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BOUNDARY CONTROL FOR A STEFAN PROBLEM

by

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BOUNDARY CONTROL

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by D.TIBA

1. Introduction

Let Ω be a bounded domain of the Euclidean space R^N with sufficiently smooth boundary Γ and Q be the cylinder $\Omega \times]0,T[$ with lateral face $\Sigma = \Gamma \times]0,T[$.

We derive existence and approximation results for the boundary control problem:

(P) Minimize $\int_0^T \left\{ \frac{1}{2} | y - y_d |^2_H + \frac{1}{2} | u |^2_{L^2(\Gamma)} \right\} dt$

over the set all functions $y \in L^2(0,T ; H)$ and $u \in L^2(\Sigma)$ subject to:

(1.1) $v_t(t,x) - \Delta y(t,x) = 0$ a.e.Q,

$v(t,x) \in \beta(y(t,x))$ a.e.Q,

(1.2.) $\frac{\partial y}{\partial n} = u$ a.e.Σ

(1.3) $y(0,x) = y_0(x)$ a.e. Ω

Here $H = L^2(\Omega)$, $y_d \in L^2(0,T;H)$, $|\cdot|_X$ denotes the norm of the space X and $\beta \subset R \times R$ is a maximal monotone graph. When β is given by :

$$(1.4) \qquad \beta(r) = \begin{cases} r - r_0 & \text{if } r > r_0 \\ [-\delta, 0] & \text{if } r = r_0 \\ K(r-r_0) - \delta & \text{if } r < r_0 \end{cases}$$

where K, δ are positive constants, we obtain a two phases Stefan free boundary problem (see J.L.Lions [8] , p.196).

Starting from some problems arising in metallurgy, a similar control process was considered by C.Saguez [13] , Ch.4, by means of a semi-discretisation method. Our approach is different and can be compared with the papers by V.Barbu [2], [3] . It consists in replacing the problem (P) by a family of smooth problems and afterwards to tend to the limit in the approximate optimality conditions.

The case of the distributed control for the system (1.1)-(1.3) was studied by Z.Meike and D.Tiba [12] .

Denote $V = H^1(\Omega)$ and $A = V \to V^*$ the linear continuous operator defined by

(1.5) $$(Ay,z) = \int_\Omega \text{grad}\, y . \text{grad}\, z \; dx \qquad y,z \in V.$$

Consider $\varphi : H \quad]-\infty, +\infty]$ the convex, lower - semicontinuous proper function given by

(1.6) $$\varphi(y) = \begin{cases} \int_\Omega j(y(x) \; dx & \text{if } j(y) \in L^1(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

and $B : H \to H$ the maximal monotone operator, $B = \partial\varphi$.

Here $j : R \to]-\infty, +\infty]$ is the convex, lower-semicontinuous function such that $\beta = \partial j$ and $(. , .)$ is the pairing between V and V^*.

Equation (1.1) - (1.3) can be written as:

(1.7) $$\frac{dv}{dt} + Ay = f \qquad \text{a.e. } [o, T]$$

$$v(t) \in B(y(t)) \qquad \text{a.e. } [o, T]$$

(1.8) $$y(o) = y_0 \in H,$$

where $f \in L^2(o,T ; V^*)$ satisfies:

(1.9) $$\int_0^T (f(t), \Psi(t))\, dt = \int_0^T \int_\Gamma u . \Psi \, d\Gamma \; dt$$

for every $\Psi \in L^2(o,T ; V)$ and u from (1.2).

There is an extensive literature treating equation (1,7),(1,8) under various compactness or boundedness assumptions on operators A and B (which can both be nonlinear), but requiring differentiability properties for f. See e.g. O.Grange and F.Mignot [10], V. Barbu [4] , C. Saguez [13] .

The case $f \in L^2(o, T; V^*)$ is considered in a recent paper by E. Di Benedetto and R.E. Showalter [7] also allowing A and B to be nonlinear. Our <u>Proposition 3.1</u> can be deduced from their results. However we indicate a direct approach which enables us to prove in

Proposition 3.4. a weak continuous dependence on the right hand side for Eqs. (1,7), (1.8).

Our work is organized as follows. Section 2 is devoted to some preliminaries. In Section 3 we establish the existence of an optimal control and Section 4 contains an approximation process for problem (P). In the last section we add some remarks on the necessary optimality conditions.

The main results of the paper are Theorem 3.5. and Theorem 4.6.

Preliminaries

All the spaces are real. If E is a Banach space, then $L^P(0, T; E)$, $1 \leq p \leq \infty$, is the space of all p-integrable, E - valued functions; $C(o, T; E)$ is the Banach space of continuons E - valued functions and

$$W^{1,P}(o,T;E)=\left\{y \in L^P(o,T;E),\ \frac{dy}{dt} \in L^P(o,T;E)\right\}.$$

We also denote by $H^K(\Omega)$, $W^{K,P}(\Omega)$, $H^s(\Gamma)$ usual Sobolev spaces of real functions.

Let $\varphi : E \rightarrow]-\infty, +\infty]$ be a convex, lower-semicontinuous function. We write $\partial\varphi(x) \subset E^*$ (the dual space) for the set of all subgradients of φ at x

$$\partial\varphi(x)= \left\{ x^* \in E^*; \varphi(x) \leq \varphi(y) + (x^*, x-y), \forall y \in E \right\}.$$

When φ is Gâteaux differentiable, then $\partial\varphi(x)$ is single valued, $\partial\varphi(x) = \nabla\varphi(x)$.

The following two theorems will be used in the sequel :

Theorem 2.1 (V.Barbu and Th.Precupanu, [6] , p117).

Let E be a real reflexive Banach space and H be a real Hilbert space. Let $\varphi : E \rightarrow]-\infty, +\infty]$ be a convex function defined by $\varphi(x) = f(Ax)$ for $x \in E$, where f is lower - semicontinuous proper convex function defined on H, while A is a linear continuous operator from E into H.

Suppose that $\operatorname{int} D(f) \cap R(A) \neq \Phi$. Then, for every $x \in E$, Consider now two Hilbert spaces V and H such that $V \subset H$ and the inclusion mapping of V into H is continuous. We are given a linear continuous operator $A : V \to V^*$ which is assumed to satisfy:

(2.1) $(Ay, z) = (Az, y) \qquad y, z \in V$

(2.2) $(Ay, y) \geq \omega |y|_V^2 \qquad \forall y \in V, \omega > o.$

Let $\varphi : V \to]-\infty, +\infty]$ be a lower-semicontinuous convex function nonidentically $+\infty$.

We consider the following variational problem

(2.3) $\frac{dy}{dt} + Ay + \partial\varphi(\frac{dy}{dt}) \ni f \qquad$ a.e. $]o, T[$

(2.4) $y(o) = y_o$.

Theorem 2.2. (V.Barbu [5], p. 213)

We are given $y_o \in V$, $f \in W^{1,2}(o, T; V^*)$.

Then problem (2.1), (2.2) has a unique solution $y \in C(o,T; V)$ which satisfies

(2.5.) $\sqrt{t}\frac{dy}{dt} \in L^2(o,T; V)$

(2.6) $\frac{dy}{dt} \in L^2(o,T; H)$.

3. The Existence of the Optimal Control

We remark that A given by (1.5) satisfies (2.1), but not (2.2). Instead we have ;

(3.1) $(Ay + \varepsilon y, y) \geq \varepsilon|y|_V^2$, $\forall y \in V$, for all $\varepsilon \geq o$.

Taking into account (1.6), (1.4) and assuming for convenience that $K \geq 1$, we notice the following properties of nonlinear operator B :

(3.2) $B = I + D$

where $D \subset H \times H$ is a cyclically maximal monotone operator;

(3.3) dom (B) = H and

$$|B(y)|_H \leq C_1 |y|_H + C_2 .$$

We denote $w(t) = \int_o^t y(s)\,ds$ and transform (1.7),(1.8) to obtain the pseudo-parabolic equation:

(3.4) $B\left(\frac{dw}{dt}\right) + Aw \ni g$ a.e. $]o,T[$,

$w(o) = 0$

where $g \in W^{1,2}(o, T; V^*)$ is given by

(3.5.) $g(t) \in \int_0^t f(s)\,ds + B\,y_o$.

Let $i : V \to H$ be the canonical injection and $\bar{\varphi} : V \to]-\infty,+\infty]$ be given by $\bar{\varphi}(v) = \varphi(i(v)) = \varphi|_V(v)$. By Theorem 2.1 we get:

(3.6) $\bar{B}\,v = \partial\bar{\varphi}(v) = i^* \partial\varphi(i(v)), \forall v \in V,$

(3.7) $\bar{B}(v) = B\,|_V\,(v).$

Proposition 3.1 Equation(3.4) has a unique solution $w \in C(o,T; V)$, $\frac{dw}{dt} \in L^2(o,T;H)$ and $B\left(\frac{dw}{dt}\right) \in L^2(o,T;H)$.

Proof

By Theorem 2.2 the approximate equation

(3.8) $\bar{B}\left(\frac{dw_\varepsilon}{dt}\right) + \varepsilon w_\varepsilon + A w_\varepsilon \ni g$ a.e. $]o,T[$, $w_\varepsilon(o) = 0.$

has a unique solution $w_\varepsilon \in C(o,T; V)$ such that, $\frac{dw_\varepsilon}{dt} \in L^2(o,T;H)$, $\sqrt{t}\cdot\frac{dw_\varepsilon}{dt} \in L^2(o,T; V)$.

We use (3.7) and (3.2) and multiply (3.8) by $\frac{dw}{dt}$:

(3.9) $\int_0^t \left|\frac{dw_\varepsilon}{dt}\right|_H^2 + \frac{1}{2}\left(Aw_\varepsilon(t), w_\varepsilon(t)\right) \leq \int_0^t \left(g+D(o), \frac{dw_\varepsilon}{dt}\right)$.

Next we integrate by parts in the right hand side and use the inequality :

(3.1o) $\frac{1}{T}\left|w_\varepsilon(t)\right|_H^2 \leq \int_0^t \left|\frac{dw_\varepsilon}{dt}\right|_H^2$

to get $\{w_\varepsilon\}$, $\left\{\frac{dw_\varepsilon}{dt}\right\}$ bounded in $L^\infty(o,T; V)$, respectively $L^2(o,T;H)$. From (3.3) it yields $\left\{B\left(\frac{dw_\varepsilon}{dt}\right)\right\}$ bounded in $L^2(o,T;H)$.

We subtract, two equations (3.8) and multiply by $\frac{dw_\varepsilon}{dt} - \frac{dw_\lambda}{dt}$:

$$\int_0^t \left|\frac{dw_\varepsilon}{dt} - \frac{dw_\lambda}{dt}\right|_H^2 + \int_0^t \left(\varepsilon w_\varepsilon - \lambda w_\lambda, \frac{dw_\varepsilon}{dt} - \frac{dw_\lambda}{dt}\right) +$$

$$+ \int_0^t \left(Aw_\varepsilon - Aw_\lambda, \frac{dw_\varepsilon}{dt} - \frac{dw_\lambda}{dt}\right) \leq 0 .$$

Then, we conclude

$\frac{dw_\varepsilon}{dt} \to \frac{dw}{dt}$ strongly in $L^2(o, T; H)$,

$w_\varepsilon \to w$ strongly in $C(o,T;V)$,

$D(\frac{dw_\varepsilon}{dt}) \to D(\frac{dw}{dt})$ weakly in $L^2(o,T;H)$ by the demiclosedness of maximal monotone operators.

Definition 3.2. Function $y = \frac{dw}{dt} \in L^2(o,T;H)$ will be called the generalized solution of the Stefan problem (1.1) - (1.3).

Remark 3.3. From the next proposition we'll see that in fact $y \in L^2(o,T;V)$.

Denote by $\theta : L^2(\Sigma) \to L^2(Q)$ the mapping $u \to y$, where y is given by Definition 3.4.

Proposition 3.4. Let $u_n \to u$ weakly in $L^2(\Sigma)$. Then $y_n = \theta(u_n) \to y = \theta(u)$ weakly in $L^2(o,T;V)$.

Proof

From (3.5) and (1.9) we obtain $g_n \to g$ weakly in $W^{1,2}(o,T;V^*)$.

We have

(3.11) $$\frac{dw_n}{dt} + D(\frac{dw_n}{dt}) + A w_n \ni g_n \qquad \text{a.e.}]o,T[$$

$$w_n(o) = 0,$$

and as in the preceding proof we infer $\{w_n\}, \{\frac{dw_n}{dt}\}, \{D(\frac{dw_n}{dt})\}$ bounded in $L^\infty(o,T;V)$, $L^2(o,T;H)$ respectively.

Put $\bar{A} : V \to V^*$ given by $\bar{A}v = v + Av$ and let $v_n \in L^2(o,T;V)$ be such that

(3.12) $$\bar{A} v_n = g_n - Aw_n$$

Then $v_n = h_n - w_n$, where $h_n = \bar{A}^{-1}(g_n + w_n)$ is weakly convergent $h_n \to h$ in $W^{1,2}(o,T;V)$.

From (3.11) we get

(3.13) $$\frac{dv_n}{dt} + (I + D)^{-1}\bar{A}v_n = \frac{dh_n}{dt}$$

$$v_n(o) = v_o \in \bar{A}^{-1}(By_o).$$

But $(I + D)^{-1} = B^{-1} = \partial\varphi^*$ where the convex, lower-semicontinuous function $\varphi^* : H \to]-\infty, +\infty]$ is the conjugate of φ and we can assume $\varphi^*(v) \geq$ ct. since $0 \in \operatorname{dom}(\varphi)$.

Multiply (3.13) by $\bar{A} \dfrac{dv_n}{dt}$:

$$\int_o^T \left(\frac{dv_n}{dt} , \bar{A} \frac{dv_n}{dt} \right) + \int_o^T \left(\partial\varphi^{*} (\bar{A}v_n), \frac{d}{dt} \bar{A}v_n \right) = \int_o^T \left(\frac{dh_n}{dt} , \bar{A} \frac{dv_n}{dt} \right).$$

Therefore $\left\{ \dfrac{dv_n}{dt} \right\}$ is bounded in $L^2(o,T; V)$ and by consequence $\left\{ \dfrac{dw_n}{dt} \right\}$ is bounded in $L^2 (o,T;V)$.

By (3.11) we see that $g_n - Aw_n$ bounded in $L^2(o,T;H)$. We also have

$$\frac{d}{dt}(g_n - Aw_n) = \frac{dg_n}{dt} - A \frac{dw_n}{dt}$$

bounded in $L^2(o,T; V^{*})$. Since $H \subset V^{*}$ compact, the Aubin [1] theorem gives

$g_n - Aw_n \rightarrow g - Aw$ strongly in $L^2(o,T;V^{*})$ where $w \in L^{\infty} (0, T; V)$ is such that

$w_n \rightarrow w$ weakly* in $L^{\infty} (o,T; V)$

$\dfrac{dw_n}{dt} \longrightarrow \dfrac{dw}{dt}$ weakly in $L^2(o,T; V)$.

We obtain $\bar{B} \left(\dfrac{dw_n}{dt}\right) \rightarrow d$ strongly in $L^2(o,T; V^{*})$ and by the demiclosedness of maximal monotone operators it yields $d \in B \left(\dfrac{dw}{dt}\right)$ a.e. $]o,T[$.

Finally we can pass to the limit in (3.11) and obtain $y = \dfrac{dw}{dt} = \theta (u)$ which finishes the proof.

<u>Theorem 3.5.</u> <u>There is an optimal pair</u> (u^{*}, y^{*}) <u>in</u> $L^2 (\Sigma) \times L^2 (o,T; V)$ <u>for problem</u> (P).

<u>Proof</u>

The functional

$$\Pi (u) = \int_o^T \left\{ \frac{1}{2} | \theta (u) - y_d |^2_H + \frac{1}{2} | u |^2_{L^2 (\Gamma)} \right\} dt \qquad (3.13)$$

is weakly lower-semicontinuous on $L^2 (\Sigma)$ and coercive.

<u>Remark 3.6.</u> We notice that Π is no more convex since θ is nonlinear.

4. An Approximating Process

We consider the approximate control problem:

$$(P_\varepsilon) \quad \text{Minimize} \int_0^T \left\{ \frac{1}{2} | y - y_d |_H^2 + \frac{1}{2} | u |_{L^2(\Gamma)}^2 \right\} dt$$

subject to :

$$(4.1) \qquad \frac{\partial \beta^\varepsilon(y(t,x))}{\partial t} - \Delta y(t,x) = 0 \qquad \text{a.e. } Q,$$

$$(4.2) \qquad \frac{\partial y}{\partial n} = u \qquad \text{a.e. } \Sigma,$$

$$(4.3) \qquad y(o,x) = y_0(x) \qquad \text{a.e. } \Omega,$$

where we define :

$$(4.4) \qquad \beta^\varepsilon(y) = y + \int_{-\infty}^{+\infty} d_\varepsilon (y - \varepsilon^2 \theta) \rho(\theta) \, d\theta ,$$ d_ε is the Yosida approximate of the maximal monotone graph $d(y) = \beta(y) - y$ and ρ is a Friedrichs mollifier, that is $\rho \in C_0^\infty(R)$, $\operatorname{supp}\rho \subset \subset [-1, 1]$, $\rho(-\theta) = \rho(\theta)$, $\rho \geq 0$ and $\int_{-\infty}^{\infty} \rho(\theta) \, d\theta = 1$ (we continue to assume that $K \geq 1$ in (1.4)). Corresponding to β^ε we shall denote as in (1.6), (3.2), $B^\varepsilon = I + D^\varepsilon$.

The solution of (4.1) - (4.3) can be understood in the sense of Definition 3.2. and obviously problem (P_ε) has an optimal piar $[y_\varepsilon, u_\varepsilon] \in L^2(Q) \times L^2(\Sigma)$.

We denote $\theta_\varepsilon : L^2(\Sigma) \to L^2(Q)$, the mapping $u \to y$ given by Eq. (4.1) - (4.3) according to Definition 3.2.

Lemma 4.1 For all $u \in L^2(\Sigma)$, there exists a linear operator $\nabla\theta_\varepsilon(u) : L^2(\Sigma) \to L^2(Q)$ defined by :

$$(4.5) \qquad \nabla\theta_\varepsilon(u) v = \text{weak} - \lim_{\lambda \to 0} \frac{\theta_\varepsilon(u + \lambda v) - \theta_\varepsilon(u)}{\lambda}$$

for all $v \in L^2(\Sigma)$. Moreover :

$$(4.6) \qquad \frac{dz}{dt} + \nabla D^\varepsilon(\theta_\varepsilon(u)) . \frac{dz}{dt} + Az = g \qquad \text{a.e. }]o,T[,$$

$$(4.7) \qquad z(o) = 0$$

$$(4.8) \qquad \nabla\theta_\varepsilon(u) v = \frac{dz}{dt}$$

where the meaning of g is explained below and ∇D^ε is the Gâteaux differential of $D^\varepsilon : H \to H$.

Proof

Let $y_\lambda = \theta_\varepsilon(u + \lambda v)$, $y = \theta_\varepsilon(u)$. Then using the notations of Section 3, we get:

(4.9) $$\frac{dw_\lambda}{dt} + D^\varepsilon\left(\frac{dw_\lambda}{dt}\right) + Aw_\lambda = g + \lambda h \qquad \text{a.e. }]o, T[$$

(4.1o) $$\frac{dw}{dt} + D^\varepsilon\left(\frac{dw}{dt}\right) + Aw = g \qquad \text{a.e. }]o, T[$$

(4.11) $$w_\lambda(o) = w(o) = 0$$

where g, h correspond to u, respectively v, by (1.9) and (3.5).

Subtract (4.9), (4.1o) and multiply by $\frac{dw_\lambda}{dt} - \frac{dw}{dt}$:

$$\int_o^t \left| \frac{dw_\lambda}{dt} - \frac{dw}{dt} \right|_H^2 ds + \frac{1}{2}\left(A(w_\lambda(t) - w(t)), w_\lambda(t) - w(t)\right) \leqslant$$

$$\leqslant \lambda \int_o^t \left(h, \frac{dw_\lambda}{dt} - \frac{dw}{dt} \right) ds.$$

Then $\frac{dw_\lambda}{dt} \to \frac{dw}{dt}$ and $w_\lambda \to w$ strongly in $L^2(o,T;H)$, $C(o,T;V)$. We put : $z_\lambda = \frac{w_\lambda - w}{\lambda}$

that is :

$$\int_o^t \left| \frac{dz_\lambda}{dt} \right|_H^2 + \frac{1}{2}\left(Az_\lambda(t), z_\lambda(t)\right) \leqslant \int_o^t \left(g, \frac{dz_\lambda}{dt} \right) ds.$$

By (3.1o), integrating by parts in the right hand side we get $\{z_\lambda\}$, $\left\{\frac{dz_\lambda}{dt}\right\}$ bounded in $L^\infty(o,T;V)$, $L^2(o,T;H)$.

Since D^ε is Lipschitz of constant $\frac{1}{\varepsilon}$, the Lebesgue theorem shows that

$$\frac{D^\varepsilon\left(\frac{dw_\lambda}{dt}\right) - D^\varepsilon\left(\frac{dw}{dt}\right)}{\lambda} = \frac{D^\varepsilon\left(\frac{dw_\lambda}{dt}\right) - D^\varepsilon\left(\frac{dw}{dt}\right)}{\frac{dw_\lambda}{dt} - \frac{dw}{dt}} \cdot \frac{dz_\lambda}{dt}$$

is weakly convergent in $L^2(o,T;H)$ to $\nabla D^\varepsilon\left(\frac{dw}{dt}\right) \cdot \frac{dz}{dt}$ where z is such that $z_\lambda \to z$ weakly * in $L^\infty(o,T;V)$ and strongly in $C(o,T;H)$.

We can pass to the limit and obtain (4.6) - 4.8) to finish the proof.

Lemma 4.2. For every $\varepsilon > 0$ there is $p_\varepsilon \in L^\infty(0,T;V)$, $\frac{dp_\varepsilon}{dt} \in L^2(0,T;H)$, such that it verifies together with u_ε, y_ε the approximate optimality conditions (4.1) - (4.3) and :

(4.12) $$\frac{dp_\varepsilon}{dt} + \nabla D^\varepsilon(y_\varepsilon) \cdot \frac{dp_\varepsilon}{dt} - Ap_\varepsilon = y_\varepsilon - y_d \quad \text{a.e. }]0,T[$$

(4.13) $$p_\varepsilon(T) = 0$$

and we have

(4.14) $$u_\varepsilon = p_\varepsilon\big|_\Sigma \ .$$

Proof

We denote $p_\varepsilon = -\nabla\theta_\varepsilon(u_\varepsilon)^*(y_\varepsilon - y_d)$ (in fact its extension to the whole Ω) and use (4.6) - (4.8), (1.9), (3.5) and the definition of the adjoint operator to prove (4.12), (4.13). Let

(4.15) $$\Pi_\varepsilon(u) = \int_0^T \left\{ \frac{1}{2} |\theta_\varepsilon(u) - y_d|^2_H + \frac{1}{2}|u|^2_{L^2(\Gamma)} \right\} dt$$

Then $\nabla\Pi_\varepsilon$ vanishes at point u_ε and from the above notation we see that (4.14) is true.

Lemma 4.3. Suppose that $u_\varepsilon \to u$ weakly in $L^2(\Sigma)$, then $\theta_\varepsilon(u_\varepsilon) \to \theta(u)$ weakly in $L^2(0,T;V)$.

Proof

This is a variant of Proposition 3.4.

Remark 4.4. As a consequence we get $\theta_\varepsilon(u) \to \theta(u)$ weakly in $L^2(0,T;V)$ for every u in $L^2(\Sigma)$. Moreover, by the inequality

(4.16) $$(D_\varepsilon(\theta_\varepsilon(u)) - D_\lambda(\theta_\lambda(u)),\ \theta_\varepsilon(u) - \theta_\lambda(u)) \geq$$
$$\geq (D_\varepsilon(\theta_\varepsilon(u)) - D_\lambda(\theta_\lambda(u)),\ \varepsilon D_\varepsilon(\theta_\varepsilon(u)) - \lambda D_\lambda(\theta_\lambda(u)))$$

one can see that $\{\theta_\varepsilon(u)\}$ is a Cauchy sequence and

(4.17) $$\theta_\varepsilon(u) \to \theta(u) \text{ strongly in } L^2(0,T;H).$$

Proposition 4.5. On a subsequence, we have the convergences :

(4.18) $$u_\varepsilon \to u^* \quad \text{strongly in } L^2(0,T;L^2(\Gamma)),$$

(4.19) $$y_\varepsilon \to y^* \quad \text{strongly in } L^2(0,T;H),$$

(4.20) $$p_\varepsilon \to p^* \quad \text{strongly in } C(0,T;H),$$

where $[y^*, u^*]$ is an optimal pair for problem (P).

Proof

By the minimum property we obtain:

$$\Pi_\varepsilon(u_\varepsilon) \leqslant \int_0^T \left\{ \frac{1}{2} |\theta_\varepsilon(u) - y_d|_H^2 + \frac{1}{2} |u|^2_{L^2(\Gamma)} \right\} dt \tag{4.21}$$

for any $u \in L^2(\Sigma)$. From (4.17) and (4.15) we get $\{u_\varepsilon\}$ bounded in $L^2(\Sigma)$, so $u_\varepsilon \to \bar{u}$ weakly in $L^2(\Sigma)$ on a certain subsequence.

Next, using Lemma 4.3 ,(4.17) and the weakly lower-semicontinuity of the norm, it is possible to pass to the limit in (4.21) and prove :

$$\int_0^T \left\{ \frac{1}{2} |\theta(\bar{u}) - y_d|_H^2 + \frac{1}{2} |\bar{u}|^2_{L^2(\Gamma)} \right\} dt \leqslant$$

$$\leqslant \int_0^T \left\{ \frac{1}{2} |\theta(u) - y_d|_H^2 + \frac{1}{2} |u|^2_{L^2(\Gamma)} \right\} dt$$

for any $u \in L^2(\Sigma)$, that is $\bar{u}$ is an optimal control.Henceforth we denote it u^*.

Then $u_\varepsilon \to u^*$, $y_\varepsilon \to y^*$ weakly in $L^2(\Sigma)$, respectively $L^2(0,T;V)$.

Multiply (5.12) by $\frac{dp_\varepsilon}{dt}$ and integrate over $[t, T]$:

$$\int_t^T \left|\frac{dp_\varepsilon}{dt}\right|_H^2 + \frac{1}{2}(Ap_\varepsilon(t), p_\varepsilon(t)) \leqslant C \left(\int_t^T \left|\frac{dp_\varepsilon}{dt}\right|_H^2 \right)^{\frac{1}{2}}.$$

It follows $\{p_\varepsilon\}$, $\left\{\frac{dp_\varepsilon}{dt}\right\}$ bounded in $L^\infty(o,T;V)$, $L^2(o,T;H)$. The Aubin theorem shows

$$p_\varepsilon \to p^* \text{ strongly in } L^2(o,T;H^{\frac{3}{4}}(\Omega)) \tag{4.22}$$

on a certain subsequence . Then from (4.14) and the trace theorem, Lions - Magenes [9] , we get (4.18).

Using now the same argument as in the proof of Proposition 3.4 and inequality (4.16) we obtain (4.19). Relation (4.2o) is an easy consequence of the above boundedness.

<u>Theorem 4.6. The sequence</u> $\Pi(u_\varepsilon)$ <u>is convergent to the optimal value of problem (P), which we denote S, when</u> $\varepsilon \to 0$.

<u>Proof</u>

By <u>Proposition 4.5.</u> from any subsequence of $\{u_\varepsilon\}$ we can extract another subsequence $\{u_{\varepsilon'}\}$ with properties (4.18), (4.19). Then $\Pi_{\varepsilon'}(u_{\varepsilon'}) \to S$. Therefore the initial sequence satisfies.

(4.23) $$\Pi_\varepsilon(u_\varepsilon) \to S \quad \text{as} \quad \varepsilon \to 0 .$$

New we estimate $\theta_\varepsilon(u_\varepsilon) - \theta(u_\varepsilon)$. We have :

(4.24) $$B\left(\frac{dw^\varepsilon}{dt}\right) + A\,w^\varepsilon \ni g_\varepsilon \qquad \text{a.e. }]0, T[,$$

(4.25) $$B^\varepsilon\left(\frac{dw_\varepsilon}{dt}\right) + A\,w_\varepsilon = g_\varepsilon \qquad \text{a.e. }]0, T[,$$

(4.26) $$w^\varepsilon(0) = w_\varepsilon(0) = 0$$

where $\frac{dw_\varepsilon}{dt} = \theta_\varepsilon(u_\varepsilon)$, $\frac{dw^\varepsilon}{dt} = \theta(u_\varepsilon)$ and g_ε is obtained from u_ε by (3.5), (1.9).

We subtract (4.24), (4.25) and multiply by $\frac{dw^\varepsilon}{dt} - \frac{dw_\varepsilon}{dt}$

(4.27) $$\left|\frac{dw^\varepsilon}{dt} - \frac{dw_\varepsilon}{dt}\right|_H^2 + \left(D\left(\frac{dw^\varepsilon}{dt}\right) - D^\varepsilon\left(\frac{dw^\varepsilon}{dt}\right), \frac{dw^\varepsilon}{dt} - \frac{dw_\varepsilon}{dt}\right) + \left(A\,w^\varepsilon - A\,w_\varepsilon, \frac{dw^\varepsilon}{dt} - \frac{dw_\varepsilon}{dt}\right) = 0 .$$

We use the inequalities :

(4.28) $$\left(D\left(\frac{dw^\varepsilon}{dt}\right) - D_\varepsilon\left(\frac{dw_\varepsilon}{dt}\right), \frac{dw^\varepsilon}{dt} - \frac{dw_\varepsilon}{dt}\right) \geqslant$$
$$\geqslant -\varepsilon\left(D\left(\frac{dw^\varepsilon}{dt}\right) - D_\varepsilon\left(\frac{dw_\varepsilon}{dt}\right), D_\varepsilon\left(\frac{dw_\varepsilon}{dt}\right)\right),$$

(4.29) $$|D^\varepsilon(y) - D_\varepsilon(y)| \leqslant C\,.\,\varepsilon$$

to infer from (4.27) that

$$\int_0^T \left|\frac{dw^\varepsilon}{dt} - \frac{dw_\varepsilon}{dt}\right|_H^2 - C\,.\,\varepsilon + \int_0^T \frac{d}{dt}\left(A\,(w^\varepsilon - w_\varepsilon), w^\varepsilon - w_\varepsilon\right) \leqslant 0$$

and to conclude

(4.30) $$|\theta(u_\varepsilon) - \theta_\varepsilon(u_\varepsilon)|_{L^2(0,T;\,H)} \leqslant C\,.\,\varepsilon^{\frac{1}{2}} .$$

By the definition of Π_ε, Π we get

(4.31) $$|\Pi_\varepsilon(u_\varepsilon) - \Pi(u_\varepsilon)| \leqslant C\,.\,\varepsilon$$

that is, by (4.23), we finish the proof.

Remark 4.7. In order to compute the optimal control for problem (P) we have to choose ε sufficiently small in (4.12) - (4.14) and to find the corresponding u_ε . The result of Theorem 5.5 shows that the performance given by u_ε is as close as necessary to the optimal performance.

5. Final Remarks

We give a partial answer at the question: "What are the equations verified by u^* , y^* , p^* ? "

In the case of control systems governed by variational inequalities this question was posed by Mignot [11] .

We impose the additional assumption :

(5.1) $$\operatorname{mes}\left\{ (t,x) \in Q \; ; \; y^*(t,x) = r_0 \right\} = 0$$

where r_0 is given in (1.4).

We can prove the following result :

Proposition 5.1. Under the above hypotheses, we have :

(5.2) $$(y_\varepsilon - r_0) . \nabla\beta^\varepsilon(y_\varepsilon) . \frac{dp_\varepsilon}{dt} \to (y^* - r_0) . \nabla\beta(y^*) . \frac{dp^*}{dt}$$

weakly in $L^1(Q)$.

Proof

From (1.4) and (4.4) we notice :

(5.3.) $$\nabla d_\varepsilon(r) = \begin{cases} 0 & r > r_0 - \varepsilon r_0 \\ \frac{1}{\varepsilon} & r_0 - \varepsilon\nu < r < r_0 - \varepsilon r_0 \\ \frac{m}{1+\varepsilon m} & r < r_0 - \varepsilon\nu \end{cases}$$

where $m = K - 1 \geq 0$, $\nu = r_0 + \delta$.

We deduce :

(5.4.) $$(r - r_0) \nabla d_\varepsilon (r) = d_\varepsilon(r) - g_\varepsilon(r)$$

where :

(5.5) $$g_\varepsilon(r) = \begin{cases} - r_0 & r > r_0 - \varepsilon r_0 \\ 0 & r_0 - \varepsilon\nu < r < r_0 - \varepsilon r_0 \\ - \frac{\nu}{1+\varepsilon m} & r < r_0 - \varepsilon\nu \end{cases}$$

It follows :

$$(r - r_0)\nabla\beta^{\varepsilon}(r) = (r - r_0)\,(1 + \nabla d^{\varepsilon}(r)) = r - r_0 +$$

$$+ \varepsilon^2 \int_{-1}^{1} \nabla d_{\varepsilon}(r - \varepsilon^2\theta)\,\theta\,\rho(\theta) + D^{\varepsilon}(r) - g^{\varepsilon}(r) \text{ and}$$

(5.6) $$g^{\varepsilon}(r) = \int_{-1}^{1} g_{\varepsilon}(r - \varepsilon^2\theta)\,\rho(\theta)\,d\theta \,.$$

Since d_{ε} is Lipschitz we have $|\varepsilon\nabla d_{\varepsilon}(r)| \leqslant 1$, hence :

$$h^{\varepsilon}(r) = \varepsilon^2 \int_{-1}^{1} \nabla d_{\varepsilon}(r - \varepsilon^2\theta)\,\theta\,\rho(\theta)\,d\theta \to 0$$

uniformly in r. Next we can write :

(5.7) $$(y_{\varepsilon} - r_0)\nabla\beta^{\varepsilon}(y_{\varepsilon}) = \beta^{\varepsilon}(y_{\varepsilon}) - r_0 - g^{\varepsilon}(y_{\varepsilon}) + h^{\varepsilon}(y_{\varepsilon}).$$

The term $g^{\varepsilon}(y_{\varepsilon})$ is bounded in $L^{\infty}(Q)$ by (5.6), (5.5). As concerns $\beta^{\varepsilon}(y_{\varepsilon})$, we see from (3.3), (4.19) that it is bounded in $L^2(o, T; H)$, so :

(5.8) $$\beta^{\varepsilon}(y_{\varepsilon}) \to \beta(y^*) \text{ weakly in } L^2(o,T; H).$$

Because, by (4.19), $y_{\varepsilon} \to y^*$ a.e. Q, we can deduce easily that $\beta^{\varepsilon}(y_{\varepsilon}) \to \beta(y^*)$ a.e. Q.

Here we use (5.1) and (1.4) essentially.

Now, it is obvious that $\beta^{\varepsilon}(y_{\varepsilon}) \to \beta(y^*)$ strongly in $L^2(o,T; H)$.

From (5.7) we see that :

(5.9) $$(y_{\varepsilon} - r_0)\nabla\beta^{\varepsilon}(y_{\varepsilon}).\frac{dp_{\varepsilon}}{dt} \to (\beta(y^*) - r_0).\,p^*_t + q$$

weakly in $L^1(Q)$, where q is the weak limit in $L^2(Q)$ (on a subsequence) of $g^{\varepsilon}(y_{\varepsilon}).\frac{dp_{\varepsilon}}{dt}$.

Using again (5.1) and the boundedness of $g^{\varepsilon}(y_{\varepsilon})$ in $L^{\infty}(Q)$, we prove $g^{\varepsilon}(y_{\varepsilon}) \to g(y^*)$ strongly in $L^2(Q)$, where

$$g(y^*) = \begin{cases} -r_0 & y^*(t,x) > r_0 \\ -\delta & y^*(t,x) < r_0 \end{cases}$$

is defined a.e. Q .

Then $q(t,x) = g(y^*(t,x)).\frac{dp^*}{dt}(t,x)$ a.e. Q .

After a short calculation we arrive at

(5.10) $$(\beta(y^*) - r_o).\frac{dp^*}{dt} + q = (y^* - r_o).\nabla\beta(y^*).\frac{dp^*}{dt}$$

and the proof is finished.

Remark 5.2. The argument we have used above is similar to the one given by Z. Meike and D. Tiba [12], Theorem 4.1.

Remark 5.3. We know that $y_\varepsilon \to y^*$ weakly in $L^2(o,T;V)$ and, from (4.12):

(5.11) $$\nabla\beta^\varepsilon(y_\varepsilon).\frac{dp_\varepsilon}{dt} \to 1 \text{ weakly in } L^2(o,T;V^*).$$

Combining these facts with Proposition 5.1 we formulate the conjecture $1 = \nabla\beta(y^*).\frac{dp^*}{dt}$, that is u^*, y^*, p^* should satisfy the state system (1.1) - (1.3) and the adjoint state system

$$\nabla\beta(y^*).\frac{dp^*}{dt} - Ap^* = y^* - y_d \quad \text{a.e. } o, T$$

$$p^*(T) = 0.$$

By (4.14), the optimal control can be computed as $u^* = p^*\big|_\Sigma$.

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