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# Topologically trivial algebraic 2-vector bundles on ruled surfaces.II.

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## Introduction.

It is a classical result that on compact analytic surfaces the continuous complex vector bundles of rank 2 are well-determined by their Chern classes $c_1, c_2$ (Wu [13]). In particular such a bundle is trivial iff $c_1=0$ and $c_2=0$. For a nonsingular projective surface it follows by a result of Schwarzenberger [11] that on the topologically trivial 2-vector bundle there are nontrivial algebraic structures. In fact these algebraic bundles form a very large family (that is not bounded ).

In this paper we continue the study started in [3] on the structure of topologically trivial algebraic 2-vector bundles on a ruled surface. On $\mathbb{P}^2$ the problem was studied in [1],[10],[12]. For stable bundles on ruled surfaces see [4], [8].

The first problem, in our case, is to find numerical invariants such that if one considers those algebraic bundles with fixed invariants they form an algebraic family. In section 1 we introduce for these bundles two numerical invariants d and r and we define the set M(d,r) of classes of isomorphism of bundles with fixed invariants d and r . The integer d is given by the splitting of the bundle on the general fibre and the integer r is given by some normalization of the bundle. The main result is theorem 1, which states that M(d,r) carries a natural structure of an algebraic variety and that there exists, locally, a tautological bundle. After some preparatory work in section 2, the sections 3 and 4 are devoted to the proof of the theorem 1. In section 5 we show that does not always exist, globally, a tautological bundle (theorem 2 ). Finally the section 6 is concerned with the case of rational ruled surfaces, when the algebraic structure on M(d,r) is more precisely described (theorem 3 and also [3]).

We wish to thank Constantin Bănică to introduce us to this subject and for discussions during the preparation of this paper.

## 1. The numerical invariants d and r.

The notations and the terminology are those of [7].

Let C be a nonsingular curve of genus g over the complex numbers field and let $\pi : X \longrightarrow C$ be a (geometrically) ruled surface over C. One can write $X \simeq \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a normalized locally free sheaf of rank 2 on C. Let denote by $\mathfrak{e}$ the divisor on C corresponding to the invertible sheaf $\wedge^2 \mathcal{E}$ and $e=-\deg \mathfrak{e}$. We fix a section $C_o$ of $\pi$ with $\mathcal{O}_X(C_o) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and $p_o$ a point of C. Let $f_o = \pi^{-1}(p_o)$. Any element of Num $X=H^2(X,\mathbb{Z})$ can be written $aC_o + bf_o$ with $a,b \in \mathbb{Z}$ and $C_o^2=-e$, $C_o.f_o=1$, $f_o^2=0$. Since the canonical divisor $K_X$ on X is given by $K_X \sim -2C_o + \pi^*(K_C + \mathfrak{e})$, hence for the numerical equivalence we have $K_X \equiv -2C_o + (2g-2-e)f_o$ (cf. Hartshorne [7], Ch.V ).

We will denote by $\mathcal{O}_C(1)$ the invertible sheaf associated to the the divisor $p_o$ on C. If L is an element of Pic C, we shall write $L = \mathcal{O}_C(k) \otimes L_o$, where k=deg L and $L_o \in \mathrm{Pic}^o C$. We also denote by

$\mathcal{F}(aC_o+bf_o) = \mathcal{F}\otimes\mathcal{O}_X(a)\otimes\pi^*(\mathcal{O}_C(b))$ for any sheaf $\mathcal{F}$ on X and any $a,b\in\mathbb{Z}$ ( $\mathcal{O}_X(a)\simeq\mathcal{O}_X(aC_o)$).

Let E be a topologically trivial algebraic 2-vector bundle on X ,i.e. $c_1(E)=(0,0)$ and $c_2(E)=0$. Since the fibres of $\pi$ are isomorphic to $\mathbb{P}^1$ , we can speak about the generic splitting type of E and we put $E|_f\simeq\mathcal{O}_f(d)\oplus\mathcal{O}_f(-d)$ for a general fibre f,where $d\geqslant 0$ (cf.[6] and semi-continuity theorem).

The second numerical invariant r is obtained by the following normalization:

$-r=\inf\{\ell \mid$ there exists $L\in \mathrm{Pic}\ C, \deg L=\ell$ s.t. $H^0(X,E(-dC_o)\otimes\pi^*(L))\neq 0\}$

One has $H^0(X,E(-dC_o)\otimes\pi^*(L))\simeq H^0(C,\pi^*(E(-dC_o))\otimes L)$ and moreover $H^0(C,\pi_*(E(-dC_o))\otimes L)$ does not vanish when $\deg L\gg 0$ and is zero when $\deg L\ll 0$ (for, use a suitable filtration of $\pi_*(E(-dC_o))$ with subbundles and Riemann-Roch for divisors on C ). Therefore, there exists such an integer r.

We shall call a topologically trivial algebraic 2-vector bundle with numerical invariants d and r,simply, a 2-vector bundle of type (d,r).

Let us denote by M(d,r) the set of classes of isomorphism of 2-vector bundles on X of fixed type (d,r).

Our purpose is to prove the following

<u>Theorem 1.</u> Suppose $d>0$.Then:

(1) The set M(d,r) carries a natural structure of algebraic variety.

(2) There exists,locally relative to M(d,r), a tautological bundle (i.e. for every affine open subset V of M(d,r) there is a bundle $\mathcal{F}$ on $X\times V$ such that for each $t\in V$ there is an isomorphism $\mathcal{F}/\mathfrak{m}_t\mathcal{F}\simeq E_t$, where $E_t$ denotes a 2-vector bundle corres-

ponding to the class t ).

Remarks. 1. For the case d=0 see the remark at the end of lemma in section 2.

2. In general does not exist globally a tautological bundle; see theorem 2.

3. In the case of rational ruled surfaces the algebraic structure on M(d,r) is more precisely described; see[3] and theorem 3.

## 2. Some properties of the bundles of M(d,r).

We want to prove the following

Lemma. (1) Every 2-vector bundle E of type (d,r) is given by an extension of the form

$$0\to \mathcal{O}_X(dC_0+rf_0)\otimes \pi^*(L_2)\longrightarrow E\longrightarrow \mathcal{J}_Y\otimes \mathcal{O}_X(-dC_0-rf_0)\otimes\pi^*(L_1)\to 0,$$

where $L_1,L_2\in \mathrm{Pic}^0C$ and Y is a locally complete intersection of codimension 2 in X with deg Y=d(2r-de).

(2) Every algebraic 2-vector bundle given by an extension like above is of type (d,r).

(3) For every fixed data $Y,L_1,L_2$, where $Y\subset X$ is a locally complete intersection of codimension 2 with deg Y=d(2r-de) and $L_1,L_2\in \mathrm{Pic}^0C$, there exist bundles appearing as extensions like above and these extensions are uniquely determined modulo $\mathbb{C}^*$, by the isomorphic classes of bundles.

(4) Suppose $d>0$ and let E be a 2-vector bundle of type (d,r). Then the data $Y,L_1,L_2$ from the corresponding extension are uniquely determined by E ($L_1$ and $L_2$ up to an isomorphism).

Proof. (1) By the definition of r there exists $L_2\in \mathrm{Pic}^0C$, such that $H^0(X, E(-dC_0-rf_0)\otimes \pi^*(L_2^{-1}))\neq 0$ (in fact its dimension is 1). We choose a non-zero section and apply Serre's method ([9], Ch.I, §5 ) we obtain E as an extension of the desired form (see [3] for more details ).

(2) Obviously an algebraic 2-vector bundle given by such an extension has $c_1(E)=(0,0)$ and $c_2(E)=0$. By restricting the exact sequence above to a fibre f provided that $f\cap Y=\emptyset$ we get that the splitting type of E is (d,-d); a simple argument shows that the integer r from the given extension is the second numerical invariant of E, hence E is of type (d,r).

(3) Let us denote $\mathcal{L}_1=\mathcal{O}_X(-dC_o-rf_o)\otimes\pi^*(L_1)$ and $\mathcal{L}_2=$ $=\mathcal{O}_X(dC_o+rf_o)\otimes\pi^*(L_2)$. Consider the spectral sequence of term

$E_2^{p,q}=H^p(X,\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{I}_Y\otimes\mathcal{L}_1,\mathcal{L}_2))$, which converges to

$E^{p+q}=\mathrm{Ext}^{p+q}(\mathcal{I}_Y\otimes\mathcal{L}_1,\mathcal{L}_2)$. On the other hand, $\mathcal{E}xt^o(\mathcal{I}_Y\otimes\mathcal{L}_1,\mathcal{L}_2)\simeq$ $\simeq\mathcal{L}_2\otimes\mathcal{L}_1^{-1}$ and $\mathcal{E}xt^1(\mathcal{I}_Y\otimes\mathcal{L}_1,\mathcal{L}_2)\simeq\mathcal{E}xt^2(\mathcal{O}_Y\otimes\mathcal{L}_1,\mathcal{L}_2)\simeq\mathcal{O}_Y$ (for the last isomorphism we use the connexion between the dualizing sheaves $\omega_Y$ and $\omega_X$ and that dim Y=0). But one can easily see that $H^2(X,\mathcal{L}_2\otimes\mathcal{L}_1^{-1})=0$, hence the exact sequence of lower degree terms becomes:

$$0\longrightarrow H^1(X,\mathcal{L}_2\otimes\mathcal{L}_1^{-1})\longrightarrow \mathrm{Ext}^1(\mathcal{I}_Y\otimes\mathcal{L}_1,\mathcal{L}_2)\longrightarrow H^o(Y,\mathcal{O}_Y)\longrightarrow 0.$$

Now, by a result due to Serre ([9],Ch,I,§5), any element belonging to $\mathrm{Ext}^1(\mathcal{I}_Y\otimes\mathcal{L}_1,\mathcal{L}_2)$ which has an invertible image in $H^o(Y,\mathcal{O}_Y)$, defines an extension of the desired form. One has $H^o(X,E\otimes\mathcal{L}_2^{-1})\cong\mathbb{C}$ and the second statement follows by a well-known argument.

(4) $L_2$ is determined being $\pi_*(E(-dC_o-rf_o))$. Y is determined as the zero-set of the unique (mod $\mathbb{C}^*$) non-zero global section of $E\otimes\mathcal{L}_2^{-1}$. Using again $H^o(X,E\otimes\mathcal{L}_2^{-1})\simeq\mathbb{C}$, $\mathcal{I}_Y\otimes\mathcal{L}_1^{-1}$ will be well-

determined and therefore (by removability) $\mathcal{L}_1$, that is $L_1$ will be well-determined by the bundle E.

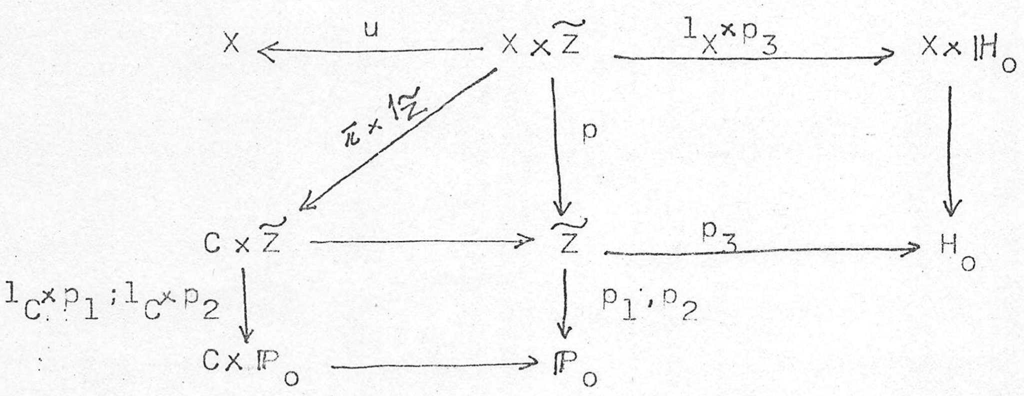
Remark. For d=0 it follows from the part (1) that every 2-vector bundle E of type (0,r) is of the form $\pi^*(F)$, where F is an algebraic 2-vector bundle on the curve C with $c_1(F)=0$. Moreover, there is an one to one correspondence between the set of isomorphism classes of these bundles on the surface X and the set of isomorphism classes of 2-vector bundles on the curve C with $c_1=0$. Thus the classification of these bundles means the classification of the corresponding bundles on curves.

## 3. The algebraic structure of M(d,r).

In this section we shall prove the part (1) of the theorem 1. Suppose $d>0$ and let E be a 2-vector bundle of type (d,r). It follows by the lemma that E defines an element $\xi_E \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{J}_Y \otimes \mathcal{L}_1, \mathcal{L}_2)$, where $\mathcal{L}_1 = \mathcal{O}_X(-dC_o - rf_o) \otimes \pi^*(L_1)$, $\mathcal{L}_2 = \mathcal{O}_X(dC_o + rf_o) \otimes \pi^*(L_2)$ and the data $L_1, L_2$, Y are uniquely determined by E. If $Y \neq \emptyset$ (i.e. E is non-uniform; see [3]) then $\xi_E \neq 0$; if $Y=\emptyset$ (i.e. 2r=de and E is uniform) we agree to expel the single decomposable bundle of type (d,r). It follows again by the lemma that the class of E in M(d,r) defines a unique point in the projective space

$$(\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{J}_Y \otimes \mathcal{L}_1, \mathcal{L}_2) \setminus \{0\})/\mathbb{C}^*.$$

In order to parametrize the whole set M(d,r) we have to move $L_1, L_2$ and Y and thus to look for the variation of Ext. Let $\mathbb{P}_o$ be the Picard variety of isomorphism classes of line bundles of zero degree on C and let $\mathbb{L}_o$ be the universal Poincaré-bundle on $C \times \mathbb{P}_o$. Let $\mathbb{H}_o$ be the Hilbert variety of zero-dimensional locally complete intersections of degree d(2r-de) in X and let $\mathcal{I}_o$ be the ideal sheaf of the universal subspace $Y_o \subset X \times \mathbb{H}_o$. Recall that $\mathbb{H}_o$ is smooth connected, quasi-projective of dimension 2d(2r-de)(see [5]). Let us denote by $\widetilde{Z}$ the variety $\mathbb{P}_o \times \mathbb{P}_o \times \mathbb{H}_o$. We have the following diagram with natural maps:




We use the notations:

$\mathbb{L}_i = (1_C \times p_i)^*(\mathbb{L}_o)$ i=1,2; $\mathcal{I}_{\mathcal{Y}} = (1_X \times p_3)^*(\mathcal{I}_o)$, where $\mathcal{Y} = (1_X \times p_3)^{-1}(Y_o)$.

Now it is natural to consider the relative $\mathcal{E}xt$:

$$\widetilde{\mathcal{R}} = \mathcal{E}xt^1_p(u^*(\mathcal{O}_X(-dC_o - rf_o)) \otimes (\pi \times 1_{\widetilde{Z}})^*(\mathbb{L}_1) \otimes \mathcal{I}_{\mathcal{Y}}, u^*(\mathcal{O}_X(dC_o + rf_o)) \otimes (\pi \times 1_{\widetilde{Z}})^*(\mathbb{L}_2)).$$

Take the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{I}_Y \otimes \mathcal{L}_1, \mathcal{L}_2)),$$

and notice that

$$E_2^{o,2} = H^o(X, \mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{I}_Y \otimes \mathcal{L}_1, \mathcal{L}_2)) \simeq H^o(X, \mathcal{E}xt^3_{\mathcal{O}_X}(\mathcal{O}_Y \otimes \mathcal{L}_1, \mathcal{L}_2)) = 0$$

$$E_2^{1,1} = H^1(X, \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{I}_Y \otimes \mathcal{L}_1, \mathcal{L}_2)) \simeq H^1(X, \mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{O}_Y \otimes \mathcal{L}_1, \mathcal{L}_2)) \simeq H^1(Y, \mathcal{O}_Y) = 0$$

$$E_2^{2,o} = H^2(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_Y \otimes \mathcal{L}_1, \mathcal{L}_2)) \simeq H^2(X, \mathcal{L}_2 \otimes \mathcal{L}_1^{-1}) = 0.$$

It follows that $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{I}_Y \otimes \mathcal{L}_1, \mathcal{L}_2) = 0$ and if we denote by z the point of $\widetilde{Z}$ determined by $L_1, L_2$ and Y, we deduce by means of a result of [2] that

$$\widetilde{\mathcal{R}}_z / m_z \widetilde{\mathcal{R}}_z \simeq \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_Y \otimes \mathcal{L}_1, \mathcal{L}_2).$$

But, generally, $\widetilde{\mathcal{R}}$ is only a coherent sheaf and not a locally free one. This occurs because, although Z is reduced ( in fact nonsingular) the function

$$z \longrightarrow \dim \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_Y, \mathcal{L}_2 \otimes \mathcal{L}_1^{-1}) \quad (z=(L_1, L_2, Y))$$

is not constant. The jump of dim $\operatorname{Ext}^1$ happens already in the case when C is an elliptic curve (see an example in [3]).

We take the stratification given by $\dim \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_Y, \mathcal{L}_2 \otimes \mathcal{L}_1^{-1}) =$ $=$constant and we get (by semicontinuity theorem) finitely many strata

$$Z_i = \{ z \in \widetilde{Z} \mid \dim \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{I}_Y \otimes \mathcal{L}_1, \mathcal{L}_2) = i,\ z=(L_1, L_2, Y) \},$$

which are Zariski locally closed subsets of $\widetilde{Z}$. We choose on $Z_i$ the reduced structure and let $\mathcal{R}_i$ be the corresponding relative $\mathcal{E}xt$. Now every sheaf $\mathcal{R}_i$ is locally free. Let Z be the direct sum of $Z_i$ and $\mathcal{R}$ the corresponding locally free sheaf on Z. It follows from previous considerations that there exists a natural injective map $M(d,r) \longrightarrow \mathbb{P}(\mathcal{R}^*)$ and we shall identify the set $M(d,r)$ with its image in the projective bundle of the dual of $\mathcal{R}$, $\mathbb{P}(\mathcal{R}^*)$. It is not difficult to see that $M(d,r)$ is a Zariski open subset of $\mathbb{P}(\mathcal{R}^*)$ (see [1] Lemma 3) and so $M(d,r)$ carries a natural structure of an algebraic variety.

## 4. The local existence of the tautological bundle.

In this section we shall prove the part (2) of the theorem 1. Clearly we may assume $Z = Z_i$ (one stratum) and $\mathcal{R} = \mathcal{R}_i$. Let $p: X \times Z \longrightarrow Z$ be the canonical projection and let denote $\mathcal{T}_1 = u^*(\mathcal{O}_X(-dC_o - rf_o)) \otimes (\bar{\pi} \times 1_Z)^*(\mathbb{L}_1)$ and $\mathcal{T}_2 = u^*(\mathcal{O}_X(dC_o + rf_o)) \otimes (\bar{\pi} \times 1_Z)^*(\mathbb{L}_2)$. Then we have that $\mathcal{R} = \mathcal{E}xt^1_p(\mathcal{I}_{\mathcal{Y}} \otimes \mathcal{T}_1, \mathcal{T}_2)$ is a locally free sheaf on Z and commutes with base change.

Let $\mathbb{P}(\mathcal{R}^*)$ be the projective bundle of the dual of $\mathcal{R}$ and let $q: \mathbb{P}(\mathcal{R}^*) \longrightarrow Z$ be the canonical projection. Consider the cartesian diagram

$$\begin{array}{ccc} X \times \mathbb{P}(\mathcal{R}^*) & \xrightarrow{q'} & X \times Z \\ \downarrow p' & & \downarrow p \\ \mathbb{P}(\mathcal{R}^*) & \xrightarrow{q} & Z \end{array}$$

and let $\mathcal{O}_{\mathbb{P}(\mathcal{R}^*)}(1)$ be the tautological invertible sheaf on $\mathbb{P}(\mathcal{R}^*)$. We use the notations: $\mathcal{T}'_1 = q'^*(\mathcal{T}_1)$, $\mathcal{T}'_2 = q'^*(\mathcal{T}_2)$ and $\mathcal{Y}' = q'^*(\mathcal{Y})$. The canonical surjection $q^*(\mathcal{R}^*) \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{R}^*)}(1)$ gives a morphism

$$\mathcal{O}_{\mathbb{P}(\mathcal{R})}(-1) \longrightarrow q^*(\mathcal{R}) \simeq \mathcal{E}xt^1_{p'}(\mathcal{I}_{\mathcal{Y}'} \otimes \mathcal{T}'_1, \mathcal{T}'_2),$$

hence a global section $\xi$ in $\mathcal{E}xt^1_{p'}(\mathcal{I}_{\mathcal{Y}'} \otimes \mathcal{T}'_1, \mathcal{T}_2 \otimes p'^*(\mathcal{O}_{\mathbb{P}(\mathcal{R}^*)}(1)))$.

Let now V be an affine open subset in $M(d,r) \subset \mathbb{P}(\mathcal{R}^*)$ and $V' = p'^{-1}(V) = X \times V$. We have the isomorphism

$$\mathrm{Ext}^1(V'; \ldots) \xrightarrow{\sim} H^0(V, \mathcal{E}xt^1_{p'}(\ldots))$$

and let $\eta \in \mathrm{Ext}^1(V'; \mathcal{I}_{\mathcal{Y}'} \otimes \mathcal{T}'_1, \mathcal{T}'_2 \otimes p'^*(\mathcal{O}_{\mathbb{P}(\mathcal{R}^*)}(1)))$ be the corresponding element of $\xi|_V$. The element $\eta$ gives an extension on $V'$

$$0 \longrightarrow \mathcal{T}'_2 \otimes p'^*(\mathcal{O}_{\mathbb{P}(\mathcal{R}^*)}(1)) \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_{\mathcal{Y}'} \otimes \mathcal{T}'_1 \longrightarrow 0,$$

where the sheaf $\mathcal{F}$ is obviously flat over V. For each $t \in V$, the reduction modulo $\mathfrak{m}_t$ of the above extension is naturally equivalent modulo $\mathbb{C}^*$ with the extension

$$0 \longrightarrow \mathcal{L}_2 \longrightarrow E_t \longrightarrow \mathcal{I}_Y \otimes \mathcal{L}_1 \longrightarrow 0,$$

where $E_t$ is a 2-vector bundle corresponding to the very class of t. Then we have the isomorphism $\mathcal{F}/\mathfrak{m}_t\mathcal{F} \simeq E_t$ and since $\mathcal{F}$ is flat over V one gets that the sheaf $\mathcal{F}$ is locally free. $\mathcal{F}$ will be a tautological bundle on $X \times V$.

## 5. The global non-existence of the tautological bundle.

We shall prove the following

Theorem 2. Let X be the rational ruled surface $\mathbb{F}_1$ (e=1). There is not globally a tautological bundle relative to M(1,1).

Proof. Let us consider the following cartesian diagram in the general case:

$$\begin{array}{ccc} X \times M(d,r) & \xrightarrow{\quad q' \quad} & X \times Z \\ \Big\downarrow p' & & \Big\downarrow p \\ M(d,r) & \xrightarrow{\quad q \quad} & Z \end{array}$$

Further we shall preserve the previous notations.

Let us suppose that there exists a bundle on $X \times M(d,r)$ such that for each $t \in M(d,r)$ there is an isomorphism $\mathcal{F}/\mathfrak{m}_t\mathcal{F} \simeq E_t$, where $E_t$ denotes a 2-vector bundle belonging to the class of t. The sheaf $p'_*(\mathcal{F}\otimes\mathcal{T}'^{-1}_2)$ is an invertible sheaf since M(d,r) is reduced and $\dim H^0(X, E_t\otimes\mathcal{L}_2^{-1})=1$ for any $E_t$. The bundle

$\mathcal{F}' = \mathcal{F}\otimes p'^*(p'_*(\mathcal{F}\otimes\mathcal{T}'^{-1}_2))$ is, again, tautological and

$p'_*(\mathcal{F}'\otimes\mathcal{T}'^{-1}_2) \simeq \mathcal{O}$ . Consequently, there is a section

$\sigma \in H^0(X\times M(d,r), \mathcal{F}'\otimes\mathcal{T}'^{-1}_2)$ with non-zero image through the identification: $p'_*(\mathcal{F}'\otimes\mathcal{T}'^{-1}_2)_t/\mathfrak{m}_t p'_*(\mathcal{F}'\otimes\mathcal{T}'^{-1}_2)_t \simeq H^0((\mathcal{F}'\otimes\mathcal{T}'^{-1}_2)/\mathfrak{m}_t(\mathcal{F}'\otimes\mathcal{T}'^{-1}_2)) \simeq$ $\simeq \mathbb{C}$, The corresponding map $\mathcal{T}'_2 \longrightarrow \mathcal{F}'$ is injective modulo $\mathfrak{m}_t$ for each $t \in M(d,r)$. Then $\mathcal{T}'_2 \longrightarrow \mathcal{F}$ is injective and its cokernel $\mathcal{C}$ is flat over M(d,r). We have the exact sequences:

$$0 \longrightarrow \mathcal{L}_2 \longrightarrow \mathcal{F}'/\mathfrak{m}_t\mathcal{F}' \longrightarrow \mathcal{C}/\mathfrak{m}_t\mathcal{C} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{L}_2 \longrightarrow E_t \longrightarrow \mathcal{I}_{Y}\otimes\mathcal{L}_1 \longrightarrow 0,$$

where $q(t)=z=(L_1,L_2,Y)$, $\mathcal{L}_1 = \mathcal{O}_X(-C_0-f_0)\otimes\pi^*(L_1)$ and

$\mathcal{L}_2 = \mathcal{O}_X(C_o+f_o)\otimes\pi^*(L_2)$. Also $\mathcal{C}/\mathfrak{m}_t\mathcal{C}\simeq\mathcal{I}_Y\otimes\mathcal{L}_1$. $\mathcal{C}$ and $\mathcal{I}_{\mathcal{Y}'}\otimes\mathcal{T}_1'$ are flat with respect to $p'$ and moreover isomorphic, locally relative to M(d,r), since $h^0 End(\mathcal{I}_Y\otimes\mathcal{L}_1)=h^0 End(\mathcal{I}_Y)=1$ when Y is 0-dimensional in X ([2], Korollar 5). As $\mathcal{E}nd(\mathcal{I}_{\mathcal{Y}'})\simeq\mathcal{O}$, hence there is an invertible sheaf $\mathcal{L}$ on M(d,r) such that $\mathcal{C}\simeq\mathcal{I}_{\mathcal{Y}'}\otimes\mathcal{T}_1'\otimes p'^*(\mathcal{L}^*)$.

Therefore we have the extension:

$$0\longrightarrow\mathcal{T}_2'\otimes p'^*(\mathcal{L})\longrightarrow\mathcal{F}''\longrightarrow\mathcal{I}_{\mathcal{Y}'}\otimes\mathcal{T}_1'\longrightarrow 0,$$

where $\mathcal{F}''=\mathcal{F}'\otimes p'^*(\mathcal{L})$ is again tautological and let $\xi\in Ext^1(X\times M(d,r);\dots)$ be the corresponding class. Since

$$\mathcal{E}xt^1_{p'}(\mathcal{I}_{\mathcal{Y}'}\otimes\mathcal{T}_1',\mathcal{T}_2'\otimes p'^*(\mathcal{L}))\simeq\mathcal{E}xt^1_{p'}(\mathcal{I}_{\mathcal{Y}'}\otimes\mathcal{T}_1',\mathcal{T}_2')\otimes\mathcal{L}\simeq q^*(\mathcal{R})\otimes\mathcal{L},$$

hence the image of $\xi$ by $Ext^1\longrightarrow H^0(\mathcal{E}xt^1)$ gives rise to a map $q^*(\mathcal{R}^*)\longrightarrow\mathcal{L}$ and as $\mathcal{F}''$ is tautological, it follows (considering again the fibres of $\pi'$) that $\mathcal{L}\simeq\mathcal{O}_{\mathbb{P}(\mathcal{R}^*)}(1)$. Consequently, we deduce that if there were globally a tautological bundle $\mathcal{F}$, then by tensoring it with an invertible sheaf, the new tautological bundle $\mathcal{F}''$ would be given globally by the following extension:

$$0\longrightarrow\mathcal{T}_2'\otimes p'^*(\mathcal{O}_{\mathbb{P}(\mathcal{R}^*)}(1))\longrightarrow\mathcal{F}''\longrightarrow\mathcal{I}_{\mathcal{Y}'}\otimes\mathcal{T}_1'\longrightarrow 0.$$

In the case of the rational ruled surface $X=\mathbb{F}_1$ for M(1,1) we have: $\mathbb{P}_0=Pic^0C=$ one point, deg Y=1, hence the Hilbert variety $\mathbb{H}^0=\mathbb{F}_1$, the universal subspace $\mathcal{Y}$ is the diagonal $\Delta$ and $\dim Ext^1_{\mathcal{O}_X}(\mathcal{I}_Y,\mathcal{L}_2\otimes\mathcal{L}_1^{-1})=1$. It follows that the variety Z consists of a single stratum and $M(1,1)\simeq\mathbb{F}_1$. The previous diagram becomes:

$$\begin{array}{ccc}
\mathbb{F}_1\times\mathbb{F}_1 & \xrightarrow{\ \text{id.}\ } & \mathbb{F}_1\times\mathbb{F}_1 \\
\downarrow p & & \downarrow p=p_2 \\
\mathbb{F}_1 & \xrightarrow{\ \text{id.}\ } & \mathbb{F}_1
\end{array}$$

We have $\mathcal{R} \simeq p_*(\mathcal{E}xt^1(\mathcal{J}_\Delta \otimes \mathcal{T}_1, \mathcal{T}_2))$, but $\mathcal{E}xt^1(\mathcal{J}_\Delta \otimes \mathcal{T}_1, \mathcal{T}_2) \simeq$

$\simeq \mathcal{E}xt^2(\mathcal{O}_\Delta \otimes \mathcal{T}_1, \mathcal{T}_2) \simeq \omega_\Delta \otimes \mathcal{T}_1^{-1} \otimes \mathcal{T}_2 \otimes \omega^{-1}_{\mathbb{F}_1 \times \mathbb{F}_1}$, $\omega_{\mathbb{F}_1 \times \mathbb{F}_1} \simeq$

$\simeq p_1^*(\omega_{\mathbb{F}_1}) \otimes p_2^*(\omega_{\mathbb{F}_1})$, $\omega_{\mathbb{F}_1} = \mathcal{O}_{\mathbb{F}_1}(-2C_o-3f_o)$, $\mathcal{T}_1 = p_1^*(\mathcal{O}_{\mathbb{F}_1}(-C_o-f_o))$,

$\mathcal{T}_2 = p_1^*(\mathcal{O}_{\mathbb{F}_1}(C_o+f_o))$ and by means of the identification $\Delta \simeq \mathbb{F}_1$

we get $\mathcal{R} \simeq \mathcal{O}_{\mathbb{F}_1}(4C_o+5f_o)$. Then $\mathcal{O}_{\mathbb{P}(\mathcal{R}^*)}(1) \simeq \mathcal{R}^* \simeq \mathcal{O}_{\mathbb{F}_1}(-4C_o-5f_o)$

and a global tautological bundle will be given by the extension:

$$0 \to p_1^*(\mathcal{O}_{\mathbb{F}_1}(C_o+f_o)) \otimes p_2^*(\mathcal{O}_{\mathbb{F}_1}(-4C_o-5f_o)) \to \mathcal{F}'' \to \mathcal{J}_\Delta \otimes p_1^*(\mathcal{O}_{\mathbb{F}_1}(-C_o-f_o)) \to 0$$

We shall prove that

$$\mathrm{Ext}^1(\mathcal{J}_\Delta \otimes p_1^*(\mathcal{O}_{\mathbb{F}_1}(-C_o-f_o)), p_1^*(\mathcal{O}_{\mathbb{F}_1}(C_o+f_o)) \otimes p_2^*(\mathcal{O}_{\mathbb{F}_1}(-4C_o-5f_o))) = 0$$

and we will derive a contradiction. We show that

$H^3(\mathbb{F}_1 \times \mathbb{F}_1, \mathcal{J}_\Delta \otimes p_1^*(\mathcal{O}_{\mathbb{F}_1}(-4C_o-5f_o)) \otimes p_2^*(\mathcal{O}_{\mathbb{F}_1}(2C_o+2f_o))) = 0$ and by duality we conclude.

The exact sequence $0 \to \mathcal{J}_\Delta \to \mathcal{O}_{\mathbb{F}_1 \times \mathbb{F}_1} \to \mathcal{O}_\Delta \to 0$ gives the exact sequence:

$$H^2(p_1^*(\mathcal{O}_{\mathbb{F}_1}(-4C_o-5f_o)) \otimes p_2^*(\mathcal{O}_{\mathbb{F}_1}(2C_o+2f_o))) \xrightarrow{\alpha} H^2(\mathcal{O}_\Delta(-2C_o-3f_o)) \to$$

$$\to H^3(\mathcal{J}_\Delta \otimes p_1^*(\mathcal{O}_{\mathbb{F}_1}(-4C_o-5f_o)) \otimes p_2^*(\mathcal{O}_{\mathbb{F}_1}(2C_o+2f_o))) \longrightarrow$$

$$\longrightarrow H^3(p_1^*(\mathcal{O}_{\mathbb{F}_1}(-4C_o-5f_o)) \otimes p_2^*(\mathcal{O}_{\mathbb{F}_1}(2C_o+2f_o))).$$

Because $H^2(\mathcal{O}_\Delta(-2C_o-3f_o)) \simeq H^2(\mathbb{F}_1, \omega_{\mathbb{F}_1}) \simeq \mathbb{C}$ and

$$H^2(p_1^*(\mathcal{O}_{\mathbb{F}_1}(-4C_o-5f_o)) \otimes p_2^*(\mathcal{O}_{\mathbb{F}_1}(2C_o+2f_o))) \simeq$$

$$\simeq H^2(\mathbb{F}_1, \mathcal{O}_{\mathbb{F}_1}(-4C_o-5f_o)) \otimes \mathrm{Hom}(\mathcal{O}_{\mathbb{F}_1}(-4C_o-5f_o), \omega_{\mathbb{F}_1})$$

the map $\alpha$ can be naturally identified to the linear map associated to the natural pairing of Serre duality, hence it is not zero. But $H^3(p_1^*(\mathcal{O}_{\mathbb{F}_1}(-4C_o-5f_o))\otimes p_2^*(\mathcal{O}_{\mathbb{F}_1}(2C_o+2f_o)))=0$ and the proof is over.

## 6. The case of rational ruled surfaces.

Theorem 3. Assume $C=\mathbb{P}^1$ ( i.e. $X\cong\mathbb{F}_e$ a rational ruled surface). Then the set $M(d,r)$ is a nonsingular, connected, quasi-projective, rational variety of dimension indicated below:

(a) when $de=2r$ and $e\geq 1$ (uniform bundles)

$$\dim M(d,r)=\tfrac{1}{2}d(de+e-2)-1$$

(b) when $e=0$ $(r=0)$ (uniform bundles)

$$\dim M(d,0)=-1 \quad (M=\emptyset\ !)$$

(c) when $2r-de>0$ and $e\geq 1$ (non-uniform bundles) there are two posibilities: $r>de$, $\dim M(d,r)=3d(2r-de)-1$, or $r\leq de$ and $\dim M(d,r)=\tfrac{1}{2}(2d-s+1)(es+2de-4r-2)+3d(2r-de)-1$ $(s=\left[(2r+1)/e\right]+1)$.

(d) when $2r-de>0$ but $e=0$ (non-uniform bundles)

$$\dim M(d,r)=6dr-1.$$

Proof. In this particular case the $\mathcal{E}xt$-sheaf $\widetilde{\mathcal{R}}$ is locally free, its rank is well-determined by $d,r$ and $e$ (see [3] for more details) and thus, the set $M(d,r)$ is a Zariski open subset of the nonsingular, projective, rational, connected variety $\mathbb{P}(\widetilde{\mathcal{R}}^*)$.

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