: the restriction of hh_a^{-1} to a connected component of U_a is a conformal transformation; $h(N \cap N_a)$ is an open set of $\mathbb{R}^4$; $h_a h^{-1}$ is given also by conformal transformations. Analogously, for any $h, h' \in A$, it follows that $h'h^{-1}$ is given by conformal transformations.

Therefore A gives a structure of differentiable manifold to E; the coordinate transformations are given by conformal transformations.

It E admits a set A^o as above, we say that E satisfies <u>the principle of the constancy of the velocity of light propagation</u>.

2.13. PORPOSITION. <u>Let</u> $G: \mathbb{R}^4 \setminus P \longrightarrow \mathbb{R}^4$ <u>be a conformal transformation. Put</u> : $d=(0,0,0,0)$, $a_1=(1,0,0,0)$, $b_1=(2,0,0,0)$, $a_2=(1,0,1/2,1/2)$ <u>and</u> $b_2=(2,0,1,1)$. <u>Suppose that</u> G <u>maps the collinear points</u> d, a_i, b_i <u>in collinear points, for</u>

i=1,2.Then G is a Weyl transformation.

Proof.First observe that since G is a conformal transformation,the line generated by G(d) and $G(a_1)$(resp.$G(a_2)$) is not a light line.

Now the proof is based on some facts proved in [2].

Let $e_1=(1,1,0,0)$,$f_1=(1,-1,0,0)$,$e_2=(1,0,1,0)$,$f_2=(1,0,0,1)$ be a basis of light vectors in $\mathbb{R}^4$.The points d,b_1,e_1,f_1 (resp.d,b_2,e_2,f_2) are the vertices of a light 2-prism π (resp. τ).By Lemma 2.3 from [2], π (or τ) is mapped on a 2-plane or on a hyperboloid H.Any straight line except the light lines which generate H,meets H in at most two points (see formula 2.1o in [2]);since G(d), $G(a_1)$,$G(b_1)$ $\in$ H,this contradicts the hypothesis that they are collinear.Therefore we can find in π and τ the parallel light lines used in Corollary 1.3 from [2], which being mapped by G in parallel light lines assure us that G is a Weyl transformation.

2.14.Let A be as in 2.12.We say that E,A satisfies the Einstein principle of inertia if there is a subset B^o of A s.t.:the domains of the charts from B^o cover E;for any $h_1,h_2 \in B^o$ with N=dom $h_1 \cap$ dom $h_2 \neq \emptyset$,the function $h_2h_1^{-1} : h_1(N) \longrightarrow \mathbb{R}^4$ is a Weyl transformation.The set B^o extends canonically to a Weyl atlas B of E i.e.an atlas for which,by definition,the coordinate transformations are Weyl.As above and using also A2 and A4 it can be proved that the charts of B are luminal and adequate.Observe that for $h_1,h_2 \in$ A,the function $h_2h_1^{-1}$ is a Weyl transformation if it satisfies conditions similar to those given in 2.13.

2.15.Remark.From the fact that E,A satisfies the Einstein principle of inertia it does not follow that the global structure of E is isomorphic with the 4-dimensional minkowskian space.

2.16. In § 2 the dimension of $\mathbb{R}^4$ is not essential. Similar considerations could be done for any "model space" $\mathbb{R}^m$ with $m \geq 3$.

Since our approach does not impose a global minkowskian structure to E (see 2.15), it seems to be more convenient that the approach developed in [3].

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