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STIINTIFICA SI TEHNICA

ISSN 0250 3638

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Radu-Nicolae GOLOGAN

PREPRINT SERIES IN MATHEMATICS

No.66/1982

BUCURESTI

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Radu-Nicolae GOLOGAN*)

October 1982

*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd.Pacii 220, 79622 Bucharest, Romania

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Radu-Nicolae GOLOGAN

Abstract. We extend the maximal ergodic theorem of Hopf to the case of σ- lattice cones introduced by Cornea and Licea in [1]. As consequences we prove some abstract potential theory results of maximal type and an abstract pointwise ergodic theorem.

The concept of σ- lattice cone of Cornea and Licea can be viewed as an abstract setting of the cone of positive measurable function over a measurable space. The aim of this paper is to extend the pointwise ergodic theorem in this abstract case. The large class of nontrivial examples of σ- lattice cones proves that such results could be useful.

For the begining let us recall some definitions, notations and basic results from [1] which we need in the sequell.

An ordered convex cone $(C,\leq,+)$ is called a σ- lattice cone if the following are true:

a) For any $x\in C$ we have $x\geq 0$;

b) For any $x,y\in C$ such that $x\leq y$ there exists $z\in C$ such that $x+z=y$;

c) The ordered set C is a σ- complete lattice;

d) Denoting as usual by "$\wedge$" (resp. "$\vee$") the infimum (resp. supremum) operation, for every $x\in C$ and any sequence $(x_n)_{n\in\mathbb{N}}\subset C$ we have:

$$x\vee(\bigwedge_{n\in\mathbb{N}} x_n)=\bigwedge_{n\in\mathbb{N}}(x\vee x_n)$$

$$x\wedge(\bigvee_{n\in\mathbb{N}} x_n)=\bigvee_{n\in\mathbb{N}}(x\wedge x_n)$$

$$x+\bigwedge_{n\in\mathbb{N}} x_n=\bigwedge_{n\in\mathbb{N}}(x+x_n)$$

$$x+\bigvee_{n\in\mathbb{N}} x_n=\bigvee_{n\in\mathbb{N}}(x+x_n)$$

If C is a σ-lattice cone, an element $x\in C$ is called finite if for every y, $y\leq x$ the element $z\in C$ such that $x=y+z$ is unique, that is equivalent with $\bigwedge_{n\geq 1} 1/_n x=0$. The cone of finite elements will be denoted by C_s.

The set $|C|$ defined formally by $|C|=C-C_s$ has in a natural way a lattice structure induced from that of C, in such a way that $|C|$ becomes an upper - σ-complete and conditionally lower-σ- complete lattice. The relations d) hold also in $|C|$.

C and C' being σ- lattice cones, a map $T:C\rightarrow C'$ is called a kernel if $T0=0$ and if for every sequence $(x_n)_{n\in\mathbb{N}}$ from C we have $T(\sum_{n=0}^{\infty} x_n)=\sum_{n=0}^{\infty} Tx_n$, the infinite sum being considered in order.

A kernel $T:C\rightarrow C'$ is called proper if for every $x\in C$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in C, increasing to x, such that $Tx_n\in C'_s$ for every $n\in\mathbb{N}$.

We say that a σ- lattice cone is proper if the identity map is a proper kernel.

For any $x\in C$ we denote by I_x the map $I_x:C\rightarrow C$ defined by

$$I_x y=\bigvee_{n\in\mathbb{N}}[(nx)\wedge y]$$

It is easy to see that for any $x\in C$, I_x is a kernel with the following properties:

(1) $I_x\leq$ identity

(2) $I_x^2=I_x$

(3) $I_x(\bigvee_{n\in\mathbb{N}} x_n) = \bigvee_{n\in\mathbb{N}} I_x x_n$

$I_x(\bigwedge_{n\in\mathbb{N}} x_n) = \bigwedge_{n\in\mathbb{N}} I_x x_n$

$I_{\bigvee_{n\in\mathbb{N}} x_n} = \bigvee_{n\in\mathbb{N}} I_{x_n}$, for every $(x_n)_{n\in\mathbb{N}}$ in C.

We shall say that I_x is the indicator of x.

Moreover, if for $z\in|C| = C-C_s$ we put $z^+=z\vee 0$, $z^-=-z\wedge 0$ (in $|C|$) we have $z=z^+-z^-$ and for every $x\in C$, $y\in C_s$.

$I_{(x-y)^+}(x-y)^- = 0,$

and

$I_{(x-y)^+}\, x \geq I_{(x-y)^+}\, y$

A measure on C is a kernel $\mu : C \to \overline{\mathbb{R}}_+$. The set of measures on C is a σ- lattice cone that is complete.

If T is a kernel on C, an element $x\in C$ (respectively a measure μ on C) is called T-supermedian if $Tx \leq x$ (respectively $\mu(Tx)\leq\mu(x)$ for every $x\in C$). An element $x\in C$ (resp. a measure μ) will be called T-invariant if equalities hold in the last relations.

If $x\in C_s$ is T-supermedian the Riesz decomposition theorem asserts that there exist unique u, $v\in C_s$ such that

$$x = G_T u+v,$$

where $G_T=I+T+\ldots+T^n+\ldots$ and $v=\bigwedge_{n\geq 0} T^n x$ satisfies $Tv=v$.

We need also the following natural construction.

Let μ be a measure on the σ- lattice cone C and let us denote by C_o^μ the σ- complete subcone of C of those elements $x\in C$ having zero μ- measure, that is $\mu(x)=0$.

Defining in C the equivalence relation $\sim$ by: $x\sim y$ iff there exists $x_o\in C_o^\mu$ such that $x\leq y+x_o$ and $y\leq x+x_o$, the set of classes C/C_o^μ becomes a σ-lattice cone. If we denote by $\dot{x}$ the class of $x\in C$, the following are true:

(1) $\dot{x}\in(C/C_o^\mu)_s$ iff $\bigwedge_{n\in\mathbb{N}} (1/n)x\in C_o^\mu$;

(2) $\dot{\mu} : C/C_o^\mu \to \overline{\mathbb{R}}_+$ defined by $\dot{\mu}(\dot{x})=\mu(x)$ is a measure on

C/C_0^μ and $\dot{\mu}(\dot{x})=0$ implies $\dot{x}=\dot{0}$;

(3) if μ is T-supermedian the map $\dot{T}$ on C/C_0^μ defined by $\dot{T}\dot{x}=\widehat{Tx}$ is a kernel on C/C_0^μ.

Two elements $x,y\in C$ are called μ-almost everywhere (a.e.) equal if $\dot{x}=\dot{y}$.

For a sequence $(x_n)_{n\in\mathbb{N}}$ in C we shall define as usual the upper limit and the lower limit by:

$$\limsup_{n\to\infty} x_n = \bigwedge_n \bigvee_{m\geq n} x_m$$

$$\liminf_{n\to\infty} x_n = \bigvee_n \bigwedge_{m\geq n} x_m$$

We shall say that the limit of the sequence $(x_n)_{n\in\mathbb{N}}$ exists if $\limsup x_n = \liminf x_n$ and that the limit exists μ-a.e. if $\widehat{\limsup} x_n = \widehat{\liminf} x_n$. In particular if $\mu(\limsup x_n)<\infty$ and $\mu(\liminf x_n)=\mu(\limsup x_n)$ the limit exists μ- a.e.

Finally, two elements $x,y\in C$ are said to have the same support μ- a.e. if $I_{\dot{x}}=I_{\dot{y}}$ as kernels in C/C_0^μ

The results of the paper can now be formulated.

The first one is the natural extension of Hopf's maximal ergodic lemma.

If T is a kernel on the σ- lattice cone C satisfying $TC_s\subset C_s$ and $x\in C$ let us denote by $r_n(x,T)=r_n(x)$ the element $1/_n(x+Tx+\ldots+T^{n-1}x)$ for every $n\geq 1$

Proposition. (Maximal ergodic lemma). Let C be a σ-lattice cone, T a kernel on C satisfying $TC_s\subset C_s$ and μ a proper T-supermedian measure. If $x=x'-x''\in|C|$ with $x'\in C$, $x''\in C_s$ and $X_N=\bigvee_{n=1}^{N} r_n(x,T)$ for $N\geq 1$ we have:

$$\mu(I_{X_N^+}x')\geq\mu(I_{X_N^+}x'') \text{ for every } N\geq 1 .$$

<u>Proof.</u> The proof will be on the line of that of Garcia for the classical ergodic lemma ([2]). Let us suppose first that

$\mu(x')$ and $\mu(x'')$ are finite. From the fact that $X_N^+ \geq r_n(x,T)$ we have that $TX_N^+ \geq Tr_n(r,T)$ for every $n=0,\ldots,N-1$ considering $r_o=0$. Adding x in both parts of the last inequality we obtain that $TX_N^+ \geq x_{n+1}$ for $n=0,1,\ldots,N-1$, that is:

$$TX_N^+ + x \geq X_N$$

or:

$$TX_N^+ + x' + X_N^+ \geq x'' + X_N^+$$

Applying the kernel $I=I_{X_N^+}$ to the last inequality we have:

$$ITX_N^+ + Ix' \geq Ix'' + IX_N^+ = Ix'' + X_N^+$$

This implies that:

$$\mu(ITX_N^+) + \mu(Ix') \geq \mu(Ix'') + \mu(X_N^+)$$

But $I \leq id$ and μ is T-supermedian which together with $\mu(X_N^+) < \infty$ implies the desired inequality.

If $x' \in C$ or $x'' \in C_s$ have not finite measure it will suffice to use the fact that μ is proper: there are increasing sequences $(x'_n)_{n \geq 0}$ and $(x''_n)_{n \geq 0}$ such that $\vee x'_n = x$, $\vee x''_n = x''$ and $\mu(x'_n) < \infty$, $\mu(x''_n) < \infty$ for every $n \in \mathbb{N}$. We can apply the preceeding proof for $x_{n,m} = x'_n - x''_m$ and then use a standard upper limit argument.

The following consequences of the abstract ergodic lemma can be viewed as abstract potential theory results.

<u>Theorem I.</u> Let C,T and μ be as above and $x,y \in C_s$ such that y is T-invariant. The following then hold:

(i) $y \geq \bigwedge_{n=1}^{\infty} 1/_n r_n(x,T)$ implies $\mu(y) \geq \mu(I_y x)$

(ii) $y \leq \bigvee_{n=1}^{\infty} 1/_n r_n(x,T)$ implies $\mu(y) \leq \mu(I_y x)$.

<u>Proof.</u> We shall apply the preceeding proposition for $z = \varepsilon y - x$, where $\varepsilon > 1$ is arbitrary. We have:

$$\mu(I_{z_N^+} \varepsilon y) \geq \mu(I_{z_N^+} x);$$

where $Z_N^+ = \bigvee_{n=1}^{N} r_n(z,T)$. Making N to tend to infinity, the sequence Z_N^+ being increasing, we obtain:

$$\mu(I_{Z^+}\varepsilon y) \geq \mu(I_{Z^+}x), \qquad (*)$$

where

$$I_{Z^+} = I_{\bigvee_{n=1}^{\infty} r_n(z,T)} = I_{[\varepsilon y - \bigwedge_{n=1}^{\infty} 1/_n (x+Tx+\ldots+T_x^{n-1})]^+},$$

the last equality being an easy consequence of the T-invariance of y and the distributivity laws in $|C|$.

Moreover the inequalities $y \geq \bigwedge_{n=1}^{\infty}(1/_n r_n(x,T))$ and $\varepsilon > 1$ implie, as a direct consequence of the definition of the indicator kernel, that $I_{Z^+} = I_y$. Thus the inequality (*) can be writen:

$$\varepsilon\mu(y) = \mu(I_y \varepsilon y) \geq \mu(I_y x)$$

In order to obtain inequality (i) it is sufficient to make $\varepsilon \downarrow 1$.

The proof of (ii) runs in the same way if we apply the ergodic lemma for $x - \varepsilon y$ where $0 < \varepsilon < 1$.

The following is an immediate consequence of theorem 1.

<u>Corollary 1</u> . Let C,T and μ be as above and let $x \in C_s$ have finite μ- measure. Then every T-invariant finite element $y \in C$ satisfying $x \leq y \leq \bigvee_{n=1}^{\infty} 1/_n r_n(x,T)$ equals x μ-a.e. In the same manner every T-invariant element $y \in C$ having μ- a.e. the same support as x (that is: $\mu(I_y x) = \mu(x)$) and satisfying $\bigwedge_{n=1}^{\infty} 1/_n r_n(x,T) \leq y \leq x$ equals x μ- a.e.

<u>Proof</u>: For the first part, we have from Theorem 1(i) that $\mu(y) \leq \mu(I_y x)$. But $\mu(I_y x) \leq \mu(x)$ so $\mu(x) = \mu(y)$, which combined with $y \geq x$ and $\mu(x) < \infty$ concludes the proof.

The proof of the second part makes use, in the same way of Theorem 1 (ii).

It is interesting to reed this corollary in the case when C is the cone of positive measurable functions over a σ-finite measure space, T being the extension of a $L_1(X,\mathcal{X},\mu)$-positive contraction. For example if $f\in L_1$ is positive and $\sup_{n\geq 1} 1/_n(f+Tf+\ldots+T^{n-1}f)=\infty$ μ-a.e., our results asserts that there exists no T-invariant finite positive measurable function greater than f μ-a.e. Also if $f\gneq 0$ is in L_1 and $\inf_{n\geq 1} 1/_n(f+Tf+\ldots+T^{n-1}f)=0$ μ-a.e., than there exists no T-invariant measurable positive function less than f μ- a.e. and having μ- a.e. the same support as f.

The second corollary can be viewed as a disjointness result in the Riesz decomposition.

Corollary 2. Let C, T and μ be as above. Suppose that $x\in C_s$ is T-supermedian and $x=G_Tu+v$ is the Riesz decomposition. Then:

$$\mu(v) = \mu(I_V x)$$

In particular if $\mu(x)<\infty$ we have $\mu(I_V G_T u)=0$, that is the invariant part and the potential part have μ-a.e. disjoint supports.

Proof. From theorem 1(i) we have that $\mu(v)\geq\mu(I_V x)$ because v is invariant and $v=\bigwedge_{n\geq 1}^{\infty} T^n x=\bigwedge_{n=1}^{\infty} 1/_n r_n(T,x)$, the oposite inequality being obvious. For the second part apply the kernel I_V and the measure μ to $x=G_Tu+v$.

Our generalisation of the pointwise ergodic theorem is also a consequence of theorem 1. However the abstract setting and absence of units involves some more assumptions.

Theorem 2 (Ergodic theorem). Let C, T be as above and let μ be a T-invariant proper measure. Let $x\in C$ and suppose that $\mu(\limsup_{n\to\infty} r_n(T,x))<\infty$. Then the following are equivalent:

a) $\limsup 1/_n r_n(T,x)$ and $\liminf 1/_n r_n(T,x)$ have μ-a.e. the same support;

b) the limit of $1/_n r_n(T,x)$ exists μ-a.e. Moreover in every case we have:

$$\mu(\liminf 1/_n r_n(T,x))=\mu(\limsup 1/_n r_n(T,x))=\mu(I_{\liminf 1/_n r_n(x,T)}x)$$

Proof. Let us use the following notations:

$$x^{*}=\limsup 1/_n r_n(T,x)$$
$$x_{*}=\liminf 1/_n r_n(T,x)$$

By standard arguments we have $Tx_{*}\leq x_{*}$ and $Tx^{*}\geq x^{*}$, that is by the T-invariance of the measure μ and the supposition that x^{*} has μ- finite measure that in $(C/C_0^{\mu})_s$ $\dot{x}_{*}$ and $\dot{x}^{*}$ are T-invariant.

The implication b) => a) being obvious, let us remark, in proving the oposite one, that $\dot{x}^{*}\leq \bigvee_{n=1}^{\infty} 1/_n r_n(\dot{T},\dot{x})$ and $\dot{x}_{*}\geq \bigwedge_{n=1}^{\infty} 1/_n r_n(T,x)$, so by theorem 1 used in C/C_0^{μ}, we have:

$$\dot{\mu}(\dot{x}_{*})\geq\dot{\mu}(I_{\dot{x}_{*}}\dot{x})$$

and

$$\dot{\mu}(\dot{x}^{*})\leq\dot{\mu}(I_{\dot{x}^{*}}\dot{x})$$

As, by usual arguments, it is easily seen that $\dot{\mu}(I_{\dot{x}_{*}}\dot{x})=\mu(I_{x_{*}}x)$ and $\dot{\mu}(I_{\dot{x}^{*}}\dot{x})=\mu(I_{x^{*}}x)$, by combining the two inequalities we obtain the desired result.

Finally, let us remark that in the classical L_1-case discussed above, theorem 2 gives necessary and sufficient conditions that for $f\in L_1$, $f\geq 0$ the ergodic average converges μ- a.e., knowing that $\limsup_{n\to\infty} 1/_n(f+Tf+\ldots+T^{n-1}f)$ is integrable, without knowing the L_∞-behaviour of T.

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