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by

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1. Introduction. Let D be a bounded strictly pseudoconvex domain in C^n with smooth boundery and let $A^m(D)$ be the set of holomorphic functions in D and of differentiability class C^m in $\bar{D}$.

In [5] M. Hakim and N. Sibony prove that a closed subset E of ∂D, which is locally a peak set for $A^\infty(D)$ is locally contained in a totally real submanifold M of ∂D of dimension $n - 1$, which is complex-tangential at every point of E.

A main tool in the proof is the theorem of F. R. Harvey and R. O. Wells Jr. about the zero-set of a non-negative strictly plurisubharmonic function [6].

In [5] there is also proved that a closed subset of a totally real submanifold M of ∂D of dimension $n - 1$, which is complex-tangential at every point of M is locally a peak set for $A^\infty(D)$. In [3], J. Chaumat and A. M. Chollet prove that the necessary condition and the sufficient condition given above are equivalent.

In [9] we prove that a CR submanifold M of C^n of CR dimension ℓ is the zero-set of a non-negative strictly ℓ-pseudoconvex function, which is plurisubharmonic on M.

Here we prove that the zero-set Z of a non-negative strictly $n - q$ - pseudoconvex function which is plurisubharmonic on Z is locally contained in a generic submanifold of C^n of codimension q, result which represents a generalization of the theorem of F. R. Harvey and R. O. Wells Jr. [6] (theorem 1).

We also give a generalization of the conditions given by M. Hakim and N. Sibony [5] for weakly pseudoconvex domains (theorem 2 and theorem 3).

2. Preliminaries. Let M be a smooth real submanifold of C^n. For a point p of M, we denote by $T_p(M)$ the tangent space of M at p and with $TC_p(M)$ the maximal complex subspace of $T_p(M)$ with the complex structure induced by the natural inclusion $T_p(M) \subset C^n$. If $\dim_C TC_p(M) = m =$ constant for each point p and M we say that M is a CR manifold and CR $\dim(M) = m$.

We denote by $H_p(M)$, respectively $A_p(M)$, the subspace of $T_p(M) \otimes C$ given by the holomorphic tangent vectors to M at p, respectively the antiholomorphic tangent vectors to M at p. We have $TC_p(M) \otimes C = H_p(M) \oplus A_p(M)$ and $TC_p(M)$ and $H_p(M)$ are naturally isomorphic [4] . If $\dim_R M = k$ and CR - $\dim(M) = m$, M is called a generic manifold if $m = k - n$. For simplicity we shall call M a subgeneric manifold if $m = k - n + 1$. M is called totally real if CRdim$(M) = 0$.

Let M be a CR submanifold of C^n of dimension k and CR-dimension m and $p \in M$. Let $s = k - 2m$, $r = n + m - k$. After a complex-linear change of coordinates in C^n, M may be represented in the neighborhood of p by the equations:

$$\begin{aligned} z_j &= t_j + i g_j(t,w) & j &= 1,\dots,s \\ z_{j+s} &= h_j(t,w) & j &= 1,\dots,r \\ z_{j+s+r} &= w_j & j &= 1,\dots,m \end{aligned}$$

where p corresponds to the origin $(t,w) = (t_1,\dots,t_s,w_1,\dots,w_m)$ are coordinates in the neighborhood of the origin in $R^s \times C^m$ and $\{g_j\}$, $\{h_j\}$ are real and complex-valued functions respectively vanishing to second order at the origin [14] . If M is a CR-submanifold of C^n, a smooth function f on M is called a CR-function on M if there exists an extension $\tilde{f}$ of f to C^n such that $\bar\partial \tilde{f}|_M = 0$ (we take "smooth" to mean C^∞). Let D be a bounded domain in C^n with C^2-boundery. For a point p of ∂D we consider a defining function for D, i.e. a C^2-function ρ defined in the neighborhood of p such that $D \cap U = \{z \in U \mid \rho(z) < 0\}$ and $d\rho \neq 0$ on $\partial D \cap U$. We say that D is strictly q-pseudoconvex at p if the complex Hessian of ρ has at least n-q positive eigenvalues with eigenvectors in $TC_p(\partial D)$.

If φ is a C^2 real valued function on D we say that φ is a strictly q-pseudoconvex function if the complex Hessian of φ has at least n-q strictly positive eigenvalues in D.

We denote by $A^k(D)$ the set of holomorphic functions in D which have a C^k extension to $\overline{D}$. A closed subset E of ∂D is a peak set for $A^k(D)$ if there exists $f \in A^k(D)$ such that f=1 on E and $|f|<1$ on $\overline{D} \setminus E$. It is easy to see that a closed subset E of ∂D is a peak set for $A^k(D)$ if and only if there exists $f \in A^k(D)$ such that $f|E = 0$ and $\mathrm{Re} f<0$ on $\overline{D} \setminus E$. A closed subset E of ∂D is locally a peak set for $A^k(D)$ at $p \in E$, if there exists a neighborhood V of p such that $E \cap \overline{V}$ is a peak set for $A^k(D)$ and it is a locally peak set for $A^k(D)$ if it is locally a peak set for $A^k(D)$ at every point of E.

A smooth submanifold M of ∂D is called complex tangential at a point p of M if $T_p(M) \subset TC_p(\partial D)$.

3. <u>Zero Sets of Non-Negative Strictly q Pseudoconvex Functions.</u>

<u>Lemma 1</u>. Let φ be a real-valued C^2 strictly n-q-pseudoconvex function defined on a neighborhood of $0 \in C^n$. Suppose $\varphi(0)=0$, $\mathrm{grad}\,\varphi(0)=0$ and the complex Hessian of φ has n - q eigenvalues which vanish at the origin. Then, there exists a complex-linear change of coordinates $z = (z_1,\dots,z_n)$ in C^n such that:

$$\varphi(z) = \sum_{j=1}^{q}(1+\lambda_j)x_j^2 + \sum_{j=1}^{q}(1-\lambda_j)y_j^2 + \sum_{i=1}^{q}\sum_{j=q+1}^{n}(a_{ij}x_ix_j + b_{ij}y_iy_j + c_{ij}x_iy_j + d_{ij}x_jy_i) +$$
$$+ \sum_{i,j=q+1}^{n}(\alpha_{ij}x_ix_j + \beta_{ij}y_iy_j + \gamma_{ij}x_iy_j) + o(|z|^2)$$

where $\lambda_j \geq 0$, $z = x + iy$, x and y are the real and the imaginary parts of the coordinates $z \in C^n$.

Proof.

We denote $x' = (x_1,\dots,x_q)$, $x'' = (x_{q+1},\dots,x_n)$, $y' = (y_1,\dots,y_q)$.

$y'' = (y_{q+1},\ldots,y_n)$, $z' = x' + iy'$, $z'' = x'' + iy''$.

We have $\varphi(z) = \mathrm{Re}\sum_{i,j=1}^{n} \frac{\partial^2\varphi}{\partial z_i \partial z_j}(0) z_i z_j + \sum_{i,j=1}^{n} \frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}(0) z_i \bar{z}_j + o(|z|^2)$.

By making a complex-linear change of coordinates in C^n we may suppose that

$$\left[\frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}(0)\right]_{1\leq i,\, j\leq n} = \begin{array}{c c} & \begin{array}{cc} \overbrace{\qquad}^{q} & \overbrace{\qquad}^{n-q} \end{array} \\ \begin{array}{c} q\left\{\right. \\ n-q\left\{\right. \end{array} & \left[\begin{array}{cc} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ 0 & 0 \end{array}\right] \end{array}$$

and $\varphi(z) = |z'|^2 + \mathrm{Re}({}^t z S z) + o(|z|^2)$ where $S = \left[\frac{\partial^2\varphi}{\partial z_i \partial z_j}(0)\right]_{1\leq i,j\leq n}$

Let $s = \begin{bmatrix} x \\ y \end{bmatrix}$ be a real $2n$ - vector in R^{2n}, where $x,y \in R^n$, $E' = \left\{ s \in R^{2n} \mid x''=0, y''=0 \right\}$, $E'' = \left\{ s \in R^{2n} \mid x' = 0, y' = 0 \right\}$. We shall identify E' with R^{2q} and E'' with $R^{2(n-q)}$. E' and E'' are complex subspaces of $C^n = E' \oplus E''$ and for $s \in C^n$ we obtain $s = s' + s''$ with $s' \in E'$, $s'' \in E''$. With these notations we obtain that

$$\varphi(s) = |s'|^2 + {}^t sTs + o(|s|^2) = |s'|^2 + \langle Ts,s\rangle + o(|s|^2)$$

where $\langle , \rangle$, is the inner product in R^{2n} and $T = \begin{bmatrix} A & -B \\ -B & -A \end{bmatrix}$ with $S = A + iB$, A and B real symmetric matrices.

We consider A', B' the $n\times q$ matrices which have the first q columns of A and B respectively, A'', B'' the $n \times n - q$ matrices which have the last $n-q$ columns of A and B respectively. We obtain :

$$T = T' + T'' \quad \text{where} \quad T' = \begin{array}{c} \begin{array}{cccc} \overbrace{\quad}^{q} & \overbrace{\quad}^{n-q} & \overbrace{\quad}^{q} & \overbrace{\quad}^{n-q} \end{array} \\ \begin{bmatrix} A' & 0 & -B' & 0 \\ -B & 0 & -A' & 0 \end{bmatrix} \end{array}, \quad T'' = \begin{array}{c} \begin{array}{cccc} \overbrace{\quad}^{q} & \overbrace{\quad}^{n-q} & \overbrace{\quad}^{q} & \overbrace{\quad}^{n-q} \end{array} \\ \begin{bmatrix} 0 & A'' & 0 & -B'' \\ 0 & -B'' & 0 & -A'' \end{bmatrix} \end{array}$$

We have $T's = T's'$ and $T''s = T''s''$, so we may assume :

$T' : E' \longrightarrow E' \oplus E''$, $T'' : E'' \longrightarrow E' \oplus E''$ and $T' = T'_1 + T'_2$, $T'' = T''_1 + T''_2$

where $T'_1 : E' \longrightarrow E'$, $T'_2 : E' \longrightarrow E''$, $T''_1 : E'' \longrightarrow E'$, $T''_2 : E'' \longrightarrow E''$.

Using the notations above, $\langle Ts,s\rangle = \langle (T'+T'')s, s\rangle = \langle T's,s\rangle +$
$+ \langle T''s,s\rangle = \langle T's',s\rangle + \langle T''s'',s\rangle = \langle T'_1 s',s\rangle + \langle T'_2 s',s\rangle +$
$+ \langle T''_1 s'',s\rangle + \langle T''_2 s'',s\rangle = \langle T'_1 s',s'\rangle + \langle T'_2 s',s''\rangle + \langle T''_1 s'',s'\rangle +$
$+ \langle T''_2 s'',s''\rangle$

$$T_1' = \begin{array}{r} \\ q\{ \\ n-q\{ \\ q\{ \\ n-q\{ \end{array} \begin{array}{c} \begin{array}{cccc} \overbrace{\;q\;} & \overbrace{n-q} & \overbrace{\;q\;} & \overbrace{n-q} \end{array} \\ \begin{bmatrix} A_1' & 0 & -B_1' & 0 \\ 0 & 0 & 0 & 0 \\ -B_1' & 0 & -A_1' & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

where A_1', B_1' are the $q \times q$ matrices obtained by taking the first q rows of A' and B' respectively.

Let J be the real orthogonal matrix representing the multiplication by $i = \sqrt{-1}$ i.e. $J\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$. If $v' \in E'$ is an eigenvector for T_1' with eigenvalue λ, then Jv' is an eigenvector for T_1' with eigenvalue $-\lambda$. Because A and B are symmetric matrices, it follows that A_1', B_1' are symmetric matrices, thus T_1' is a symmetric matrix.

We may consider $\left\{ v_1', \ldots, v_q', Jv_1', \ldots, Jv_q' \right\}$ an orthonormal basis for E' consisting of eigenvectors of T_1'. If λ_j is the eigenvalue of v_j, by interchanging v_j and Jv_j if necessary, we may assume each $\lambda_j \geqslant 0$.

We consider an orthonormal basis of R^{2n} of the form $\left\{ v_1', \ldots, v_q', v_{q+1}'', \ldots, v_n'', Jv_1', \ldots, Jv_q', Jv_{q+1}'', \ldots, Jv_n'' \right\}$.

If $\left\{ e_1, \ldots, e_{2n} \right\}$ is the standard basis in R^{2n} and we pass at the basis obtained above, because $Je_i = e_{i+n}$, we have in fact a complex - linear change of coordinates in C^n.

Let

$$e_i = \sum_{j=1}^{q} c_{ij}' v_j' + \sum_{j=1}^{q} d_{ij}' Jv_j' \qquad 1 \leqslant i \leqslant q$$

$$e_{q+i} = \sum_{j=1}^{n-q} c_{ij}'' v_{q+j}'' + \sum_{j=1}^{n-q} d_{ij}'' Jv_{q+j}'' \qquad 1 \leqslant i \leqslant n-q$$

$$e_{n+i} = \sum_{j=1}^{q} -d_{ij}' v_j' + \sum_{j=1}^{q} c_{ij}' Jv_j' \qquad 1 \leqslant i \leqslant q$$

$$e_{n+q+i} = \sum_{j=1}^{n-q} -d_{ij}'' v_{q+j}' + \sum_{j=1}^{n-q} c_{ij}'' Jv_{q+j}'' \qquad 1 \leqslant i \leqslant n-q$$

We denote $C' = (c'_{ij})_{1\leq i,j\leq q}$, $D' = (d'_{ij})_{1\leq i,j\leq q}$,
$\hat{C}'' = (c''_{ij})_{1\leq i,j\leq n-q}$, $\hat{D}'' = (d''_{ij})_{1\leq i,j\leq n-q}$.

If the new coordinates are $\sigma = \begin{bmatrix}\xi\\ \eta\end{bmatrix}$ then $\sigma = {}^t P s$, where

$$P = \begin{bmatrix} C' & 0 & D' & 0 \\ 0 & \hat{C}'' & 0 & D'' \\ -D' & 0 & C' & 0 \\ 0 & -\hat{D}'' & 0 & C'' \end{bmatrix}$$

is an orthogonal matrix.

Thus $s = ({}^t P)^{-1}\sigma = P\sigma$ and in the new coordinates φ becomes :

$$\tilde{\varphi}(\sigma) = \varphi(P\sigma) = |(P\sigma)'|^2 + \langle T_1'(P\sigma)', (P\sigma)'\rangle + \langle T_2'(P\sigma)', (P\sigma)''\rangle +$$
$$+ \langle T_1''(P\sigma)'', (P\sigma)'\rangle + \langle T_1''(P\sigma)'', (P\sigma)''\rangle + o(|P\sigma|^2)$$

We have $P = P' + P''$ where $P' : E' \longrightarrow E'$, $\hat{P}'' : \hat{E}'' \longrightarrow \hat{E}''$,
$P' = \begin{bmatrix} C' & D' \\ -D' & C' \end{bmatrix}$ $\hat{P}'' = \begin{bmatrix} \hat{C}'' & D'' \\ -\hat{D}'' & \hat{C}'' \end{bmatrix}$ are orthogonal matrices.

Therefore, $\tilde{\varphi}(\sigma) = |P'\sigma'|^2 + \langle T_1'P'\sigma', P'\sigma'\rangle + \langle T_2'P'\sigma', P''\sigma''\rangle +$

$$+ \langle T_1''P''\sigma'', P'\sigma'\rangle + \langle T_2''P''\sigma'', P''\sigma''\rangle + o(|P\sigma|^2) =$$
$$= |\sigma'|^2 + \sum_{j=1}^{q}\lambda_j\xi_j^2 - \sum_{j=1}^{q}\lambda_j\eta_j^2 + \langle T_2'P'\sigma', P''\sigma''\rangle + \langle T_1''P''\sigma'', P'\sigma'\rangle +$$
$$+ \langle T_1''P''\sigma'', P''\sigma''\rangle + o(|\sigma|^2)$$

and the proof of the lemma is complete.

<u>Theorem 1</u>. Let φ be a non-negative C^2-function in an open set $U \subset \mathbb{C}^n$ and let $Z = \{ z \in U \mid \varphi(z) = 0 \}$. Let z_0 be a point of Z such that the complex Hessian of φ has q strictly positive eigenvalues and n-q zero eigenvalues at z_0. Then, there exists a neighborhood V of z_0 and a C^1-generic submanifold M of U, of codimension q in U, such that $Z \cap V \subset M$.

Proof.

Since the conclusion of the theorem is local, it suffices to assume that $z_0 = 0$ and $\varphi(0) = 0$. Because each point of Z is a relative minimum for φ, grad φ vanishes on Z. Therefore in a neighborhood of the origin we can apply lemma 1 and obtain the Taylor expansion:

$$\varphi(z)=\sum_{j=1}^{q}(1+\lambda_j)x_j^2+\sum_{j=1}^{q}(1-\lambda_j)y_j^2+\sum_{i=1}^{q}\sum_{j=q+1}^{n}(a_{ij}x_ix_j+b_{ij}y_iy_j+c_{ij}x_iy_j+d_{ij}x_jy_i)+$$
$$+\sum_{i,j=1}^{q}(\alpha_{ij}x_ix_j+\beta_{ij}y_iy_j+\gamma_{ij}x_iy_j)+o(|z|^2)$$

with $\lambda_j \geqslant 0$, $j = 1,\dots,n$.

Let $M = \left\{z\in V \,\middle|\, \frac{\partial\varphi}{\partial x_1}(z)= \cdots = \frac{\partial\varphi}{\partial x_q}(z) = 0\right\}$

We have $\frac{\partial\varphi}{\partial x_i}=2(1+\lambda_i)x_i+\sum\limits_{j=q+1}^{n}(a_{ij}x_j+c_{ij}y_j)+o(|z|) \qquad 1\leq i\leq q$

We denote by $\psi_i=\frac{\partial\varphi}{\partial x_i}$, $i = 1,\dots,q$ and we obtain

$$\frac{\partial\psi_i}{\partial x_j}(0) = \begin{cases} 2(1+\lambda_i) & \text{if } i = j \\ 0 & \text{if } i \neq j,\ 1\leq j\leq q \\ a_{ij} & \text{if } q<j\leqslant n \end{cases}$$

$$\frac{\partial\psi_i}{\partial y_j}(0) = \begin{cases} 0 & \text{if } j\leqslant q \\ c_{ij} & \text{if } q<j\leqslant n \end{cases}$$

Thus $\frac{\partial(\psi_1,\dots,\psi_q)}{\partial(x_1,\dots,x_n,y_1,\dots,y_n)}(0)$ has maximal rank q and it follows that M is a C^1-submanifold of real codimension q in the neighborhood of the origin. But $\frac{\partial(\psi_1,\dots,\psi_q)}{\partial(z_1,\dots,z_n)}(0)$ has also maximal rank q and this proves that M is generic at the origin, hence in a neighborhood of the origin too.

<u>Corrolary</u>. Let M be a C^1-submanifold of an open set $U\subset C^n$ of real codimension q. M is a generic submanifold of U iff there exists a non-negative C^2-strictly n-q -pseudoconvex function φ defined in a neighborhood U' of M in U such that the complex Hessian of φ has n-q zero eigenvalues on M and $M = \{z\in U' \mid \varphi(z) = 0\}$.

Proof.

If M is a generic submanifold of U then ℓ = CR dim(M) = $\dim_R M - n =$ $= 2n - q - n = n - q$ and we obtain the result from [9] .

If φ is given , M is locally contained in a generic submanifold of codimension q.

4. Peak sets in weakly pseudoconvex domains.

We shall prove the following results :

Theorem 2. Let D be a domain in $\mathbb{C}^n$ with C^2-boundary ∂D. We suppose that $D = \{z \mid \rho(z) < 0\}$ where ρ is a C^2-function in the neighborhood of ∂D and $d\rho \neq 0$ on ∂D. Let E be a closed subset of ∂D which is locally a peak set for $A^2(D)$ and p a point of E. We suppose that the complex Hessian of ρ at p has q strictly positive eigenvalues and $n-q-1$ zero eigenvalues with eigenvectors in $TC_p(\partial D)$. Then E is locally contained in the neighborhood of p in a subgeneric CR-submanifold M of ∂D, of C^1-differentiability class, of dimension $2n-q-2$, CR-dimension $n-q-1$, such that M is complex-tangential at every point of E.

Proposition 1. Let D be a domain in $\mathbb{C}^n$ with a smooth boundary; let $D = \{ z \mid \rho(z) < 0 \}$ where ρ is a smooth function with $d\rho \neq 0$ on ∂D. We suppose that there exists a subgeneric submanifold M of ∂D of CR-dimension $n-q-1$ which is complex-tangential at every point of M. Let p be a point of M such that D is pseudoconvex at p, strictly $n-q-1$-pseudoconvex at p. Then there exists a neighborhood U of p and $\psi \in C^\infty(U)$ such that :

a) $\psi = 0$ on $M \cap U$

b) $\operatorname{Re}\psi < 0$ on $\bar{D} \cap U \setminus M$

c) $\bar{\partial}\psi$ vanishes to infinite order on $M \cap U$

d) $\bar{\partial}(\frac{1}{\psi})$ extended with zero to $M \cap U$ is in $C^\infty_{(0,1)}(\bar{D} \cap U)$.

Theorem 3. Let D be a bounded pseudoconvex domain with C^∞-boundary. In the conditions of proposition 1, we suppose that there exists a closed subset E of M which contains the point p, such that there exists a compact neighborhood V of p, $V \subset U$ (where U is given by proposition 1) and there exists a CR-function s on $U \cap M$ such that :

i) s vanishes on $E \cap V$ to fourth order .

ii) Re $s<0$ on $U\cap M\setminus E\cap V$

Then E is locally a peak set at p for $A^{\infty}(D)$.

Remark 1. If D is strictly pseudoconvex, i.e. $q = n - 1$ we obtain the theorems 1 and 2 from [5].

Remark 2. Let $D\subset C^n$, $D=\{z\in C^n \mid \rho(z)<0\}$ where ρ is of C^1 differentiability class and $d\rho\neq 0$ on ∂D. There are no generic submanifolds M of C^n such that $M\subset\partial D$ and M is complex-tangential at every point of M.

Indeed, let suppose that M is a generic submanifold of ∂D. Let p be a point of M and we may assume that M is represented in the neighborhood of p by the equations

$$
\begin{aligned}
z_1 &= x_1 + ih^1(x,w)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
z_r &= x_r + ih^r(x,w)\\
z_{r+1} &= w_1\\
&\cdots\cdots\cdots\\
z_n &= w_m
\end{aligned}
$$

with $x=(x_1,\dots,x_r)\in R^r$ and $w=(w_1,\dots,w_m)\in C^m$, where p corresponds to the origin and $\{h^j\}_{j=1,\dots,r}$ are real functions vanishing to second order at the origin. Then $T_o(M)=\{z \mid y_1=\dots=y_r=0\}$, $TC_o(\partial D)=\left\{z \mid \sum_{i=1}^{n} z_i\frac{\partial\rho}{\partial z_i}(0)=0\right\}$

Because $T_o(M)\subset TC_o(\partial D)$ we have $\sum_{k=1}^{r} x_k\frac{\partial\rho}{\partial z_k}(0)+\sum_{k=r+1}^{n} z_k\frac{\partial\rho}{\partial z_k}(0)=0$ for every real $x_1,\dots,x_r$ and complex $z_{r+1},\dots,z_n$. It follows that $\frac{\partial\rho}{\partial z_k}(0)=0$ for $k=1,\dots,n$ and these contradict the assumption that $d\rho\neq 0$ on ∂D.

Proof of theorem 2.

We choose local coordinates in C^n such that p corresponds to the origin and ρ is given in the neighborhood of the origin by the equation $\rho = u+\rho_1(z,w)$ where $z\in C^{n-1}$, $w\in C$, $w=u+iv$ and ρ_1 vanishes to second order at the origin.

Because p is a point of N, there exists a neighborhood V of p $f\in A^2(D)$ such that $f=0$ on $E\cap V$ and $\mathrm{Re}\, f<0$ on $\overline{D}\setminus E\cap V$.

Let $\tilde{f}$ be an extension of C^2-differentiability class of f to C^n, $g = \mathrm{Re}\,\tilde{f}$, $h = \mathrm{Im}\,\tilde{f}$.

By the Hopf lemma we obtain that $\frac{\partial \tilde{f}}{\partial w}(0) \neq 0$. But

$$\left|\frac{\partial \tilde{f}}{\partial w}(0)\right|^2 = \left|\frac{\partial g}{\partial u}(0)\right|^2 + \left|\frac{\partial g}{\partial v}(0)\right|^2 = \det\begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial h}{\partial u} \\ \frac{\partial g}{\partial v} & \frac{\partial h}{\partial v}\end{vmatrix}(0) > 0$$

It follows that the set $\Gamma = \left\{(z,w) \mid \tilde{f}(z,w) = 0\right\}$ is in the neighborhood of the origin a real manifold of dimension $2(n-1)$. Becaude g has a local maximum at 0 relative to D we obtain $\frac{\partial g}{\partial z_j}(0) = 0$, $j = 1,\dots, n-1$ and since the Cauchy-Riemann equations hold at 0 we conclude that $\frac{\partial h}{\partial z_j}(0) = 0$, $j = 1,\dots,n-1$. Therefore the tangent plane at 0 to Γ is given by the equation $w = 0$ and we can solve $\tilde{f}(z,w) = 0$ in the neighborhood of the origin to obtain $w = H(z)$ where H is a function of C^2-differentiability class, vanishing to second order at the origin. If $(z_o,w_o) \in E \cap V$ we have

$$\begin{cases} f(z_o,w_o) = 0 \\ D^\alpha \bar{\partial} f(z_o,w_o) = 0 \text{ for } |\alpha| \leq 1 \\ w_o = H(z_o) \end{cases} \qquad (1)$$

Because $f(z,H(z)) = 0$ we have $\bar{\partial} f(z,H(z)) = 0$ or

$$\frac{\partial f}{\partial \bar{z}_j}(z,H(z)) + \frac{\partial f}{\partial \bar{w}}(z,H(z))\frac{\partial \bar{H}}{\partial \bar{z}_j} + \frac{\partial f}{\partial w}(z,H(z))\frac{\partial H}{\partial \bar{z}_j}(z) = 0$$

$j = 1,\dots,n-1$.

But $\frac{\partial f}{\partial \bar{z}_j}(z_o,w_o) = \frac{\partial f}{\partial \bar{w}}(z_o,w_o) = 0$ and since $\frac{\partial f}{\partial w} \neq 0$ in the neighborhood of the origin we obtain $\bar{\partial} H(z_o) = 0$. From (1), by recurrence we obtain $D^\alpha \bar{\partial} H(z_o) = 0$ for $|\alpha| \leq 1$.

The tangent plane at Γ in (z_o,w_o) is $w = H(z_o) + \sum_{j=1}^{n-1} \frac{\partial H}{\partial z_j}(z_o)(z_j - z_{oj})$

We have $\frac{\partial}{\partial z_i}(\rho(z,H(z))) = \frac{\partial \rho}{\partial z_i}(z,H(z)) + \frac{\partial \rho}{\partial w}\cdot\frac{\partial H}{\partial z_i} + \frac{\partial \rho}{\partial \bar{w}}\cdot\frac{\partial \bar{H}}{\partial z_i}$

and
$$\frac{\partial}{\partial z_i \partial \bar{z}_j}\rho(z,H(z)) = \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(z,H(z)) + \frac{\partial^2 \rho}{\partial z_i \partial w}\frac{\partial H}{\partial \bar{z}_j} + \frac{\partial^2 \rho}{\partial z_i \partial \bar{w}}\cdot\frac{\partial \bar{H}}{\partial \bar{z}_j} +$$

$$+ \frac{\partial}{\partial \bar{z}_j}\left(\frac{\partial \rho}{\partial w}\right)\frac{\partial H}{\partial z_i} + \frac{\partial \rho}{\partial w}\cdot\frac{\partial^2 H}{\partial z_i \partial \bar{z}_j} + \frac{\partial}{\partial \bar{z}_j}\left(\frac{\partial \rho}{\partial \bar{w}}\right)\cdot\frac{\partial \bar{H}}{\partial z_i} + \frac{\partial \rho}{\partial \bar{w}}\cdot\frac{\partial^2 \bar{H}}{\partial z_i \partial \bar{z}_j}$$

Because H vanishes to second order in $z = 0$ and from the relations (1) we conclude that $\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \rho(z, H(z)) \Big|_{z=0} = \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(z,w) \Big|_{\substack{z=0\\ w=0}}$

It follows that in the neighborhood of $0 \in C^{n-1} = \left\{ (z,w) \,\middle|\, w = 0 \right\} = TC_0(\partial D)$ the function $\Theta(z) = \rho(z, H(z))$ has q strictly positive eigenvalues.

Since $f = 0$ on $E \cap V$ and $\mathrm{Re}\, f < 0$ on $\bar{D} \setminus E \cap V$ and since $f(z, H(z)) = 0$ we obtain that $\Theta(z) = \rho(z, H(z)) \geq 0$. If $\Theta(z_0) = 0$, z_0 is a point of local minimum for Θ and grad $\Theta(z_0) = 0$, or

$$\frac{\partial \rho}{\partial z_j}(z_0, w_0) + \frac{\partial \rho}{\partial w}(z_0, w_0) \frac{\partial H}{\partial z_j}(z_0) = 0 \qquad j = 1, \ldots\ldots, n-1 \qquad (2)$$

We have $TC_{(z_0, w_0)}(\partial D) = \left\{ (z,w) \,\middle|\, \sum_{j=1}^{n-1} (z_j - z_{0j}) \frac{\partial \rho}{\partial z_j}(z_0, w_0) + \right.$

$\left. + (w - w_0) \frac{\partial \rho}{\partial w}(z_0, w_0) = 0 \right\}$,

$$T_{(z_0, w_0)}(\Gamma) = \left\{ (z,w) \,\middle|\, w - H(z_0) = \sum_{j=1}^{n-1} (z_j - z_{0j}) \frac{\partial H}{\partial z_j}(z_0) \right\}$$

Using (2) we conclude that $TC_{(z_0, w_0)}(\partial D) = T_{(z_0, w_0)}(\Gamma)$. But $E \cap V = \left\{ (z,w) \in V \,\middle|\, \Theta(z) = 0,\ w = H(z) \right\}$ in the neighborhood of the origin and the projection E_1 of $E \cap V$ on $\left\{ (z,w) \,\middle|\, w = 0 \right\}$ is in the neighborhood of the origin the zero-set of a non-negative strictly $n - q$ -pseudoconvex function. Thus, by theorem 1, E_1 is locally contained in a generic submanifold M_1 of C^{n-1} of dimension $2(n-1) - q = 2n - q - 2$.

We solve now locally the equation $\rho(z,w) = 0$ and we obtain $\mathrm{Re}\, w = \varphi(z, \mathrm{Im}\, w)$. Let define $\psi(z) = \varphi(z, \mathrm{Im}\, H(z)) + i\, \mathrm{Im}\, H(z)$ and $M = \left\{ (z,w) \,\middle|\, z \in M_1, w = \psi(z) \right\}$.

If $(z,w) \in E \cap V$, $f(z,w) = 0$, thus $w = H(z)$ and $\rho(z, H(z)) = 0$. Therefore, $\mathrm{Re}\, w = \varphi(z, \mathrm{Im}\, w)$ or $\mathrm{Re}\, H(z) = \varphi(z, \mathrm{Im}\, H(z))$ and $\mathrm{Im}\, w = \mathrm{Im}\, H(z)$, i.e. $z \in M_1$ and $w = \psi(z)$. It follows that $E \cap V \subset M \subset \partial D$.

If $z_0 \in E_1$, we denote $w_0 = H(z_0) = \psi(z_0)$. Because $\rho(z, \psi(z)) = 0$, by computing grad $\psi(z_0)$ from $\rho(z_0, \psi(z_0)) = 0$ we obtain from (2) that grad $H(z_o) = \text{grad}\,\psi(z_o)$. Since $M \in \{(z,w) \mid w = \psi(z)\}$ and H and ψ have the same derivatives at z_o, we conclude that

$T_{(z_o,w_o)}(M) \subset T_{(z_o,w_o)}(\Gamma) = TC_{(z_o,w_o)}(\partial D)$ if $z_o \in E_1$.

If we suppose that $M_1 \subset C^{n-1}$ is given in the neighborhood of the origin by the equations $\rho_1(z) = \dots \rho_q(z) = 0$ with $d\rho_1 \wedge \dots \wedge d\rho_q|_o \neq 0$ and $\bar\partial\rho_1 \wedge \dots \wedge \bar\partial\rho_q|_o \neq 0$, M is given by the equations $\rho_1 = \dots = \rho_q = 0$, $\rho_{q+1} = u - \mathrm{Re}\,\psi = 0$, $\rho_{q+2} = v - \mathrm{Im}\,\psi = 0$. We have

$d\rho_1 \wedge \dots \wedge d\rho_q \wedge d\rho_{q+1} \wedge d\rho_{q+2}|_o = d\rho_1 \wedge \dots \wedge d\rho_q \wedge du \wedge dv|_o \neq 0$ and

$$\left(\frac{\partial\rho_i}{\partial(z_j,w)}\right)_{\substack{1\le i\le q+2\\ 1\le j\le n-1}} = \begin{array}{c} \overbrace{\hphantom{xxxxxx}}^{n-1} \\ \begin{bmatrix} \dfrac{\partial\rho_i}{\partial z_j} & 0 \\ -\dfrac{\partial \mathrm{Re}\,\psi}{\partial z_j} & \dfrac{1}{2} \\ -\dfrac{\partial \mathrm{Im}\,\psi}{\partial z_j} & \dfrac{i}{2} \end{bmatrix} \end{array}$$

Because $\mathrm{rk}\left(\dfrac{\partial\rho_i}{\partial z_j}(z)\right)_{\substack{1\le i\le q\\ 1\le j\le n-1}} = q$ it follows that

$\mathrm{rk}\left(\dfrac{\partial\rho_i}{\partial(z_j,w)}(z,w)\right)_{\substack{1\le i\le q+2\\ 1\le j\le n-1}} = q+1$, thus M is a CR manifold of dimension $2n-q-2$ and CR-dimension $n-q-1$.

For the proof of proposition 1, we shall need the following 2 lemmas :

<u>Lemma 2</u>. Let $D = \{z \in C^n \mid \rho(z) < 0\}$ where ρ is a C^2-function. We suppose that there exists a CR submanifold M of ∂D of CR-dimension m which is complex-tangential at every point of M. Let p be a point of M such that D is pseudoconvex at p. Then the complex Hessian of ρ has m zero eigenvalues with eigenvectors in $TC_p(M)$.

<u>Remark 3</u>. If D is pseudoconvex, strictly m-pseudoconvex, the lemma follows from [1] and [14] .

Proof of lemma 2.

Let ξ be a section of the subbundle TC(M) of TC(∂D). By [2] (pag. 175) we know that $([J\xi,\xi]_p, \tau_p) + ([\xi,\xi]_p, \tau_p) = 2i\,L_p(\xi,\xi)$ where J defines the complex structure of TC(M), [ ] is the Lie bracket, $\tau = J(\text{grad}\,\rho)$, (,) is the inner product on $T(C^n)$, and if $\xi = \mathrm{Re}\sum_{j=1}^{n} a_j \frac{\partial}{\partial z_j}$, $L_p(\xi,\xi) = -2i\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(p) a_i \bar{a}_j$

Because $TC_p(M) = T_p(M) \cap JT_p(M)$ it follows that $J\xi$ is also a section of $TC(M) \subset T(M)$, thus $[J\xi, \xi]_p \in T_p(M) \subset TC_p(\partial D)$.

But $T_p(\partial D) = R[\gamma_p] \oplus TC_p(\partial D)$ ([2]) and we obtain that $([J\xi,\xi]_p, \gamma_p) = 0$. Because $([\xi,\xi]_p, \gamma_p) = 0$ we obtain $L_p(\xi,\xi) = 0$ for each section of $TC(M)$. Thus the Levi form of ρ vanishes on a complex subspace of $TC_p(\partial D)$ of complex dimension m and because D is pseudoconvex at p, the lemma follows.

Lemma 3. Let $D \subset C^n$, $D = \{ z \in C^n \mid \rho(z) < 0 \}$, ρ of class C^1. Let M be a subgeneric submanifold of ∂D, which is complex-tangential at every point of M. Then, for each point p of M, the projection M' of M on $TC_p(\partial D)$ is in the neighborhood of p a generic submanifold of $TC_p(\partial D)$.

Proof of lemma 3.

After a complex-linear change of coordinates in C^n, we may assume that $p = 0$ and M is represented in the neighborhood of p by the equations :

$$\begin{aligned}
z_1 &= x_1 + ih^1(x,w) \\
&\cdots\cdots \\
z_q &= x_q + ih^q(x,w) \\
z_{q+1} &= w_1 \\
&\cdots\cdots \\
z_{n-1} &= w_{n-q+1} \\
z_n &= g(x,w)
\end{aligned}$$

where $x = (x_1,\ldots,x_q) \in R^q$, $w = (w_1,\ldots,w_{n-q-1}) \in C^{n-q-1}$, $z_j = x_j + iy_j$, $j = 1,\ldots,q$, h^j, g are real, respectively complex functions in the neighborhood of the origin in $R^q \times C^{n-q-1}$ vanishing to second order at the origin.

We have $T_o(M) = \{ z \mid y_1 = \ldots = y_q = 0,\ z_n = 0 \} \subset TC_o(\partial D) =$
$= \left\{ z \,\middle|\, \sum_{i=1}^{n} \frac{\partial \rho}{\partial z_i}(0) z_i = 0 \right\}$

As in remark 2 we obtain that $\frac{\partial \rho}{\partial z_i}(0) = 0$ for $i = 1,\ldots,n-1$ thus $TC_0(\partial D) = \{ z | z_n = 0 \}$.

Let $\rho_1 = y_1 - h^1(x,w),\ldots,\rho_q = y_q - h^q(x,w), \rho_{q+1} = x_n - \mathrm{Re}\, g(x,w)$, $\rho_{q+2} = y_n - \mathrm{Im}\, g(x,w)$ and the projection M' of M onto $\{ z | z_n = 0 \}$ is given by $\{ z \in C^{n-1} | \rho_1(z) = \ldots = \rho_q(z) = 0 \}$. Now it is easy to observe that $d\rho_1 \wedge \ldots \wedge d\rho_q|_0 \neq 0$ and $\partial\rho_1 \wedge \ldots \wedge \partial\rho_q|_0 \neq 0$, i.e. M' is a generic submanifold of C^{n-1}.

Remark 4. M' is called the associated generic manifold of M and g is a CR function on M' ([7],[8]).

Proof of proposition 1.

With the notations from the proof of lemma 2 let V be a neighborhood of the origin in $R^q \times C^{n-q-1}$ and $\gamma : V \longrightarrow C^{n-1}$, $\gamma(x,w) = (x_1 + ih^1(x,w),\ldots,x_q + ih^q(x,w), w_1,\ldots,w_{n-q-1})$.

We extend γ to $\Gamma : W \longrightarrow C^{n-q-1}$ taking $\Gamma(z,w) = \sum_\alpha \frac{1}{\alpha!} D_x^\alpha \gamma(x,w)(iy)^\alpha \chi(\lambda_{|\alpha|} y)$ where W is a neighborhood of the origin in $C^q \times C^{n-q-1}$, where χ is in $C^\infty(R^q)$ with compact support $\chi = 1$ in the neighborhood of the origin and $\lambda_{|\alpha|}$ is a sequence of positive numbers increasing sufficiently quick to infinity (as in the proof of Borel's theorem).

We have $\frac{\partial \Gamma}{\partial x_j} = \sum_\alpha \frac{1}{\alpha!} D^\alpha_{x x_j} \gamma(x,w)(iy)^\alpha \chi(\lambda_{|\alpha|} y)$

If $y = 0$, $\frac{\partial \Gamma}{\partial x_j}(x,0,w) = \frac{\partial \gamma}{\partial x_j}(x,w)$ and analoguously $\frac{\partial \Gamma}{\partial y_j}(x,0,w) = i\frac{\partial \gamma}{\partial x_j}(x,w)$. It follows that the matrix $\left(\frac{(\partial \mathrm{Re}\,\Gamma_i, \mathrm{Im}\,\Gamma_i)}{\partial(x_j, y_j, u_k, v_k)}(0) \right)$ is non-singular and Γ is a C^∞ isomorphism in the neighborhood of the origin.

Thus, we obtained a C^∞-change of coordinates $\zeta' = (\zeta_1,\ldots,\zeta_{n-1}) = \Gamma(z,w)$ near the origin in C^{n-1}. The associated generic manifold M' is given in the new coordinates by the equations $y_1 = \ldots = y_q = 0$ and it is easy to see that $\frac{\partial \Gamma}{\partial \bar{z}_j}(x,0,w) = 0$ to infinite order.

Because g is a CR function on M' we may choose an extension $\tilde{g}(\zeta')$ of g to C^{n-1} such that $\bar{\partial}\,\tilde{g}$ vanishes to infinite order on

M' [12] .

We denote by $\Gamma_1 : C^n \longrightarrow C^n$ the map $\Gamma_1(z', z_n) = (\Gamma(z'), z_n)$ which is a C^∞-isomorphism near the origin in C^n. We have $\Gamma_1(z', z_n) = (\Gamma(z'), \tilde{g}_1(z')) + (0, z_n - \tilde{g}_1(z'))$ where $\tilde{g}_1(z') = \tilde{g}(\Gamma(z'))$.

We denote by ϕ the map $\phi(z') = (\Gamma(z'), \tilde{g}_1(z'))$ defined on a neighborhood of 0 in C^{n-1} with values in C^n and we may write :

$$\rho(\Gamma_1(z',z_n)) = \rho(\phi(z') + (0, z_n - \tilde{g}_1(z'))) =$$
$$\rho(\phi(z')) + 2\mathrm{Re}\frac{\partial\rho}{\partial z_n}(\phi(z'))(z_n - \tilde{g}_1(z')) + \mathrm{Re}\frac{\partial^2\rho}{\partial z_n^2}(\phi(z'))(z_n - \tilde{g}_1(z')) +$$
$$+ \frac{\partial^2\rho}{\partial z_n \partial \bar{z}_n}(\phi(z'))|z_n - \tilde{g}_1(z')|^2 + O(|z_n - \tilde{g}_1(z')|^2)$$

Remark 5. Let M be given by the parametric equations $\gamma_j : V \longrightarrow C$, $j = 1,\dots,n$, where V is an open set in R^{2n-q-2} and

$$\mathrm{rk}\left(\frac{\partial\gamma_i, \bar{\gamma}_i}{\partial t_j}\right)_{\substack{1\le i\le n\\ 1\le j\le 2n-q-2}} = 2n-q-2$$

Then, for each point p in M and $k = 1,\dots,2n - q - 2$

$$\sum_{j=1}^{n} \frac{\partial\gamma_j}{\partial t_k}(\gamma^{-1}(p))\left(\frac{\partial}{\partial z_j}\right)_p + \sum_{j=1}^{n} \frac{\partial\bar{\gamma}_j}{\partial t_k}(\gamma^{-1}(p))\left(\frac{\partial}{\partial \bar{z}_j}\right)_p \in T_p(M)\otimes C$$

But $T_p(M) \otimes C \subset TC_p(\partial D) \otimes C = H(T_p(\partial D)) \oplus A(T_p \partial D))$. Therefore

$$\sum_{j=1}^{n} \frac{\partial\gamma_j}{\partial t_k}(\gamma^{-1}(p))\left(\frac{\partial}{\partial z_j}\right)_p \in H(T_p(\partial D))$$ and we obtain $\sum_{j=1}^{n} \frac{\partial\gamma_j}{\partial t_k}\cdot\frac{\partial\rho}{\partial z_j} = 0$

We follow now the proof of proposition 1.

We denote $z'' = (z_{q+1},\dots,z_n)$ and we have $\rho(\phi(z')) = \rho(\phi(x,z'')) + \sum_{i=1}^{q} \frac{\partial(\rho\circ\phi)}{\partial y_i}(x,z'')y_i + \frac{1}{2}\sum_{i,j=1}^{q} \frac{\partial^2(\rho\circ\phi)}{\partial y_i \partial y_j}(x,z'')y_i y_j + O(|y|^3)$.

The functions $\phi_j(x,z'')$, $j = 1,\dots,n$ are local parametric equations for M, so $\rho(\phi(x,z'')) = 0$.

$$\frac{\partial(\rho\circ\phi)}{\partial y_i}(x,z'') = \sum_{k=1}^{n}\left[\frac{\partial\rho}{\partial\xi_k}\cdot\frac{\partial\phi_k}{\partial y_i} + \frac{\partial\rho}{\partial\bar{\xi}_k}\frac{\partial\bar{\phi}_k}{\partial y_i}\right] =$$

$$= \sum_{k=1}^{n} 2\mathrm{Re}\frac{\partial\rho}{\partial\xi_k}\cdot\frac{\partial\phi_k}{\partial y_i} = 2\mathrm{Re}\sum_{k=1}^{n-1}\frac{\partial\rho}{\partial\xi_k}\cdot\frac{\partial\Gamma_k}{\partial y_i} + 2\mathrm{Re}\frac{\partial\rho}{\partial\xi_n}\cdot\frac{\partial\tilde{g}_1}{\partial y_i} =$$

$$= 2\mathrm{Re}\, i\sum_{k=1}^{n-1}\frac{\partial\rho}{\partial\xi_k}\cdot\frac{\partial\gamma_k}{\partial x_i} + 2\mathrm{Re}\frac{\partial\rho}{\partial\xi_n}\cdot\frac{\partial\tilde{g}_1}{\partial y_i} =$$

$$= 2\mathrm{Re}\, i\left[\sum_{k=1}^{n-1}\frac{\partial\rho}{\partial\xi_k}\cdot\frac{\partial\gamma_k}{\partial x_i} + \frac{\partial\rho}{\partial\xi_n}\cdot\frac{\partial\tilde{g}_1}{\partial x_i}\right] + 2\mathrm{Re}\frac{\partial\rho}{\partial\xi_n}\left[\frac{\partial\tilde{g}_1}{\partial y_i} - i\frac{\partial\tilde{g}_1}{\partial x_i}\right]$$

By the remark 5 we have $\sum_{k=1}^{n-1} \frac{\partial \rho}{\partial \xi_k} \cdot \frac{\partial \gamma_k}{\partial x_i} + \frac{\partial \rho}{\partial \xi_n} \cdot \frac{\partial \tilde{g}_1}{\partial x_i} = 0$

Thus $\frac{\partial(\rho\circ\phi)}{\partial y_i}(x,z'') = 2\mathrm{Re}\left[-i\frac{\partial \rho}{\partial \xi_n}\left(\frac{\partial \tilde{g}_1}{\partial x_i} + i\frac{\partial \tilde{g}_1}{\partial y_i}\right)\right] = -4\mathrm{Im}\frac{\partial \rho}{\partial \xi_n}\frac{\partial \tilde{g}_1}{\partial \bar{z}_i}$

and $\frac{\partial \tilde{g}_1}{\partial \bar{z}_i}(x,z'') = \frac{\partial(\tilde{g}\circ\Gamma)}{\partial \bar{z}_i}(x,z'') = \sum_{j=1}^{n-1}\left[\frac{\partial \tilde{g}}{\partial \xi_j}(\Gamma(x,z''))\cdot\frac{\partial \Gamma_j}{\partial \bar{z}_i}(x,z'') + \right.$

$$\left. + \frac{\partial \tilde{g}}{\partial \bar{\xi}_j}(\Gamma(x,z''))\frac{\partial \bar{\Gamma}_j}{\partial \bar{z}_i}(x,z'')\right]$$

But $\frac{\partial \Gamma_j}{\partial \bar{z}_i}(x,z'') = 0$ for $i = 1,\dots,q$ and $\frac{\partial \tilde{g}}{\partial \xi_j}(\Gamma(x,z'')) = 0$ for $j = 1,\dots,n-1$, therefore $\frac{\partial \tilde{g}_1}{\partial \bar{z}_i}(x,z'') = 0$ for $i = 1,\dots,q$.

Finally we obtain that

$$\rho(\Gamma_1(z',z_n)) = \frac{1}{2}\sum_{i,j=1}^{q}\frac{\partial^2(\rho\circ\phi)}{\partial y_i \partial y_j}(x,z'')y_i y_j + 2\mathrm{Re}\frac{\partial \rho}{\partial z_n}(\phi(z'))(z_n - \tilde{g}_1(z')) +$$

$$+\mathrm{Re}\frac{\partial^2 \rho}{\partial z_n^2}(\phi(z'))(z_n - \tilde{g}_1(z'))^2 + \frac{\partial^2 \rho}{\partial z_n \partial \bar{z}_n}(\phi(z))|z_n - \tilde{g}_1(z')|^2 +$$

$$+ o(|z_n - \tilde{g}_1(z')|^3 + o(|y|^3)$$

By taking $w = \frac{1}{2}\frac{\partial \rho}{\partial z_n}(0)(z_n - \tilde{g}_1(z'))$ we have :

$$\rho(\Gamma_1(z', w)) = \mathrm{Re}\, w + \frac{1}{2}\sum_{i,j=1}^{q}\frac{\partial^2(\rho\circ\phi)}{\partial y_i \partial y_j}(x,z'')y_i y_j + \mathrm{Re}\, bw^2 + c|w|^2 +$$

$$+ o(|w|) + o(|w|^3) + o(|y|^3).$$

We define $\theta_1(z', w) = \rho(\Gamma_1(z', w)) - \mathrm{Re}\, w$ and by

$$\theta(z') = \rho(\Gamma(z'), 0) = \frac{1}{2}\sum_{i,j=1}^{q}\frac{\partial^2(\rho\circ\phi)}{\partial y_i \partial y_j}(x,z'')y_i y_j + o(|y|^3).$$

We observe that grad $\theta = 0$ and we have :

$$\frac{\partial \theta}{\partial z_i} = \sum_{k=1}^{n-1}\left(\frac{\partial \rho}{\partial \xi_k}\cdot\frac{\partial \Gamma_k}{\partial z_i} + \frac{\partial \rho}{\partial \bar{\xi}_k}\cdot\frac{\partial \bar{\Gamma}_k}{\partial z_i}\right)$$

$$\frac{\partial^2 \theta}{\partial z_i \partial \bar{z}_j} = \sum_{k,l=1}^{n-1}\frac{\partial^2 \rho}{\partial \xi_k \partial \bar{\xi}_l}\cdot\frac{\partial \bar{\Gamma}_l}{\partial \bar{z}_j}\cdot\frac{\partial \Gamma_k}{\partial z_i} + \sum_{l=1}^{n-1} a_l \frac{\partial \Gamma_l}{\partial \bar{z}_j} + \sum_{k=1}^{n-1} b_k \frac{\partial \rho}{\partial \xi_k} +$$

$$+ \sum_{k=1}^{n-1} c_k \frac{\partial \bar{\Gamma}_k}{\partial z_i} + \sum_{k=1}^{n-1} d_k \frac{\partial \rho}{\partial \bar{\xi}_k}$$

$$\frac{\partial^2 \theta}{\partial z_i \partial \bar{z}_j}(0) = \sum_{k,l=1}^{n-1}\frac{\partial^2 \rho}{\partial \xi_k \partial \bar{\xi}_l}(0)\frac{\partial \bar{\Gamma}_l}{\partial \bar{z}_j}\cdot\frac{\partial \Gamma_k}{\partial z_i} = \frac{\partial^2 \rho}{\partial \xi_i \partial \bar{\xi}_j}(0),\ i = 1,\dots,q.$$

By lemma 2 the complex Hessian of ρ has in the origin q strictly positive eigenvalues and n - q - 1 zero eigenvalues with eigenvectors in $TC_0(M) = \{\xi \mid \xi_1 = \cdots = \xi_q = \xi_n = 0\}$

Thus $\left(\frac{\partial^2 \theta}{\partial z_i \partial \bar{z}_j}(0)\right)_{1\leq i,j\leq q}$ is strictly positive definite. But $\frac{\partial^2 \theta}{\partial z_i \partial \bar{z}_j}(0) = \frac{1}{4}\frac{\partial^2(\rho\circ\phi)}{\partial y_i \partial y_j}(0)$ and by continuity there exists a neighborhood of the origin such that $\theta(z') \geqslant K\|y\|^2$.

If we denote $\rho_1(z', w) = \rho(\Gamma_1(z'), w)$ we have :

$\rho_1(z', w) = \operatorname{Re} w + \theta_1(z', w)$ with $\theta_1(z', 0) \geqslant K\|y\|^2$ (3).

Thus $\rho_1(z', w) = \theta_1(z', 0) + uA(z') + vB(z') + u^2C(z', w) + uvD(z', w) + v^2E(z', w)$ with $A(0) = 1$, $B(0) = 0$ or $\rho_1(z', w) = \theta_1(z', 0) + u(1 + O(|z'|) + uC(z', w) + vD(z', w)) + O(1)v^2 \geqslant$
$\geqslant K\|y\|^2 + u(1 + O(|z'|) + O(|u|) + O(|v|)) - Cv^2$.

If $\rho_1 \leq 0$ we have $u(1 + O(|z'|) + O(|u|) + O(|v|)) \leq Cv^2 - K\|y\|^2$ and for $|z'| + |u| + |v|$ small enough we obtain

$u \leq C_1 v^2 - K_1\|y\|^2 \leq C_1 v^2 \leq C_1(u^2 + v^2)$.

It means that the projection of D in the w-plane is on the outside of the circle $C_1(u^2 + v^2) - u = 0$. We shall define ψ by $\psi(w) = \frac{w}{1-2C_1 w}$ which maps the outside of the circle $C_1(u^2 + v^2) - u = 0$ in the inside of the circle $C_1(u^2 + v^2) + u = 0$. Thus if $\rho_1 \leq 0$ we have $\operatorname{Re}\psi \leq 0$ and $\operatorname{Re}\psi = 0$ if and only if $w = 0$. But from (3) we have $0 \geqslant \rho_1(z', 0) = \theta_1(z', 0) \geqslant K\|y\|^2$ and it follows that $y_1 = \dots = y_q = 0$. We conclude that for a sufficiently small neighborhood of the origin $\psi = 0$ on $M \cap U$ and $\operatorname{Re}\psi < 0$ on $\bar{D} \cap U \setminus M$.

In order to prove the conditions c) and d) we must write ψ in the coordinates ξ :

$$\psi(\xi) = \frac{\frac{1}{2}a(z_n - \tilde{g}_1(z'))}{1-2C_1(z_n - \tilde{g}_1(z'))} = \frac{\frac{1}{2}a(z_n - \tilde{g}(\Gamma(z')))}{1-2C_1(z_n - \tilde{g}(\Gamma(z')))} = \frac{\frac{1}{2}a(\xi_n - \tilde{g}(\xi'))}{1-2C_1(\xi_n - \tilde{g}(\xi'))}$$

Because $\bar{\partial}\tilde{g}$ vanishes to infinite order on M' we obtain c) .

We have to prove now that $\lim\limits_{\substack{\xi \in \bar{D} \\ \xi \to M\cap U}} \frac{\bar{\partial}\psi}{\psi^k} = 0$ for each $k \in \mathbb{N}$.

It is sufficient to prove that $\bar\partial\psi = O(|w|^n)$ for each $n \in \mathbb{N}$. Since $\bar\partial\psi$ vanishes to infinite order on $M \cap U$ we have $\bar\partial\psi = O(|y|^n + |w|^n)$ (4) for each $n \in \mathbb{N}$.

If $\xi \in \bar{D}$, $\rho_1(\Gamma_1^{-1}(\xi)) = \rho_1(z', w) \leq 0$, thus $0 \geq \rho_1(z', w) = \operatorname{Re} w + \theta_1(z', 0) + O(|w|) \geq K\|y\|^2 - C|w|$ and we obtain that $|w| \geq K_1\|y\|^2$.

By the relations (4) the condition d) is also verified.

For the proof of theorem 3 we shall need the following lemma :

Lemma 4. In the hypotheses of theorem 3, there exists an extension $\tilde{s}$ of s in the neighborhood of p such that $\bar\partial\tilde{s}$ vanishes to infinite order on M and $\tilde{s}$ vanishes to fourth order on E.

Proof of lemma 4.

For the proof we refer to [12] and [13] .

If M is given in the neighborhood of p by $\left\{ z \,\middle|\, \rho_1(z) = \dots = \rho_m(z) = 0 \right\}$ with $d\rho_1 \wedge \dots \wedge d\rho_m \neq 0$, then we define the extension

$$\tilde{s}_r = s_0 + \sum_{l=1}^{r} \frac{(-1)}{l!} \sum_{\|I\|=l} h_I \rho^I \qquad \text{such that } \bar\partial\tilde{s}_r = \sum_{\|I\|=r} (\bar\partial h_I)\rho^I$$

where s_0 is defined to be the extension of s by translation in the normal directions to M at p, $\rho^I = \rho_1^{i_1} \dots \rho_k^{i_k}$, where $I = (i_1, \dots, i_k)$ and $\left\{\bar\partial\rho_1, \dots, \bar\partial\rho_k\right\}$ is a maximal subset of $\left\{\bar\partial\rho_1, \dots, \bar\partial\rho_m\right\}$ of linear independent vectors at p, $\bar\partial s_0 = \sum_{i=1}^{k} h_i \bar\partial\rho_i$ and $\bar\partial h_I = \sum_{j=1}^{k} h_{I_j} \bar\partial\rho_j$

Then $\tilde{s} = \tilde{s}_r + \sum_{k=r+1}^{\infty} (\tilde{s}_{k+1} - \tilde{s}_k - f_k)$ where f_k vanishes in the neighborhood of p and $\| s_{k+1} - s_k - f_k \|_r$ is sufficiently small ($\| \; \|_r$ is a seminorm in the topology of uniform convergence of the derivatives up to order r on compact sets).

It is sufficient to prove that h_I vanishes to order $4 - j$ where $j = \|I\|$ and $1 \leq I \leq 3$. But $\bar\partial s_0 = 0$ on E, thus $h_i = 0$ on E. For $|\alpha| = 1$, $D^\alpha \bar\partial s_0 = \sum_{i=1}^{k} D^\alpha h_i \bar\partial\rho_i + \sum_{i=1}^{k} h_i D^\alpha \bar\partial\rho_i$ and it follows that $D^\alpha h_i = 0$ on E. By recurrence h_i vanishes on E to third order.

Similarly, we prove the assertion for $\|I\| > 1$.

Proof of theorem 3.

With the notations from the proof of proposition 1, we shall modify the function ψ in order to obtain a function φ defined in a neighborhood U_1 of $E\cap V$ such that :

a) $\varphi = 0$ on $E\ V$

b) $\mathrm{Re}\,\varphi < 0$ on $\bar{D}\cap U_1 \setminus E\cap V$

c) $D^{\alpha}(\bar{\partial}\varphi) = 0$ for each $\alpha \in N^n$

d) $\bar{\partial}(\frac{1}{\varphi})$ extended by 0 on $E\cap V$ is in $C^{\infty}_{(0,1)}(\bar{D}\cap U_1)$.

Let $\tilde{s}$ be an extension of s givsn by lemma 4 and we define $\varphi = \psi + \lambda\tilde{s}$ with $\lambda > 0$

a) $\psi = 0$ on $M\cap U$ and $s = 0$ on $E\cap V$, thus $\varphi = 0$ on $E\cap V$.

b) $\mathrm{Re}\,\psi < 0$ on $\bar{D}\cap U\setminus M$ and $\mathrm{Re}\,\tilde{s} < 0$ on $M\cap U\setminus E\cap V$. It follows that $\mathrm{Re}\,\varphi < 0$ on $M\cap U\setminus E\cap V$. If we prove that $|\mathrm{Re}\,\psi| > C|\mathrm{Re}\,\tilde{s}|$ on $\bar{D}\cap U_1\setminus M\cap U$, where U_1 is a neighborhood of $E\cap V$, then for λ small enough we obtain b).

But in the neighborhood of $E\cap V$, $\mathrm{Re}\,\tilde{s} = O(|y|^4 + |w|^4)$ and because $|w| \geqslant K\|y\|^2$ we have $\mathrm{Re}\,\tilde{s} = O(|w|^2)$. Since $E\cap V$ is compact we obtain $|\mathrm{Re}\,\tilde{s}| \leqslant C|w|^2$ in the neighborhood of $E\cap V$.

We have $\psi = \dfrac{w}{1-2C_1w} = \dfrac{u+iv}{1-2C_1u-2C_1iv} = \dfrac{(u+iv)(1-2C_1u+2C_1iv)}{|1-2C_1w|^2}$

and $|\mathrm{Re}\,\psi| = \dfrac{2C_1|w|^2 - u}{|1-2C_1w|^2}$ on $\bar{D}\cap U$.

We know from the proof of proposition 1 that $C_1(u^2+v^2) \geqslant u$ on $\bar{D}\cap U$, so $2C_1|w|^2 - u \geqslant 2C_1|w|^2 - C_1|w|^2 = C_1|w|^2$ and $|\mathrm{Re}\,\psi| \geqslant K|w|^2$.

Thus $\left|\dfrac{\mathrm{Re}\,\psi}{\mathrm{Re}\,s}\right| \geqslant \dfrac{K}{C_1}$.

The assertion c) is clear from the definition of $\tilde{s}$.

As in the proof of proposition 1 we obtain that $\bar{\partial}\varphi = O(|w|^n)$ for each $n\in N$ and to prove d) it is enough to prove that $|\varphi| \geqslant C|w|$ in the neighborhood of $E\cap V$. But $\varphi = \psi + \lambda\tilde{s} = w + O(|w|^2) +$

$+\lambda O(|y|^4+|w|^4) = w + O(|w|^2)$ and $|\varphi| \geqslant C|w|$.

Let V_1 be a compact neighborhood of $E \cap V$ which is contained in U_1 and let $\varkappa$ a positive C^∞-function on C^n with compact support contained in U_1, $\varkappa = 1$ in the neighborhood of V_1. Because $\bar\partial\varkappa = 0$ in the neighborhood of V_1, $\varphi \neq 0$ outside V_1, $\bar\partial\left(\frac{\varkappa}{\varphi}\right) = \frac{\bar\partial\varkappa}{\varphi} + \varkappa\,\bar\partial\left(\frac{1}{\varphi}\right)$, $\bar\partial\left(\frac{1}{\varphi}\right) \in C^\infty(\bar D \cap U_1)$, by extending with zero outside $\bar D \cap U_1$, we obtain that $\bar\partial\left(\frac{\varkappa}{\varphi}\right) \in C^\infty_{(0,1)}(\bar D)$.

Using a theorem of J.J. Kohn ([10], [11]) there exists $g \in C^\infty(\bar D)$ such that $\bar\partial\left(\frac{\varkappa}{\varphi}\right) = \bar\partial g$. Since $\bar D$ is compact we may add to g a sufficiently great constant such that Re $g > 0$ in $\bar D$.

We define $h = \dfrac{\varphi}{\varkappa - g\varphi} = \dfrac{1}{\frac{\varkappa}{\varphi} - g}$

Then h is in $C^\infty(\bar D)$, $\bar\partial h = 0$, $h = 0$ on $E \cap V_1$ and Re $h < 0$ on $\bar D \setminus E \cap V_1$.

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