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ABSTRACT

We prove the stability of the index and the semicontinuity of the dimensions of the cohomology groups of semi-Fredholm complexes of Banach spaces and closed linear operators with respect to perturbations of the operators and of the underlying spaces which are small with respect to the gap topologies. It seems that, even for single semi-Fredholm operators, some of the statements are more general than the current ones. The results are applied to obtain semi-continuity for joint spectra of finite systems of commuting bounded linear operators.

STABILITY OF THE INDEX OF A SEMI-FREDHOLM COMPLEX OF BANACH SPACES

Ernst Albrecht and Florian-Horia Vasilescu

1. INTRODUCTION

The aim of this work is to study the stability of the index of a semi-Fredholm complex of Banach spaces (see Section 3) of the form

$$\ldots \rightarrow X^{p-1} \xrightarrow{\alpha^{p-1}} X^{p} \xrightarrow{\alpha^{p}} X^{p+1} \xrightarrow{\alpha^{p+1}} \ldots, \qquad (1.1)$$

where α^p is a closed linear operator for every $p \in \mathbb{Z}$, with respect to the gap topology. This means that both spaces and operators may be perturbed. The main result of the present paper asserts that the index (i.e. the extended Euler characteristic) of such a complex is stable under small perturbations and the dimension of the groups of the associated cohomology is upper semicontinuous, at least for complexes of finite length (see Corollary 3.6). However, we obtain adequate information even for complexes of infinite length (see Theorems 3.5 and 3.7). In particular, we extend the corresponding result from [8], as well as the standard result of stability (in the gap topology) of the index of a semi-Fredholm operator. It seems that actually in the context of semi-Fredholm operators we obtain more general assertions than the current ones. This happens because we allow the variation of both spaces and operators. In addition, our method enables us to give in some cases effective estimates (see, for instance, the proof of Proposition 2.20).

The ocurrence of complexes of Banach (or Hilbert) spaces in rather different domains of mathematics, for instance in the theory of the $\overline{\partial}$-operator in strongly pseudoconvex manifolds [6],[7],[10],[15] etc., as well as in the spectral and Fredholm theory of several commuting operators [17],[4], [19],[5] shows that they are remarkable mathematical objects whose intrinsic systematic study (see also [12],[17],[16], [19],[20],[5] etc.) could be of interest. It is the purpose of this paper to show that a highly abstract concept of a complex of quotients of Banach subspaces has important stability properties under a general type of "small" perturbations. The choice of such a concept has been dictated by some technical reasons, mainly by the invariance under duality, as well as by some significant examples, discussed in the last section.

We shall follow, as a rule, the notation and terminology from [8]. Nevertheless, for convenience of the reader (and to point out some small differences), we shall list the symbols that are used in the present work.

Let X and Y be (complex) Banach spaces. Then $\mathcal{C}(X,Y)$ is the set of all closed linear operators that are defined on linear submanifolds of X and have values in Y. The subset of those operators from $\mathcal{C}(X,Y)$ which are everywhere defined (and hence continuous) is denoted by $\mathcal{L}(X,Y)$. We set $\mathcal{C}(X) := \mathcal{C}(X,X)$ and $\mathcal{L}(X) := \mathcal{L}(X,X)$. For an arbitrary $S \in \mathcal{C}(X,Y)$ we denote by $D(S)$, $N(S)$, $R(S)$, $G(S)$ and $\gamma(S)$ the domain of definition, the null-space, the range, the graph and the reduced minimum modulus of S, respectively. Let $\Phi_+(X,Y)$ be the set of those operators S from $\mathcal{C}(X,Y)$ such that $R(S)$ is closed and $\dim N(S) < \infty$. Similarly, $\Phi_-(X,Y)$ is the set of those operators $S \in \mathcal{C}(X,Y)$ such that $\dim Y/R(S) < \infty$ (this condition implies, in particular, that $R(S)$ is closed [8]). The set $\Phi_-(X,Y) \cup \Phi_+(X,Y)$ is the class of semi-Fredholm operators, and $\Phi(X,Y) := \Phi_-(X,Y) \cap \Phi_+(X,Y)$ is the class of Fredholm operators. The index of a semi-Fredholm operator S will be denoted by $\operatorname{ind} S$.

The direct sum $X \oplus Y$ of two Banach spaces X and Y will always be endowed with the norm $\|x \oplus y\|^2 = \|x\|^2 + \|y\|^2$ for all $x \in X$ and $y \in Y$. This convention automatically covers the case of Hilbert spaces. The family of all closed linear subspaces of a Banach space X will be denoted by $\mathcal{S}(X)$. If $Z \in \mathcal{S}(X)$, then $Z^\perp$ is the annihilator of Z in the dual X^* of X. For every $x \in X$ the symbol $d(x,Z)$ stands for the distance from x to Z.

We recall that the gap topology on $\mathcal{S}(X)$ is defined in the following way: If Y and Z are in $\mathcal{S}(X)$, then one sets

$$\delta(Y,Z) := \sup_{\substack{y \in Y \\ \|y\| \leq 1}} d(y,Z) ,$$

and $\hat{\delta}(Y,Z) := \max\{\delta(Y,Z), \delta(Z,Y)\}$. Then the mapping $\hat{\delta}(Y,Z)$, which is equivalent to the Hausdorff metric on the set of all unit balls of the spaces from $\mathcal{S}(X)$, defines the gap topology of $\mathcal{S}(X)$. One can define in a similar way the gap topology on $\mathcal{C}(X,Y)$, by setting

$$\delta(S,T) = \delta(G(S),G(T))$$

and $\hat{\delta}(S,T) = \max\{\delta(S,T), \delta(T,S)\}$ for all $S,T \in \mathcal{C}(X,Y)$.

We now give a brief description of the contents of the present work. The second section is concerned with the semicontinuity of the dimension of the kernels and cokernels of certain semi-Fredholm operators, with respect to the gap topology. Although intended to provide the auxiliary apparatus of the whole work, this section contains some results that are, we think, interesting for their own sake. Particularly, Theorem 2.2 is not only the core of the second section, but it also yields a general method of solving linear equations by successive approximations (see also Corollary 2.3). This method leads to effective estimates (see the proof of Proposition 2.20) and can even be combined with some non-linear objects (see the proof of Lemma 2.16).

The third section is concerned with the stability of the index of a semi-Fredholm (Fredholm) complex under small perturbations in the gap topology.

The fourth section contains applications of the results from the previous sections. The first example shows that the study of complexes of pairs of subspaces (a concept somehow suggested by [12]) can be reduced to the study of usual complexes. In particular, we obtain the stability of semi-Fredholm pairs of subspaces, as made in [8], even for the case when both arguments vary (see Remark IV. 4.31 from [8]). Then we discuss the invariance of the class of semi-Fredholm complexes under some natural transformations. The third example concerns a class of objects which are complexes modulo compact operators, and whose properties can be obtained from those of usual complexes, via a natural transformation. At this point the use of complexes of quotients of Banach subspaces is effective. Finally, we apply our results to obtain semicontinuity statements for joint spectra of commuting finite systems of bounded linear operators. We are especially interested in operators induced on invariant or quotients of invariant closed subspaces by globally defined linear operators.

This work has been prepared while the second named author was a guest of the University of Saarbrücken as a Humboldt Fellow. He would like to express his gratitude to the Alexander von Humboldt-Foundation for its support and to the University of Saarbrücken for its hospitality.

2. SEMICONTINUITY OF THE DIMENSION

The standard framework of this section is described by the following:

2.1. DEFINITION. Consider the Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$, and let $X_o, X \in \mathcal{S}(\mathfrak{X})$ and $Y_o, Y \in \mathcal{S}(\mathfrak{Y})$ be such that $X_o \subset X$ and $Y_o \subset Y$. For every operator $S \in \mathcal{C}(X/X_o, Y/Y_o)$ we define the following linear spaces:

$$N_o(S) := \{x \in X;\ x + X_o \in N(S)\},$$

$$R_o(S) := \{y \in Y;\ y + Y_o \in R(S)\},$$

$$G_o(S) := \{x \oplus y \in X \oplus Y; x + X_o \in D(S), y \in S(x + X_o)\}.$$

It is plain that $N(S) = N_o(S)/X_o$, $R(S) = R_o(S)/Y_o$ and $G(S)$ is naturally isomorphic to $G_o(S)/(X_o \oplus Y_o)$. In particular, $N_o(S)$ and $G_o(S)$ are closed, whereas $R_o(S)$ is closed if and only if $R(S)$ is closed. If $X_o = 0$ and $Y_o = 0$, then $N_o(S) = N(S)$, $R_o(S) = R(S)$ and $G_o(S) = G(S)$. If $\tilde{X}_o, \tilde{X} \in \mathcal{S}(\mathfrak{X})$ and $\tilde{Y}_o, \tilde{Y} \in \mathcal{S}(\mathfrak{Y})$ are such that $\tilde{X}_o \subset \tilde{X}$ and $\tilde{Y}_o \subset \tilde{Y}$, and if $\tilde{S} \in \mathcal{C}(\tilde{X}/\tilde{X}_o, \tilde{Y}/\tilde{Y}_o)$, then we set

$$\delta_o(S, \tilde{S}) := \delta(G_o(S), G_o(\tilde{S}))$$

and $\hat{\delta}_o(S, \tilde{S}) := \max\{\delta_o(S, \tilde{S}), \delta_o(\tilde{S}, S)\}$, which are computed in $\mathfrak{X} \oplus \mathfrak{Y}$. If $X_o = \tilde{X}_o = 0$ and $Y_o = \tilde{Y}_o = 0$, then $\delta_o(S, \tilde{S}) = \delta(S, \tilde{S})$ and $\hat{\delta}_o(S, \tilde{S}) = \hat{\delta}(S, \tilde{S})$.

Let S be as above and let M be another Banach space. Let also $A \in \mathcal{L}(M, Y/Y_o)$. We define an extension S_1 of S by the equation $S_1(\xi \oplus v) = S\xi + Av$ for all $\xi \in D(S)$ and $v \in M$, and hence $S_1 \in \mathcal{C}((X/X_o) \oplus M, Y/Y_o)$. Since $(X/X_o) \oplus M$ is naturally isomorphic to $(X \oplus M)/(X_o \oplus 0)$, the entities given by Definition 2.1 make sense for S_1, with $\mathfrak{X}$ replaced by $\mathfrak{X} \oplus M$. If $\dim M < \infty$, then we have

$$\dim N(S_1)/N(S) + \dim R(S_1)/R(S) = \dim M, \qquad (2.1)$$

where $\mathfrak{X}$ is identified with $\mathfrak{X} \oplus 0$. The proof of (2.1) can be found in [19], Lemma 2.7.

Let X be an arbitrary Banach space and let $Y \in \mathcal{S}(X)$ be such that $\dim X/Y \leqq m < \infty$. For every $\varepsilon > 0$ one can choose a projection P of X onto Y such that $\|P\| \leqq 1 + m + \varepsilon$ (see, for instance [14]). In the following, we shall stress the choice of such a projection P by saying that $\|P\|$ <u>depends only on</u> m.

2.2. THEOREM. <u>Let</u> S <u>and</u> $\tilde{S}$ <u>be as in Definition</u> 2.1. <u>If</u> $S \in \Phi_-(X/X_o, Y/Y_o)$ <u>and the numbers</u> $\delta_o(S,\tilde{S})$ <u>and</u> $\delta(\tilde{Y},Y)$ <u>are sufficiently small, then</u> $\tilde{S} \in \Phi_-(\tilde{X}/\tilde{X}_o, \tilde{Y}/\tilde{Y}_o)$ <u>and</u> $\dim (\tilde{Y}/\tilde{Y}_o)/R(\tilde{S}) \leqq \dim (Y/Y_o)/R(S)$.

<u>Proof.</u> We shall essentially refine an approximation method from [17], Lemma 2.1 (see also [19], Lemma 2.1). The proof will be divided into several steps.

1^o Note that the space $(Y/Y_o)/R(S)$ is isomorphic to the space $Y/R_o(S)$. Therefore, $\dim Y/R_o(S) = m$, where $m := \dim(Y/Y_o)/R(S)$. Let P be a projection of Y onto $R_o(S)$ such that $\|P\|$ depends only on m. The null-space of P will be denoted by M. Since $\dim(\tilde{Y} + M)/\tilde{Y} \leqq m$, we can choose a projection $\tilde{P}$ of $\tilde{Y} + M$ onto $\tilde{Y}$ such that $\|\tilde{P}\|$ depends only on m. Let us also note that $\gamma(S) > 0$, since $R(S)$ is closed.

2^o Let us choose some positive numbers $r > \gamma(S)^{-1}$, $\delta > \delta(\tilde{Y},Y)$ and $\delta_o > \delta_o(S,\tilde{S})$. If $\delta(\tilde{Y},Y)$ and $\delta_o(S,\tilde{S})$ are sufficiently small, then we may assume that

$$\varphi(\delta_o,\delta,r) := \|\tilde{P}\| (\delta + \delta_o(1+\delta)(1+r^2)^{1/2} \|P\|) < 1. \qquad (2.2)$$

We shall show that if (2.2) is fulfilled, then $\tilde{Y} = R_o(\tilde{S}) + \tilde{P}M$.

3^o Let $\tilde{y} \in \tilde{Y}$ be arbitrary. If $\tilde{y} \neq 0$, then by the choice of δ in 2^o, there exists $y \in Y$ such that $\|\tilde{y} - y\| < \delta \|\tilde{y}\|$, and hence $\|y\| < (1+\delta)\|\tilde{y}\|$. In the trivial case $\tilde{y} = 0$ we take $y = 0$. One has to proceed similarly in the following estimates, where we shall constantly omit the trivial cases.

4^o We can write $y = u_1 + v_1$, where $u_1 \in R_o(S)$ and $v_1 \in M$. Then there exists $\xi_1 \in D(S)$ such that $u_1 + Y_o = S\xi_1$. Moreover, the vector ξ_1 can be chosen such that $\|\xi_1\| < r\|S\xi_1\|$. Let $x_1 \in \xi_1$ be such that $\|x_1\| < r\|S\xi_1\|$. Since $u_1 = Py$, and hence $\|u_1\| < (1+\delta)\|P\|\,\|\tilde{y}\|$ by 3^o, we have

$$\|x_1\| < r\|u_1\| < r(1+\delta)\|P\|\,\|\tilde{y}\| .$$

5^o Notice that $x_1 \oplus u_1 \in G_o(S)$. Therefore, by the choice of δ_o, there exists an element $\tilde{x}_1 \oplus \tilde{u}_1 \in G_o(\tilde{S})$ such that

$$\|x_1 - \tilde{x}_1\|^2 + \|u_1 - \tilde{u}_1\|^2 < \delta_o^2 (\|x_1\|^2 + \|u_1\|^2) \leq$$
$$\leq \delta_o^2 (1+\delta)^2 (1+r^2)\|P\|^2 \|\tilde{y}\|^2,$$

by the estimates of $\|x_1\|$ and $\|u_1\|$ from 4^o. In particular,

$$\|x_1 - \tilde{x}_1\| < \delta_o (1+\delta)(1+r^2)^{1/2} \|P\|\,\|\tilde{y}\|$$

and

$$\|u_1 - \tilde{u}_1\| < \delta_o (1+\delta)(1+r^2)^{1/2} \|P\|\,\|\tilde{y}\|.$$

6^o Let us consider the vector $\tilde{y}_1 = \tilde{y} - \tilde{u}_1 - \tilde{P}v_1 \in \tilde{Y}$, where $v_1 = y - u_1 \in M$ (see 4^o). One has

$$\|\tilde{y}_1\| = \|\tilde{P}(\tilde{y} - \tilde{u}_1 - v_1)\| \leq$$
$$\leq \|\tilde{P}\|(\|\tilde{y} - y\| + \|\tilde{u}_1 - u_1\|) \leq \varphi(\delta_o, \delta, r)\|\tilde{y}\|,$$

with $\varphi(\delta_o,\delta,r)$ as in (2.2), where we have used the estimates of $\|\tilde{y}-y\|$ and $\|\tilde{u}_1-u_1\|$ from 3^o and 5^o, respectively. Since $v_1=(1-P)y$, we also have

$$\|\tilde{x}_1\| \leqq \|\tilde{x}_1-x_1\|+\|x_1\| \leqq \rho_1(\delta_o,\delta,r)\|\tilde{y}\|$$

and

$$\|v_1\| \leqq \rho_2(\delta)\|\tilde{y}\|,$$

where

$$\rho_1(\delta_o,\delta,r) := (1+\delta)(r+\delta_o(1+r^2)^{1/2})\|P\|, \tag{2.3}$$

$$\rho_2(\delta) := (1+\delta)\|1-P\|.$$

(We have used the estimates of $\|y\|$, $\|x_1\|$ and $\|x_1-\tilde{x}_1\|$ from 3^o, 4^o and 5^o, respectively.)

7^o We can now repeat the previous steps, replacing the vector $\tilde{y}$ by the vector $\tilde{y}_1$ defined in 6^o. We obtain a system of vectors $(\tilde{y}_2,\tilde{x}_2,\tilde{u}_2,v_2)$ such that

$$\tilde{y}_2=\tilde{y}_1-\tilde{u}_2-\tilde{P}v_2=\tilde{y}-\tilde{u}_1-\tilde{u}_2-\tilde{P}(v_1+v_2),$$

$$\|\tilde{y}_2\| \leqq \varphi\|\tilde{y}_1\| \leqq \varphi^2\|\tilde{y}\|,$$

$$\|\tilde{x}_2\| \leqq \rho_1\|\tilde{y}_1\| \leqq \rho_1\varphi\|\tilde{y}\|,$$

$$\|v_2\| \leqq \rho_2\|\tilde{y}_1\| \leqq \rho_2\varphi\|\tilde{y}\|$$

and $\tilde{u}_2 \in \tilde{S}(\tilde{x}_2+\tilde{x}_o)$, where $\varphi := \varphi(\delta_o,\delta,r)$, $\rho_1 := \rho_1(\delta_o,\delta,r)$ and $\rho_2 := \rho_2(\delta)$. In this manner, we obtain by induction the sequences $\{\tilde{y}_n\}_n \subset \tilde{Y}$, $\{\tilde{x}_n\}_n \subset \tilde{X}$, $\{\tilde{u}_n\}_n \subset \tilde{Y}$ and $\{v_n\}_n \subset M$ such that $\tilde{x}_n \oplus \tilde{u}_n \in G_o(\tilde{S})$,

$$\tilde{y}_n=\tilde{y}-(\tilde{u}_1+\ldots+\tilde{u}_n)-\tilde{P}(v_1+\ldots+v_n),$$

$$\|\tilde{y}_n\| \leqq \varphi^n\|\tilde{y}\|,$$

$$\|\tilde{x}_n\| \leqq \rho_1\varphi^{n-1}\|\tilde{y}\|, \tag{2.4}$$

$$\|v_n\| \leqq \rho_2\varphi^{n-1}\|\tilde{y}\|$$

for all integers $n \geqq 1$. From (2.4) and (2.2) it follows that the series Σx_n and Σv_n are convergent in $\tilde{X}$ and M to certain vectors $\tilde{x}$ and v, respectively. Since $\tilde{y}_n \to 0$ as $n \to \infty$ and $\tilde{P}$ is continuous, the series $\Sigma \tilde{u}_n$ is also convergent. Hence the series $\Sigma \tilde{S}(\tilde{x}_n + \tilde{X}_o)$ is convergent and its sum must be equal to $\tilde{S}(\tilde{x} + \tilde{X}_o)$, because $\tilde{S}$ is a closed operator. Moreover, if $\tilde{u}$ is the sum of the series $\Sigma \tilde{u}_n$, then $\tilde{x} \oplus \tilde{u} \in G_o(\tilde{S})$. Consequently, $\tilde{y} = \tilde{u} + \tilde{P}v$, where $\tilde{u} \in R_o(\tilde{S})$ and $v \in M$.

8^o We only note that the space $(\tilde{Y}/\tilde{Y}_o)/R(\tilde{S})$ is (algebraically) isomorphic to the space $\tilde{Y}/R_o(\tilde{S})$ and that $\dim \tilde{Y}/R_o(\tilde{S}) \leqq \dim \tilde{P}M \leqq \dim M = m$. In particular, $R(\tilde{S})$ is closed. This completes the proof of the theorem.

2.3. COROLLARY. __With the notation of Theorem__ 2.2, __let__ $\tilde{S}_1 \in \mathcal{C}((\tilde{X}/\tilde{X}_o) \oplus M, \tilde{Y}/\tilde{Y}_o)$ __be given by the equation__ $\tilde{S}_1(\xi \oplus v) = \tilde{S}\xi + \tilde{A}v$, __where__ $\tilde{A}v = \tilde{P}v + \tilde{Y}_o$, __for all__ $\xi \in D(\tilde{S})$ __and__ $v \in M$. __Assume that__

$$\varphi_o := \varphi(\delta_o(S,\tilde{S}), \delta(\tilde{Y},Y), \gamma(S)^{-1}) < 1. \qquad (2.5)$$

__Then the operator__ $\tilde{S}_1$ __is surjective and one has__

$$\gamma(\tilde{S}_1)^{-1} \leqq (1-\varphi_o)^{-1}(\rho_{10}^2 + \rho_{20}^2)^{1/2}, \qquad (2.6)$$

__where__

$$\rho_{10} := \rho_1(\delta_o(S,\tilde{S}), \delta(\tilde{Y},Y), \gamma(S)^{-1}),$$

$$\rho_{20} := \rho_2(\delta(\tilde{Y},Y)).$$

__Proof.__ If (2.5) is fulfilled, then we can make an appropriate choice of the numbers δ_o, δ and r such that (2.2) is also fulfilled. Then we have $\tilde{Y} = R_o(\tilde{S}) + \tilde{P}M$, and hence $\tilde{Y}/\tilde{Y}_o = R(\tilde{S}) + (\tilde{P}M + \tilde{Y}_o)/\tilde{Y}_o$, so that the operator $\tilde{S}_1$

is surjective. Let $\tilde{\eta} \in \tilde{Y}/\tilde{Y}_o$ and $\varepsilon > 0$ be arbitrary. Choose $\tilde{y} \in \tilde{\eta}$ such that $\|\tilde{y}\| \leqq (1+\varepsilon)\|\tilde{\eta}\|$. From the proof of Theorem 2.2 it follows that there exists a triple $(\tilde{x}, \tilde{u}, v)$ such that $\tilde{y} = \tilde{u} + \tilde{P}v$ and $\tilde{x} \oplus \tilde{u} \in G_o(\tilde{S})$. From (2.4) we obtain the estimates

$$\|\tilde{x}\| \leqq \sum_{n=1}^{\infty} \|\tilde{x}_n\| \leqq (1-\varphi)^{-1} \rho_1 \|\tilde{y}\|,$$

$$\|v\| \leqq \sum_{n=1}^{\infty} \|v_n\| \leqq (1-\varphi)^{-1} \rho_2 \|\tilde{y}\|.$$

If $\tilde{\xi} = \tilde{x} + \tilde{Y}_o$, then we can write

$$\|\tilde{\xi} \oplus v\| \leqq \|\tilde{x} \oplus v\| \leqq (1-\varphi)^{-1} (\rho_1^2 + \rho_2^2)^{1/2} (1+\varepsilon) \|\tilde{\eta}\|,$$

and $\tilde{S}_1(\tilde{\xi} \oplus v) = \tilde{\eta}$. Therefore

$$\gamma(\tilde{S}_1)^{-1} \leqq (1-\varphi)^{-1} (\rho_1^2 + \rho_2^2)^{1/2} (1+\varepsilon).$$

Letting $\delta_o \to \delta_o(S, \tilde{S})$, $\delta \to \delta(\tilde{Y}, Y)$, $r \to \gamma(S)^{-1}$ and $\varepsilon \to 0$, from the last inequality we derive (2.6).

2.4. Remarks. 1^o If the operator S from Theorem 2.2 is surjective, then the projections P and $\tilde{P}$ are identities and $M = 0$ (see the step 1^o of the proof of Theorem 2.2). Then the operator $\tilde{S}_1$ from Corollary 2.3 is equal to $\tilde{S}$. Note also, that in this case we have

$$\varphi(\delta_o, \delta, r) = \delta + \delta_o(1+\delta)(1+r^2)^{1/2},$$

$$\rho_1(\delta_o, \delta, r) = (1+\delta)(r + \delta_o(1+r^2)^{1/2}),$$

$$\rho_2(\delta) = 0,$$

which are derived from (2.2) and (2.3).

2^o In the case of Hilbert spaces it is not necessary to consider quotient spaces. We have, in general, $\|P\| = \|\tilde{P}\| = \|1 - P\| = 1$ both in (2.2) and (2.3).

2.5. COROLLARY. *Let* $X, Y \in \mathcal{S}(\mathfrak{X})$ *be such that* $X \subset Y$ *and* $m := \dim Y/X < \infty$. *Let also* $\tilde{X}, \tilde{Y} \in \mathcal{S}(\mathfrak{X})$ *be such that* $\tilde{X} \subset \tilde{Y}$. *If*

$$(1+m)(\delta(\tilde{Y}, Y) + \sqrt{2}(1+m)\delta(X, \tilde{X})(1+\delta(\tilde{Y}, Y))) < 1,$$

then $\dim \tilde{Y}/\tilde{X} \leq m$.

Proof. We apply Theorem 2.2 to the canonical inclusions $S : X \to Y$ and $\tilde{S} : \tilde{X} \to \tilde{Y}$. We have $\delta(S, \tilde{S}) \leq \delta(X, \tilde{X})$, $\gamma(S) = 1$, $\|P\| \leq 1 + m + \varepsilon$ and $\|\tilde{P}\| \leq 1 + m + \varepsilon$, where $\varepsilon > 0$ is as small as we desire. Then the condition from the statement implies (2.5), whence we derive the conclusion.

2.6. LEMMA. *Let* S *and* $\tilde{S}$ *be as in Definition* 2.1. *If* S *is densely defined, then* $G_o(S^*) = G_o'(-S)^{\perp}$, *where* $G_o'(-S)$ *is the set* $G_o(-S)$ *regarded as a subset of* $\mathfrak{Y} \oplus \mathfrak{X}$. *We also have* $N_o(S^*) = R_o(S)^{\perp}$ *and* $R_o(S^*) \subset N_o(S)^{\perp}$. *When* $R(S)$ *is closed, then the last inclusion is an equality. If* $\tilde{S}$ *is also densely defined, then* $\delta_o(\tilde{S}^*, S^*) = \delta_o(S, \tilde{S})$.

Proof. Notice that $S^* \in \mathcal{C}(Y_o^{\perp}/Y^{\perp}, X_o^{\perp}/X^{\perp})$, by the natural identification of $(X/X_o)^*$ with $X_o^{\perp}/X^{\perp}$ ($X_o^{\perp}$ and $X^{\perp}$ computed in $\mathfrak{X}^*$) and a similar identification of $(Y/Y_o)^*$ with $Y_o^{\perp}/Y^{\perp}$.

If $y \oplus x \in \mathfrak{Y} \oplus \mathfrak{X}$ and if $\eta = y + Y_o$, $\xi = x + X_o$, then we have $y \oplus x \in G_o'(-S)$ if and only if $\eta \oplus \xi \in G'(-S)$ (where $G'(-S)$ is $G(-S)$ with the changed order). Similarly, if $g \oplus f \in \mathfrak{Y}^* \oplus \mathfrak{X}^*$ and if $G = g + Y^{\perp}$, $F = f + X^{\perp}$, we have $g \oplus f \in G_o(S^*)$ if and only if $G \oplus F \in G(S^*)$. Since we have the following duality relation

$$\langle G \oplus F, \eta \oplus \xi\rangle = \langle G, \eta\rangle + \langle F, \xi\rangle = \langle g, y\rangle + \langle f, x\rangle = \langle g \oplus f, y \oplus x\rangle$$

for all $x \in X$, $y \in Y$, $f \in X_o^{\perp}$, $g \in Y_o^{\perp}$, and $G(S^*) = G'(-S)^{\perp}$ (see [8], III.5.5), then we easily derive that $G_o(S^*) = G_o'(-S)^{\perp}$.

Using the relations $N(S^*) = R(S)^{\perp}$ and $R(S^*) \subset N(S)^{\perp}$, as well as the fact that the last inclusion is an equality when $R(S)$ is closed (see [8], Theorem IV.5.13), we obtain readily the assertions concerning $N_o(S^*)$ and $R_o(S^*)$ from the statement.

Now, let us observe that

$$\delta_o(\tilde{S}^*, S^*) = \delta(G_o(\tilde{S}^*), G_o(S^*)) =$$
$$= \delta(G_o(\tilde{S})^{\perp}, G_o(S)^{\perp}) = \delta_o(S, \tilde{S}),$$

by the first part of the proof and Theorem IV.2.9 from [8].

We now give a dual version of Theorem 2.2.

2.7. THEOREM. <u>Let</u> S <u>and</u> $\tilde{S}$ <u>be as in Definition</u> 2.1. <u>If</u> $S \in \Phi_+(X/X_o, Y/Y_o)$ <u>and the numbers</u> $\delta(X_o, \tilde{X}_o)$ <u>and</u> $\delta_o(\tilde{S}, S)$ <u>are sufficiently small, then</u> $\tilde{S} \in \Phi_+(\tilde{X}/\tilde{X}_o, \tilde{Y}/\tilde{Y}_o)$ <u>and</u> $\dim N(\tilde{S}) \leq \dim N(S)$.

<u>Proof</u>. With no loss of generality we may assume that both S and $\tilde{S}$ are densely defined, and thus S^* and $\tilde{S}^*$ exist. Let us check that the conditions of Theorem 2.2 are fulfilled for S^* and $\tilde{S}^*$. We have $S^* \in \Phi_-(Y_o^{\perp}/Y^{\perp}, X_o^{\perp}/X^{\perp})$ in virtue of [8], Theorem IV.5.13. Notice also that $\delta(\tilde{X}_o^{\perp}, X_o^{\perp}) = \delta(X_o, \tilde{X}_o)$ and that $\delta_o(S^*, \tilde{S}^*) = \delta_o(\tilde{S}, S)$ by Lemma 2.6. Consequently, if $\delta(X_o, \tilde{X}_o)$ and $\delta_o(\tilde{S}, S)$ are sufficiently small, then, according to Theorem 2.2, $\tilde{S}^* \in \Phi_-(\tilde{Y}_o^{\perp}/\tilde{Y}, \tilde{X}_o^{\perp}/\tilde{X}^{\perp})$ and

$$\dim(\tilde{X}_o^{\perp}/\tilde{X}^{\perp})/R(\tilde{S}^*) \leq \dim(X_o^{\perp}/X^{\perp})/R(S^*),$$

which implies that $\tilde{S} \in \Phi_+(\tilde{X}/\tilde{X}_o, \tilde{Y}/\tilde{Y}_o)$ and that $\dim N(\tilde{S}) \leq \dim N(S)$.

2.8. <u>Remarks</u>. 1° Let us consider the function

$$\varphi_*(\delta_o, \delta, r) := \|\tilde{P}_*\|(\delta + \delta_o(1+\delta)(1+r^2)^{1/2}\|P_*\|), \quad (2.7)$$

where P_* and $\tilde{P}_*$ correspond respectively to the projections

P and $\widetilde{P}$ in (2.2), when S and $\widetilde{S}$ are replaced by S* and $\widetilde{S}$*. If

$$\varphi_*(\delta_o(\widetilde{S},S),\delta(X_o,\widetilde{X}_o),\gamma(S)^{-1}) < 1, \tag{2.8}$$

then the conclusion of the previous theorem holds. Indeed, in this case we can apply Theorem 2.2 to S* and $\widetilde{S}$* (note that $\gamma(S) = \gamma(S^*)$, by Theorem IV.5.13 from [8]).

Note also that the functions (2.3) can be written in this case in the following way:

$$\rho_{1*}(\delta_o,\delta,r) := (1+\delta)(r+\delta_o(1+r^2)^{1/2}\|P_*\|,$$
$$\rho_{2*}(\delta) := (1+\delta)\|1-P_*\|. \tag{2.9}$$

2^o If the operator S is injective (and hence S* is surjective), then the functions φ_* and ρ_{1*} (from (2.7) and (2.9) resp.) are equal to the function φ and ρ_1 as given by Remark 2.4.1^o, while $\rho_{2*} = 0$.

2.9. LEMMA. Let S and $\widetilde{S}$ be as in Definition 2.1, and let R(S) be closed. Then we have the estimates

$$\delta(R_o(S),R_o(\widetilde{S})) \leq (1+\gamma(S)^{-2})^{1/2}\,\delta_o(S,\widetilde{S}),$$
$$\delta(N_o(\widetilde{S}),N_o(S)) \leq (1+\gamma(S)^{-2})^{1/2}\,\delta_o(\widetilde{S},S).$$

Proof. Let $\delta > \delta_o(S,\widetilde{S})$ and $r > \gamma(S)^{-1}$ be fixed. Let also $y \in R_o(S)$. Then $y + Y_o = S\xi$, and we can choose $\xi \in D(S)$ such that $\|\xi\| < r\|S\xi\| \leq r\|y\|$. Let also $x \in \xi$ be such that $\|x\| < r\|y\|$. Since $x \oplus y \in G_o(S)$, we can find an element $\widetilde{x} \oplus \widetilde{y} \in G_o(\widetilde{S})$ such that

$$\|x-\widetilde{x}\|^2 + \|y-\widetilde{y}\|^2 < \delta^2(\|x\|^2+\|y\|^2) < \delta^2(1+r^2)\|y\|^2.$$

Therefore

$$\delta(R_o(S),R_o(\widetilde{S})) \leq \delta(1+r^2)^{1/2}.$$

Letting $\delta \to \delta_o(S,\tilde{S})$ and $r \to \gamma(S)^{-1}$, we obtain the first estimate of the statement.

Let us prove the second estimate. We assume, with no loss of generality, that both S and $\tilde{S}$ are densely defined. Since R(S) is closed, then $N_o(S)^{\perp} = R_o(S^*)$ and $R_o(\tilde{S}^*) \subset N_o(\tilde{S})^{\perp}$, by virtue of Lemma 2.6.

Hence

$$\delta(N_o(\tilde{S}),N_o(S)) = \delta(N_o(S)^{\perp},N_o(\tilde{S})^{\perp}) \leq$$
$$\leq \delta(R_o(S^*),\overline{R_o(\tilde{S}^*)}) \leq (1+\gamma(S^*)^{-2})^{1/2}\delta_o(S^*,\tilde{S}^*) =$$
$$= (1+\gamma(S)^{-2})^{1/2}\delta_o(\tilde{S},S),$$

by the first part of the proof and Lemma 2.6.

For the next statements we assume that the Banach subspaces $X_o,\tilde{X}_o,X,\tilde{X}$ are in $\mathcal{S}(\mathfrak{X})$, $Y_o,\tilde{Y}_o,Y,\tilde{Y}$ are in $\mathcal{S}(\mathfrak{Y})$ and $Z_o,\tilde{Z}_o,Z,\tilde{Z}$ are in $\mathcal{S}(\mathfrak{Z})$.

2.10. PROPOSITION. Let $S \in \mathcal{C}(X/X_o,Y/Y_o)$ and $T \in \mathcal{C}(Y/Y_o,Z/Z_o)$ be such that $R(S) \subset N(T)$, $\dim N(T)/R(S) < \infty$ and R(T) is closed. Let also $\tilde{S} \in \mathcal{C}(\tilde{X}/\tilde{X}_o,\tilde{Y}/\tilde{Y}_o)$ and $\tilde{T} \in \mathcal{C}(\tilde{Y}/\tilde{Y}_o,\tilde{Z}/\tilde{Z}_o)$ be such that $R(\tilde{S}) \subset N(\tilde{T})$. If the numbers $\delta_o(S,\tilde{S})$ and $\delta_o(\tilde{T},T)$ are sufficiently small, then

$$\dim N(\tilde{T})/R(\tilde{S}) \leq \dim N(T)/R(S)$$

and $R(\tilde{T})$ is closed.

Proof. Note that $S \in \Phi_-(X/X_o,N_o(T)/Y_o)$. From Lemma 2.9, it follows that $\delta(N_o(\tilde{T}),N_o(T))$ is as small as we desire if $\delta_o(\tilde{T},T)$ is sufficiently small. If, in addition, $\delta_o(S,\tilde{S})$ is sufficiently small, then $\tilde{S} \in \Phi_-(\tilde{X}/\tilde{X}_o,N_o(\tilde{T})/\tilde{Y}_o)$ and

$$\dim N(\tilde{T})/R(\tilde{S}) \leq \dim N(T)/R(S),$$

by virtue of Theorem 2.2.

We have only to prove that $R(\tilde{T})$ is closed. Let $T_o \in \mathcal{C}(Y/R_o(S), Z/Z_o)$ be the operator induced by T (note that $(Y/Y_o)/R(T)$ is isomorphic to $Y/R_o(S)$). Then $T_o \in \Phi_+(Y/R_o(S), Z/Z_o)$. Since $R(\tilde{S})$ (and hence $R_o(\tilde{S})$) may be supposed to be closed by the first part of the proof, we can also consider the operator $\tilde{T}_o \in \mathcal{C}(\tilde{Y}/R_o(\tilde{S}), \tilde{Z}/\tilde{Z}_o)$, induced by $\tilde{T}$ in a similar way. Then $\tilde{T}_o \in \Phi_+(\tilde{Y}/R_o(\tilde{S}), \tilde{Z}/\tilde{Z}_o)$ by Theorem 2.7, provided that $\delta(R_o(S), R_o(\tilde{S}))$ and $\delta_o(\tilde{T}_o, T_o)$ are sufficiently small. Since $\delta(R_o(S), R_o(\tilde{S}))$ is as small as we want if $\delta_o(S,\tilde{S})$ is sufficiently small (Lemma 2.9) and $\delta_o(\tilde{T}_o, T_o) = \delta_o(\tilde{T}, T)$ (note that $G_o(T_o) = G_o(T)$ and that $G_o(\tilde{T}_o) = G_o(\tilde{T})$), it follows that $R(\tilde{T}) = R(\tilde{T}_o)$ is closed if the conditions of our proposition are fulfilled.

2.11. DEFINITION. A pair of operators (S,T) as in the statement of Proposition 2.10 will be called semi-Fredholm.

Then Proposition 2.10 is a stability result for semi-Fredholm pairs of operators.

2.12. COROLLARY. With the conditions of Proposition 2.10, suppose that $R(S) = N(T)$. If

$$\delta_o \geq \max\{\delta_o(S,\tilde{S}), \delta_o(\tilde{T},T)\},$$

$$\delta \geq \max\{(1+\gamma(S)^{-2})^{1/2}\delta_o(S,\tilde{S}), (1+\gamma(T)^{-2})^{1/2}\delta_o(\tilde{T},T)\},$$

$$r \geq \max\{\gamma(S)^{-1}, \gamma(T)^{-1}\},$$

and if $\varphi(\delta_o,\delta,r) < 1$, then $R(\tilde{S}) = N(\tilde{T})$ and $R(\tilde{T})$ is closed. Moreover,

$$\max\{\gamma(\tilde{S})^{-1}, \gamma(\tilde{T})^{-1}\} \leq (1-\varphi(\delta_o,\delta,r))^{-1}\rho_1(\delta_o,\delta,r), \quad (2.10)$$

where φ and ρ_1 are as in Remark 2.4.1°.

Proof. We follow the lines of the proof of Proposition 2.10. Since $N(T) = R(S)$, then the operator $S \in \Phi_-(X/X_o, N_o(T)/Y_o)$ is surjective and we may apply Remark 2.4.1°. If $\delta_o \geq \delta_o(S,\tilde{S})$,

$$\delta \geq (1+\gamma(T)^{-2})^{1/2}\,\delta_o(\tilde{T},T) \geq \delta(N_o(\tilde{T}),N_o(T))$$

(Lemma 2.9) and $r \geq \gamma(S)^{-1}$, and if $\varphi(\delta_o,\delta,r) < 1$, then, according to Corollary 2.3, we obtain that $R(\tilde{S}) = N(\tilde{T})$ and that

$$\gamma(\tilde{S})^{-1} \leq (1-\varphi(\delta_o,\delta,r))^{-1}\,\rho_1(\delta_o,\delta,r).$$

The equality $R(S) = N(T)$ also implies that the operator $T_o \in \mathcal{C}(Y/R_o(S), Z/Z_o)$ is injective. Therefore we may use Remark 2.8.2°. If $\delta_o \geq \delta_o(\tilde{T}_o,T_o) = \delta_o(\tilde{T},T)$,

$$\delta \geq (1+\gamma(S)^{-2})^{1/2}\,\delta_o(S,\tilde{S}) \geq \delta(R_o(S),R_o(\tilde{S}))$$

(Lemma 2.9) and $r \geq \gamma(T_o)^{-1} = \gamma(T)^{-1}$ (both quantities are defined in the same way), and if $\varphi(\delta_o,\delta,r) < 1$, then $\tilde{T}_o$ is also injective and

$$\gamma(\tilde{T}_o)^{-1} = \gamma(\tilde{T})^{-1} \leq (1-\varphi(\delta_o,\delta,r))^{-1}\,\rho_1(\delta_o,\delta,r)$$

by Corollary 2.3, via Remarks 2.8.1° and 2.8.2°. Consequently, if the conditions of our corollary are fulfilled, then $R(\tilde{T})$ is closed and (2.10) holds.

2.13. Remark. A particular case of Proposition 2.10 (obtained in a different way) can be found in [5]. In the fourth section (Proposition 4.2) we shall show that Proposition 2.10 implies another type of semi-continuity of the dimension.

We need a certain version of Corollary 2.5 which does not involve any dimension. We start with some auxiliary results.

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2.14. THEOREM. Let Y be a Banach space, let $X \in \mathcal{S}(Y)$ and let $\pi : Y \to Y/X$ be the canonical mapping. Then there exists a continuous homogeneous map $\rho : Y/X \to Y$ such that $\pi\rho(y+X) = y+X$ for all $y \in Y$. Moreover, for every $\varepsilon > 0$ we can choose ρ such that $\|\rho(y+X)\| \leqq (1+\varepsilon)\|y+X\|$.

This result has been proved in [2] (see also [22]).

2.15. Remark. With the conditions of the previous theorem, we have the decomposition $Y = X + M$, where $M = \rho(Y/X)$ is not necessarily a linear space. Indeed, for every $y \in Y$ we have that $x = y - \rho(y+X) \in X$, since $\pi(x) = 0$. Hence $y = x + v$, with $v = \rho(y+X)$. Notice that this decomposition is unique. Indeed, if $y = x_1 + v_1 = x_2 + v_2$, where $x_1, x_2 \in X$ and $v_1 = \rho(\eta_1), v_2 = \rho(\eta_2) \in M$, then $v_1 - v_2 \in X$, and thus $0 = \pi(\rho(\eta_1) - \rho(\eta_2)) = \eta_1 - \eta_2$, so that $v_1 = v_2$ and $x_1 = x_2$. In this way we can define a non-linear projection P of Y onto X by the equation $Py = y - \rho(y+X)$ for all $y \in Y$. Note that

1) $\|Py\| = \|y - \rho(y+Y)\| \leqq (2+\varepsilon)\|y\|$.

We also have

2) $P(Py) = Py$,

3) $P(u+y) = u + Py$

for all $u \in X$ and $y \in Y$, by the uniqueness of the decomposition $Y = X + M$.

2.16. LEMMA. Let $X, Y \in \mathcal{S}(\mathfrak{X})$ be such that $X \subset Y$ and $\dim Y/X < \infty$. Let also $\tilde{X}, \tilde{Y} \in \mathcal{S}(\mathfrak{X})$ be such that $\tilde{X} \subset \tilde{Y}$. If

$$2(\delta(\tilde{Y}, Y) + 2\delta(X, \tilde{X})(1 + \delta(\tilde{Y}, Y))) < 1 ,$$

then $\dim \tilde{Y}/\tilde{X} < \infty$.

Proof. Let $\rho : Y/X \to Y$ be as in Theorem 2.14 and let M and P be as in Remark 2.15. We shall follow the lines of the proof of Theorem 2.2, with some changes due to the non-linearity of the method.

Let $\delta > \delta(\tilde{Y}, Y)$ and $\delta_o > \delta(X, \tilde{X})$ be fixed. Let also $\tilde{y} \in \tilde{Y}$. Then there exists $y \in Y$ such that $\|\tilde{y} - y\| < \delta \|\tilde{y}\|$. We have the decomposition $y = x_1 + v_1$, where $x_1 \in X$ and $v_1 = \rho(y + Y) \in M$. Notice that

$$\|x_1\| = \|Py\| \leqq (2+\varepsilon)\|y\| \leqq (2+\varepsilon)(1+\delta)\|\tilde{y}\|,$$

$$\|v_1\| \leqq (1+\varepsilon)\|y\| \leqq (1+\varepsilon)(1+\delta)\|\tilde{y}\|.$$

Let us choose $\tilde{x}_1 \in \tilde{X}$ such that $\|x_1 - \tilde{x}_1\| < \delta_o \|x_1\|$. Hence

$$\|x_1 - \tilde{x}_1\| < \delta_o (2+\varepsilon)(1+\delta)\|\tilde{y}\|.$$

We now consider a mapping

$$\tilde{\rho} : \overline{\tilde{Y} + \operatorname{sp} M} / \tilde{Y} \to \overline{\tilde{Y} + \operatorname{sp} M}$$

as in Theorem 2.14, where sp M is the linear space generated by M. Let also $\tilde{P}$ be the non-linear projection of $\overline{\tilde{Y} + \operatorname{sp} M}$ onto $\tilde{Y}$ that is associated to $\tilde{\rho}$ as in Remark 2.15. Then we define the elements $\tilde{v}_1 = v_1 - \tilde{\rho}(v_1 + \tilde{Y}) \in \tilde{Y}$ and $\tilde{y}_1 = \tilde{y} - \tilde{x}_1 - \tilde{v}_1$. Let us note that, by the properties of $\tilde{P}$ (Remark 2.15), we have

$$\|\tilde{y}_1\| = \|\tilde{y} - \tilde{x}_1 - \tilde{P}v_1\| = \|\tilde{P}(\tilde{y} - \tilde{x}_1 - v_1)\| \leqq$$

$$\leqq (2+\varepsilon)\|\tilde{y} - \tilde{x} - v_1\| \leqq (2+\varepsilon)(\|\tilde{y} - y\| + \|\tilde{x}_1 - x_1\|) \leqq$$

$$\leqq (2+\varepsilon)(\delta + \delta_o(2+\varepsilon)(1+\delta))\|\tilde{y}\|.$$

The condition from the statement implies that we can choose the numbers $\varepsilon > 0$, $\delta > \delta(\tilde{Y}, Y)$ and $\delta_o > \delta(X, \tilde{X})$ such that

$$q := (2+\varepsilon)(\delta + \delta_o(2+\varepsilon)(1+\delta)) < 1.$$

Consequently, as in the proof of Theorem 2.2, we may continue this approximation procedure. We find the sequence $\{\tilde{y}_n\}_n \subset \tilde{Y}$, $\{\tilde{x}_n\}_n \subset \tilde{X}$ and $\{v_n\}_n \subset M$ such that

$$\tilde{y}_n = \tilde{y} - (\tilde{x}_1 + \ldots + \tilde{x}_n) - (\tilde{P}v_1 + \ldots + \tilde{P}v_n),$$

$$\|\tilde{y}_n\| \leqq q^n \|\tilde{y}\|,$$

$$\|\tilde{x}_n\| \leqq r_1 q^{n-1} \|\tilde{y}\|,$$

$$\|v_n\| \leqq r_2 q^{n-1} \|\tilde{y}\|$$

for all integers $n \geqq 1$, where $r_1 = (2+\varepsilon)(1+\delta)$, $r_2 = (1+\varepsilon)(1+\delta)$.

We also have

$$\|\tilde{P}v_n\| \leqq (2+\varepsilon) r_2 q^{n-1} \|\tilde{y}\|,$$

by the properties of $\tilde{P}$ from Remark 2.15. Therefore the series $\Sigma\tilde{x}_n$ and $\Sigma\tilde{P}v_n$ are convergent to certain vectors $\tilde{x} \in \tilde{X}$ and $\tilde{v} \in \tilde{Y}$, respectively. Thus we have the equality $\tilde{y} = \tilde{x} + \tilde{v}$. We shall use this representation of an arbitrary element $\tilde{y} \in \tilde{Y}$ to prove the inclusion

$$\{\tilde{y} \in \tilde{Y} \; ; \|\tilde{y}\| \leqq 1\} \subset \{\tilde{x} \in \tilde{X} : \|\tilde{x}\| \leqq r_1\} + \tilde{K} \tag{2.11}$$

where $\tilde{K} \subset \tilde{Y}$ is a compact set. Let us observe that if $\|\tilde{y}\| \leqq 1$, then

$$v_1 = \rho(y + X) \in L := \rho(\{\eta \in Y/X \; ; \; \|\eta\| \leqq 1 + \delta\}),$$

and the set L is compact since Y/X is finite dimensional and ρ is continuous. Similarly, $v_n = \rho(y_{n-1} + X) \in q^{n-1} L$ for all $n \geqq 2$ (where $y_{n-1} \in Y$ and $\|y_{n-1}\| \leq (1+\delta) 2^{n-1}$), since ρ is a homogeneous map. Then we have $\tilde{v}_1 \in \tilde{L} := L + \tilde{\rho}(L + \tilde{Y})$, where $\tilde{L}$ is compact since $L + \tilde{Y}$ is compact and $\tilde{\rho}$ is continuous. Analogously, $\tilde{v}_n \in q^{n-1}\tilde{L}$ on account of the fact that $\tilde{\rho}$ is also homogeneous ($n \geqq 2$). Consequently,

$$\tilde{v} \in \tilde{L} + q\tilde{L} + q^2\tilde{L} + \ldots,$$

and the set $\tilde{K}_1 := \Sigma q^{n-1}\tilde{L}$ is compact (as a range of a compact set by a continuous mapping). Hence $\tilde{v} \in \tilde{K} := \tilde{K}_1 \cap \tilde{Y}$, so that (2.11) holds.

Now, by virtue of (2.11), the unit ball of $\tilde{Y}/\tilde{X}$ is relatively compact which implies $\dim \tilde{Y}/\tilde{X} < \infty$.

2.17. Remarks. 1^o Let us note that the method from the proof of Lemma 2.16 can also be adapted to the proof of Theorem 2.2. In such a case, one obtains an estimate of the type (2.5) which does not depend any longer on $\dim W/R(S)$. However, the conclusion of the theorem is, in this case, poorer. Namely, one only obtains that $\dim \tilde{W}/R(\tilde{S}) < \infty$.

2^o Recently, it has been shown that if in Theorem 2.14 one has $m := \dim Y/X < \infty$, then the map ρ can be chosen such that $\rho(Y/X)$ lies in a finite dimensional space, whose dimension depends, in general, both on ε and m. This yields a certain simplification of the proof of Lemma 2.16 and gives an estimate for $\dim \tilde{Y}/\tilde{X}$, [18].

2.18. PROPOSITION. Let $S \in \Phi_+(X/X_o, Y/Y_o)$ and $T \in \Phi_-(Y/Y_o, Z/Z_o)$ be such that $R(S) \subset N(T)$ and $\dim N(T)/R(S) = \infty$. If $\tilde{S} \in \mathcal{C}(\tilde{X}/\tilde{X}_o, \tilde{Y}/\tilde{Y}_o)$ and $\tilde{T} \in \mathcal{C}(\tilde{Y}/\tilde{Y}_o, \tilde{Z}/\tilde{Z}_o)$ are such that $R(\tilde{S}) \subset N(\tilde{T})$, and if the numbers $\delta(X_o, \tilde{X}_o)$, $\delta(\tilde{Z}, Z)$, $\delta_o(\tilde{S}, S)$ and $\delta_o(T, \tilde{T})$ are sufficiently small, then $R(\tilde{S})$ and $R(\tilde{T})$ are closed and $\dim N(\tilde{T})/R(\tilde{S}) = \infty$.

Proof. With no loss of generality we may assume that all involved operators are densely defined. According to Corollary 2.3, if $\delta(\tilde{Z}, Z)$ and $\delta_o(T, \tilde{T})$ satisfy (2.5) (written for this case), then there exists a Banach space M with $\dim M = m$, where $m := \dim (Z/Z_o)/R(T)$, and a surjective extension $\tilde{T}_1 \in \mathbb{C}((\tilde{X}/\tilde{X}_o) \oplus M, \tilde{Z}/\tilde{Z}_o)$ of $\tilde{T}$ such that the assignment $\tilde{T} \to \gamma(\tilde{T}_1)^{-1}$ is bounded when $\delta(\tilde{Z},Z)$ and $\delta_o(T,\tilde{T})$ tend to zero, as follows from (2.6). Let T_o be the (trivial) extension of T to $D(T) \oplus O \subset (Y/Y_o) \oplus M$. Then we have $\delta_o(T_o,\tilde{T}_1) \leqq \delta_o(T,\tilde{T})$, since $\tilde{T}_1$ extends $\tilde{T}$. Hence

$$\delta(R_o(\tilde{T}_1^*), R_o(T^*) \oplus M^*) = \delta(N_o(T_o), N_o(\tilde{T}_1)) \leqq$$
$$\leqq (1 + \gamma(\tilde{T}_1)^{-2})^{1/2} \ \delta_o(T,\tilde{T}),$$

by Lemma 2.9, so that the number $\delta(R_o(\tilde{T}_1^*), R_o(T^*) \oplus M^*)$ can be made as small as we want if $\delta_o(T,\tilde{T})$ is sufficiently small.

Let S_o and $\tilde{S}_o$ be the operators S and $\tilde{S}$ with values in $(Y/Y_o) \oplus M$ and $(\tilde{Y}/\tilde{Y}_o) \oplus M$, respectively (by canonical imbedding). If $\delta(X_o,\tilde{X}_o)$ and $\delta_o(\tilde{S}_o,S_o) = \delta_o(\tilde{S},S)$ are sufficiently small, then there exists a Banach space N such that $\dim N \leqq n$, where $n := \dim N(S)$, and a surjective extension $\tilde{S}_1^*$ of $\tilde{S}_o^*$ to the space $(\tilde{Y}_o^{\perp}/\tilde{Y}^{\perp}) \oplus M^* \oplus N$ such that the assignment $\tilde{S} \to \gamma(\tilde{S}_1^*)^{-1}$ is bounded when $\delta(X_o,\tilde{X}_o)$ and $\delta_o(\tilde{S},S)$ tend to zero. Let S_{oo}^* be the (trivial) extension of S_o^* to $D(S_o^*) \oplus O \subset (Y_o^{\perp}/Y^{\perp}) \oplus M^* \oplus N$. Then we have $\delta_o(S_{oo}^*,\tilde{S}_1^*) \leqq \delta_o(S_o^*,\tilde{S}_o^*) = \delta_o(S^*,\tilde{S}) = \delta_o(\tilde{S},S)$, since $\tilde{S}_1^*$ extends $\tilde{S}_o^*$. Therefore, by Lemma 2.9,

$$\delta(N_o(S_{oo}^*), N_o(\tilde{S}_1^*)) \leqq (1 + \gamma(\tilde{S}_1^*)^{-2})^{1/2} \ \delta_o(\tilde{S},S),$$

so that the number $\delta(N_o(S_{oo}^*), N_o(\tilde{S}_1^*))$ can be made as small as one wants if $\delta_o(\tilde{S},S)$ is sufficiently small.

Now assume that $\dim N(\tilde{T})/R(\tilde{S}) < \infty$ for some pairs $(\tilde{T},\tilde{S})$, when the numbers $\delta(X_o,\tilde{X}_o), \delta(\tilde{Z},Z), \delta_o(\tilde{S},S)$ and $\delta_o(T,\tilde{T})$ tend to zero. We have $\dim N(\tilde{T}_1)/(N(\tilde{T}) \oplus O) \leqq m$ and $\dim N(\tilde{S}_1^*)/(N(\tilde{S}_o^*) \oplus O) \leqq n$, by (2.1). Since $R(\tilde{S}_o) \subset N(\tilde{T}) \oplus O \subset N(\tilde{T}_1)$, it follows that

$$\dim N(\tilde{S}_1^*)/R(\tilde{T}_1^*)\oplus O) = \dim N(\tilde{S}_1^*)/(N(\tilde{S}_o^*)\oplus O) +$$
$$+\dim N(\tilde{S}_o^*)/R(\tilde{T}_1^*) = \dim N(\tilde{S}_1^*)/N(\tilde{S}_o^*)\oplus O) +$$
$$+\dim N(\tilde{T}_1)/(N(\tilde{T})\oplus O) + \dim N(\tilde{T})/R(\tilde{S}) < \infty.$$

Hence,

$$\dim N_o(\tilde{S}_1^*)/(R_o(\tilde{T}_1^*)\oplus O) = \dim N(\tilde{S}_1^*)/(R(\tilde{T}_1^*)\oplus O) < \infty.$$

In particular, we may apply Lemma 2.16 to the subspaces $R_o(\tilde{T}_1^*)\oplus O \subset N_o(\tilde{S}_1^*)$ and $R_o(T^*)\oplus M^*\oplus O \subset N_o(S_{oo}^*) = N(S_o^*)\oplus O$. By the previous remarks, the requirement of Lemma 2.16 is fulfilled when $\delta(X_o,\tilde{X}_o)$, $\delta(\tilde{Z},Z)$, $\delta_o(\tilde{S},S)$ and $\delta_o(T,\tilde{T})$ are sufficiently small. Consequently

$$\dim N(T)/R(S) = \dim N_o(T)/R_o(S) = \dim N_o(T_o)/R_o(S_o) =$$
$$= \dim N_o(S_o^*)/(R_o(T^*)\oplus M^*) < \infty,$$

which is a contradiction.

2.19. LEMMA. <u>Let S and $\tilde{S}$ be as in Definition 2.1, and let A, S_1 and M be as in (2.1). Then there exists an extension $\tilde{S}_1$ of $\tilde{S}$ to $(\tilde{X}/\tilde{X}_o)\oplus M$ such that</u>

$$\delta_o(S_1,\tilde{S}_1) \leq 8(m+1)\max\{\delta_o(S,\tilde{S}),\delta(Y,\tilde{Y})\},$$
$$\delta_o(\tilde{S}_1,S_1) \leqq 16(m+1)^2\max\{\delta_o(\tilde{S},S),\delta(Y,\tilde{Y})\},$$

<u>provided that $\|A\| \leqq 1$, where $m := \dim M$.</u>

<u>Proof</u>. Let $\delta_o > \delta_o(S,\tilde{S})$ and $\delta > \delta(Y,\tilde{Y})$ be fixed. Let also $\{v_1,\dots,v_m\}$ be a basis of M such that $\|v_1\| = \dots = \|v_m\| = 1$ and for each $v = \Sigma\lambda_j v_j \in M$ one has $|\lambda_j| \leqq \|v\|$ for all $j = 1,\dots,m$. The existence of such a basis follows from Auerbach's Lemma [1]. Let $\varepsilon > 0$, and let $y_j \in Av_j$ be such that $\|y_j\| \leqq (1+\varepsilon)\|Av_j\| \leqq 1+\varepsilon$ for all j. We define $A_o \in \mathcal{L}(M,Y)$ by the equation $A_o(\Sigma\lambda_j v_j) = \Sigma\lambda_j y_j$. Note that $\|A_o v\| \leqq m(1+\varepsilon)\|v\|$ for all $v \in M$. Then we choose $\tilde{y}_j \in \tilde{Y}$ such that $\|y_j - \tilde{y}_j\| \leqq \delta\|y_j\| \leqq (1+\varepsilon)\delta$. We also define $\tilde{A}_o \in \mathcal{L}(M,\tilde{Y})$ by the equation $\tilde{A}_o(\Sigma\lambda_j v_j) = \Sigma\lambda_j\tilde{y}_j$.

Let us observe that

$$\|A_o v - \tilde{A}_o v\| \leqq m\delta(1+\varepsilon)\|v\|, \qquad v \in M.$$

If $\tilde{A} \in \mathcal{L}(M, \tilde{Y}/\tilde{Y}_o)$ is given by $\tilde{A}v = \tilde{A}_o v + Y_o$, then we define the extension $\tilde{S}_1$ of $\tilde{S}$ by the equation $\tilde{S}_1(\tilde{\xi} \oplus v) = \tilde{S}\tilde{\xi} + \tilde{A}v$ for all $\tilde{\xi} \in D(\tilde{S})$ and $v \in M$.

Now let $x \oplus v \oplus w \in G_o(S_1)$ be arbitrary. Then we choose $u \in S(x + X_o)$ such that $\|u\| \leqq (1+\varepsilon)\|S(x+X_o)\|$. Since $x \oplus u \in G_o(S)$, we can find an element $\tilde{x} \oplus \tilde{u} \in G_o(\tilde{S})$ such that

$$\|x - \tilde{x}\|^2 + \|u - \tilde{u}\|^2 \leq \delta_o^2(\|x\|^2 + \|u\|^2).$$

Note that $y_o := w - u - A_o v \in Y_o$, so that $0 \oplus y_o \in G_o(S)$. Then there exists $\tilde{x}_o \oplus \tilde{y}_o \in G_o(\tilde{S})$ such that

$$\|\tilde{x}_o\|^2 + \|y_o - \tilde{y}_o\|^2 \leq \delta_o^2 \|y_o\|^2.$$

Let $\tilde{w} := \tilde{y}_o + \tilde{u} + \tilde{A}_o v$. Then $(\tilde{x} + \tilde{x}_o) \oplus v \oplus \tilde{w} \in G_o(\tilde{S})$, and we have

$$\|x \oplus v \oplus w - (\tilde{x}_o + \tilde{x}) \oplus v \oplus \tilde{w}\|^2 \leqq$$
$$\leqq (\|x-\tilde{x}\| + \|\tilde{x}_o\|)^2 + (\|y_o - \tilde{y}_o\| + \|u-\tilde{u}\| + \|A_o v - \tilde{A}_o v\|)^2 \leqq$$
$$\leqq 4(\delta_o^2(\|x\|^2 + \|u\|^2) + \delta_o^2\|y_o\|^2 + m^2\delta^2(1+\varepsilon)^2\|v\|^2) \leqq$$
$$\leqq 4(\delta_o^2(\|x\|^2 + \|u\|^2 + 4\delta_o^2(\|w\|^2 + \|u\|^2 + m^2(1+\varepsilon)^2\|v\|^2) +$$
$$+ m^2\delta^2(1+\varepsilon)^2\|v\|^2).$$

Note that $\dim \overline{R(S_1)}/\overline{R(S)} \leqq \dim R(A) \leqq m$. Therefore, there exists a projection P of $\overline{R(S_1)}$ onto $\overline{R(S)}$ such that $\|P\|$ depends only on m. Then $u + Y_o = P(w + Y_o) - PAv$, and hence

$$\|u\| \leqq (1+\varepsilon)\|u + Y_o\| \leqq (1+\varepsilon)\|P\|(\|w\| + \|v\|).$$

Consequently,

$$\|x \oplus v \oplus w - (\tilde{x} + \tilde{x}_o) \oplus v \oplus \tilde{w}\|^2 \leqq$$
$$\leqq 4(\delta_o^2\|x\|^2 + (10\delta_o^2(1+\varepsilon)^2\|P\|^2 + 4\delta_o^2)\|w\|^2 +$$
$$+ (10\delta_o^2(1+\varepsilon)^2\|P\|^2 + 4\delta_o^2 m^2(1+\varepsilon)^2 + m^2\delta^2(1+\varepsilon)^2)\|v\|^2).$$

Since we may suppose that $\|P\| \leqq m + 1 + \varepsilon$, we have

$$40\delta_o^2(1+\varepsilon)^2\|P\|^2 + 16\delta_o^2 \leqq 56\delta_o^2(1+\varepsilon)^2(m+1+\varepsilon)^2,$$

and

$$40\delta_o^2(1+\varepsilon)^2\|P\|^2 + 16\delta_o^2 m^2(1+\varepsilon)^2 + 4m^2\delta^2(1+\varepsilon)^2 \leqq$$
$$\leqq 60(1+\varepsilon)^2(m+1+\varepsilon)^2\max\{\delta_o^2,\delta^2\},$$

so that

$$\|x \oplus v \oplus w - (\tilde{x}+\tilde{x}_o) \oplus v \oplus \tilde{w}\|^2 \leqq$$
$$\leqq 64(1+\varepsilon)^2(m+1+\varepsilon)^2 \max\{\delta_o^2,\delta^2\}\|x \oplus v \oplus w\|^2 .$$

Therefore, letting $\delta_o \to \delta_o(S,\tilde{S})$, $\delta \to \delta(Y,\tilde{Y})$ and $\varepsilon \to 0$, we obtain the first estimate from the statements.

The second estimate can be obtained in a similar way, with some minor changes. We start with an arbitrary element $\tilde{x} \oplus v \oplus \tilde{w} \in G_o(\tilde{S})$ and proceed as above. We only note that there exists a projection $\tilde{P}$ of $R(\tilde{S}_1)$ onto $\overline{R(S)}$ such that $\|\tilde{P}\|$ depends only on m, that $\|\tilde{A}_o v\| \leqq m(1+\varepsilon)(1+\delta)\|v\|$ and that $\|\tilde{A}v\| \leqq \|\tilde{A}_o v\|$ for all $v \in M$. (One also has $1+\delta(Y,\tilde{Y}) \leqq 2$.) We omit the details.

We shall largely treat, in the next section, the stability of the index of a semi-Fredholm complex (in particular, of a semi-Fredholm operator) under small perturbations in the gap topology. Nevertheless, we shall end this section with a stability result for Fredholm operators, as an illustration of the resources of the preceding statements.

2.20. PROPOSITION. <u>Let</u> S <u>and</u> $\tilde{S}$ <u>be as in Definition</u> 2.1. <u>If</u> $S \in \Phi(X/X_o, Y/Y_o)$ <u>and the numbers</u> $\hat{\delta}(X_o,\tilde{X}_o)$, $\hat{\delta}(Y,\tilde{Y})$ <u>and</u> $\hat{\delta}_o(S,\tilde{S})$ <u>are sufficiently small, then</u> $\tilde{S} \in \Phi(\tilde{X}/\tilde{X}_o, \tilde{Y}/\tilde{Y}_o)$, $\dim N(\tilde{S}) \leqq \dim N(S)$, $\dim(\tilde{Y}/\tilde{Y}_o)/R(\tilde{S}) \leqq \dim(Y/Y_o)/R(S)$ <u>and</u> $\operatorname{ind}\tilde{S} = \operatorname{ind} S$.

Proof. Let $n := \dim N(S)$ and let $m := \dim (Y/Y_o)/R(S)$. We have, in particular, $S \in \Phi_+(X/X_o, Y/Y_o)$, and therefore we may apply Remark 2.8.1°. Since $\|P_*\| \leq n+1+\varepsilon$ and $\|\tilde{P}_*\| \leq n+1+\varepsilon$ in (2.7), with $\varepsilon > 0$ as small as we want, if

$$\delta := \max\{\hat{\delta}(X_o, \tilde{X}_o), \hat{\delta}(Y, \tilde{Y}), \hat{\delta}_o(S, \tilde{S})\}$$

satisfies the inequality

$$\delta(n+1)(1+(n+1)(1+\delta)(1+\gamma(S)^{-2})^{1/2}) < 1, \qquad (2.12)$$

then (2.8) is also satisfied. Hence $\tilde{S} \in \Phi_+(\tilde{X}/\tilde{X}_o, \tilde{Y}/\tilde{Y}_o)$ and $\dim N(\tilde{S}) \leq n$, by virtue of Theorem 2.7.

Assuming that (2.12) is fulfilled, let P be a projection of Y/Y_o onto $R(S)$ such that $\|P\| \leq m+1+\varepsilon$. Let M be the null-space of P. If $A \in \mathcal{L}(M, Y/Y_o)$ is the canonical inclusion, then the operator S_1 from Lemma 2.19 is surjective. Let $\tilde{S}_1$ be the extension of $\tilde{S}$ that is given by Lemma 2.19. According to this lemma, $\delta_o(S_1, \tilde{S}_1) \leq 8(m+1)\delta$. Let us observe that

$$\gamma(S_1)^{-2} \leq (1+\gamma(S)^{-2})(m+1)^2 + 1.$$

Indeed, if $r > \gamma(S)^{-1}$ and $\eta = S\xi + v \in R(S_1)$, we may assume that $\|\xi\| < r\|S\xi\|$. Therefore

$$\|\xi \oplus v\|^2 < r^2\|S\xi\|^2 + \|v\|^2 \leq r^2\|P\|^2\|S_1(\xi \oplus v)\|^2 +$$
$$+ (1+\|P\|)\|S_1(\xi \oplus v)\|^2 = ((1+r^2)\|P\|^2 + 1)\|S_1(\xi \oplus v)\|^2,$$

from which we derive easily the desired estimate.

By Theorem 2.2, if $\delta_o(S_1, \tilde{S}_1)$ and $\delta(\tilde{Y}, Y)$ are sufficiently small, then $\tilde{S}_1$ is also surjective. More precisely, if

$$\delta(1+8(m+1)(1+\delta)(2+(m+1)^2(1+\gamma(S)^{-2}))^{1/2} < 1, \qquad (2.13)$$

then, by Remark 2.4.1°, the operator $\tilde{S}_1$ is surjective. In particular,

$$\dim(\tilde{Y}/\tilde{Y}_o)/R(\tilde{S}) = \dim R(\tilde{S}_1)/R(\tilde{S}) \leq \dim M = m,$$

by (2.1).

We shall show that under certain conditions we have $\dim N(\tilde{S}_1) = \dim N(S_1)$. Let us note that

$$\delta(N_o(\tilde{S}_1), N_o(S_1)) \leqq (1+\gamma(S_1)^{-2})^{1/2}\delta_o(\tilde{S}_1, S_1) \leqq$$
$$\leqq 16(m+1)^2\delta(2+(m+1)^2(1+\gamma(S)^{-2}))^{1/2}, \tag{2.14}$$

by Lemmas 2.9 and 2.19.

Let φ_1 be the left side of (2.13). Let also

$$\rho_{11} := (1+\delta)(8\delta(m+1) + (1+8\delta(m+1))(2+(m+1)^2(1+\gamma(S)^{-2}))^{1/2}.$$

Then we have

$$\gamma(\tilde{S}_1)^{-2} \leqq (1-\varphi_1)^{-2}\rho_{11}^{\ 2},$$

by (2.6), via Remark 2.4.1° and the previous estimates for $\delta_o(S_1, \tilde{S}_1)$ and $\gamma(S_1)^{-2}$. Therefore

$$\delta(N_o(S_1), N_o(\tilde{S}_1)) \leqq 8(m+1)\delta(1+(1-\varphi_1)^{-2}\rho_{11}^{\ 2})^{1/2} \tag{2.15}$$

by Lemma 2.9. Let η_1 be the right side of (2.14) and let η_2 be the right side of (2.15). If $\eta = \max\{\eta_1, \eta_2\}$ and if

$$(n+1)(\eta + \sqrt{2}(n+1)\delta(1+\eta)) < 1, \tag{2.16}$$

then we can apply Corallary 2.5 to $X_o \subset N_o(S_1)$ and $\tilde{X}_o \subset N_o(\tilde{S}_1)$ in both directions, and we deduce that

$$\dim N(\tilde{S}_1) = \dim N_o(\tilde{S}_1)/\tilde{X}_o = \dim N_o(S_1)/X_o = \dim N(S_1).$$

Consequently

$$\operatorname{ind}\tilde{S} = \dim N(\tilde{S}) - \dim(\tilde{Y}/\tilde{Y}_o)/R(\tilde{S}) =$$
$$= \dim N(\tilde{S}_1) - \dim N(\tilde{S}_1)/N(\tilde{S}) - \dim(\tilde{Y}/\tilde{Y}_o)/R(\tilde{S}) -$$
$$- \dim R(\tilde{S}_1)/R(\tilde{S}) = \dim N(S_1) - \dim M = \dim N(S) -$$
$$- \dim(Y/Y_o)/R(S) = \operatorname{ind} S,$$

where we have used (2.1). The proof of the proposition is complete.

Let us remark that (2.12), (2.13) and (2.16) can effectively be used to find a positive number δ for which the assertion of the proposition holds.

3. STABILITY OF THE INDEX

When a complex of Banach spaces of the form (1.1) is given, we may assume with no loss of generality that every operator α^p is a closed operator in a certain fixed Banach space $\mathfrak{X}$ which contains X^p as a closed subspace for every p (take, for instance, $\mathfrak{X}$ to be the space $\oplus_p X^p$, endowed with the ℓ^2-norm). However, in this case the dual object is no longer of the same type. More precisely, the duality replaces the subspaces of $\mathfrak{X}$ with quotient spaces of $\mathfrak{X}^*$. Since the gap topology is defined on the family of subspaces of the same Banach space, a unique and natural treatment of the previous cases is provided by the following:

3.1. DEFINITION. Let $\mathfrak{X}$ be a fixed Banach space. A countable family of operators $\alpha = (\alpha^p)_{p\in\mathbb{Z}}$, where $\alpha^p \in \mathcal{C}(X^p/X_o^p, X^{p+1}/X_o^{p+1})$ and $X_o^p, X^p \in \mathcal{S}(\mathfrak{X})$, such that $R(\alpha^{p-1}) \subset N(\alpha^p)$ for all $p \in \mathbb{Z}$ will be called a complex in $\mathfrak{X}$. The set of all complexes in $\mathfrak{X}$ will be denoted by $\partial(\mathfrak{X})$. If $\overline{D(\alpha^p)} = X^p/X_o^p$ for all $p \in \mathbb{Z}$, then the complex α is said to be densely defined. For every $\alpha \in \partial(\mathfrak{X})$ we define the quantity

$$\gamma(\alpha) := \inf_{p\in\mathbb{Z}} \gamma(\alpha^p).$$

If $\gamma(\alpha) > 0$, the complex α is said to have closed range. The cohomology of the complex α is the family of linear spaces $(H^p(\alpha))_{p\in\mathbb{Z}}$, where $H^p(\alpha) := N(\alpha^p)/R(\alpha^{p-1})$ (and the latter is isomorphic to $N_o(\alpha^p)/R_o(\alpha^{p-1})$).

If S is an operator as in Definition 2.1, then S can be identified with the complex $\alpha_S = (\alpha_S^p)_{p\in\mathbb{Z}}$, where $\alpha_S^p \in \mathcal{C}(X^p/X_o^p, X^{p+1}/X_o^{p+1})$, with $X^o = X, X_o^o = X_o, X^1 = Y, X_o^1 = Y_o$, $X^p = X_o^p = 0$ if $p \notin \{0,1\}$ and $\alpha_S^o = S$.

3.2. DEFINITION. Let $\alpha = (\alpha^p)_{p\in\mathbb{Z}} \in \partial(\mathfrak{X})$ be densely defined. Then $\alpha^{p*} \in \mathcal{C}(X_o^{p+1\perp}/X^{p+1\perp}, X_o^{p\perp}/X^{p\perp})$ has the property that $R(\alpha^{p*}) \subset N(\alpha^{p-1*})$. Therefore $\alpha^* := (\alpha^{-p-1*})_{p\in\mathbb{Z}} \in \partial(\mathfrak{X}^*)$ and is called the dual of α. It is easily seen that $\gamma(\alpha^*) = \gamma(\alpha)$ and that $H^p(\alpha^*)$ is isomorphic to $H^{-p}(\alpha)$.

3.3. DEFINITION. Let $\alpha = (\alpha^p)_{p\in\mathbb{Z}} \in \partial(\mathfrak{X})$.

1^o The complex α is said to be semi-Fredholm at the step p if the pair (α^{p-1}, α^p) is semi-Fredholm (Definition 2.11).

2^o The complex α is said to be semi-Fredholm if α has closed range and at least one of the functions

$$\mathbb{Z} \ni k \to \dim H^{2k}(\alpha) \in \mathbb{Z}_+ \cup \{\infty\},$$

$$\mathbb{Z} \ni k \to \dim H^{2k+1}(\alpha) \in \mathbb{Z}_+ \cup \{\infty\}$$

is finite and has finite support.

3^o The complex α is said to be Fredholm if α has closed range and the function

$$\mathbb{Z} \ni p \to \dim H^p(\alpha) \in \mathbb{Z}_+ \cup \{\infty\}$$

is finite and has finite support.

If $\alpha \in \partial(\mathfrak{X})$ is semi-Fredholm, then we define its <u>index</u> by the equality

$$\operatorname{ind} \alpha = \sum_{p\in\mathbb{Z}} (-1)^p \dim H^p(\alpha) \quad . \tag{3.1}$$

The number (3.2) is finite if and only if the complex α is Fredholm.

Let us observe that if $\alpha \in \partial(\mathfrak{X})$ is densely defined and semi-Fredholm (Fredholm), then $\alpha^* \in \partial(\mathfrak{X}^*)$ is also semi-Fredholm (Fredholm) and $\operatorname{ind}\alpha^* = \operatorname{ind}\alpha$. The converse is also true (see the comment following Definition 3.2).

A complex $\alpha \in \partial(\mathfrak{X})$ is said to have *finite length* if $\alpha^p = 0$ for all but a finite collection of indices p. If α has finite length and $R(\alpha^p)$ is closed for all p (in particular, if $\dim H^p(\alpha) < \infty$ for all p), then α has closed range.

A complex $\alpha \in \partial(\mathfrak{X})$ is said to be *exact at the step* p if $H^p(\alpha) = 0$. If $H^p(\alpha) = 0$ for all p, then the complex α is said to be exact.

The main aim of this section is to prove the stability of the number (3.1) under small perturbations in the gap topology. From the results of the previous section, we derive first some "local" consequences.

3.4. PROPOSITION. *Let* $\alpha = (\alpha^p)_{p\in\mathbb{Z}} \in \partial(\mathfrak{X})$.

1^o *If* α *is semi-Fredholm at the step* p, *then there exists an* $\varepsilon > 0$ *such that if* $\tilde{\alpha} = (\tilde{\alpha}^q)_{q\in\mathbb{Z}} \in \partial(\mathfrak{X})$, $\delta_o(\alpha^{p-1}, \tilde{\alpha}^{p-1}) < \varepsilon$ *and* $\delta_o(\tilde{\alpha}^p, \alpha^p) < \varepsilon$, *then* $\tilde{\alpha}$ *is also semi-Fredholm at the step* p *and* $\dim H^p(\tilde{\alpha}) \leq \dim H^p(\alpha)$. *In particular, if* α *is exact at the step* p, *then* $\tilde{\alpha}$ *is also exact at the step* p.

2^o *If* α *is exact at the steps* p-1 *and* p+1, *and if* $\gamma(\alpha^{p-1}) > 0$ *and* $\gamma(\alpha^{p+1}) > 0$, *then there exists an* $\varepsilon > 0$ *such that if*

$$\max\{\delta_o(\alpha^{p-2}, \tilde{\alpha}^{p-2}), \hat{\delta}_o(\alpha^{p-1}, \tilde{\alpha}^{p-1}), \hat{\delta}_o(\alpha^p, \tilde{\alpha}^p), \delta_o(\tilde{\alpha}^{p+1}, \alpha^{p+1})\} < \varepsilon,$$

then $\tilde{\alpha}$ *is exact at the steps* p-1 *and* p+1, $\gamma(\tilde{\alpha}^{p-1}) > 0$, $\gamma(\tilde{\alpha}^{p+1}) > 0$ *and* $\dim H^p(\tilde{\alpha}) = \dim H^p(\alpha)$.

Proof. 1° The assertion is precisely the content of Proposition 2.10. When $H^p(\alpha) = 0$, we can also give some estimates (that are needed in the sequel), which are obtained directly from Corollary 2.12. Namely, if

$$\begin{aligned} &\delta_o \geqq \max\{\delta_o(\alpha^{p-1},\tilde{\alpha}^{p-1}),\delta_o(\tilde{\alpha}^p,\alpha^p)\},\\ &\delta \geqq \max\{(1+\gamma(\alpha^{p-1})^{-2})^{1/2}\delta_o(\alpha^{p-1},\tilde{\alpha}^{p-1}),(1+\gamma(\alpha^p)^{-2})^{1/2}\delta_o(\tilde{\alpha}^p,\alpha^p)\},\\ &r \geqq \max\{\gamma(\alpha^{p-1})^{-1},\gamma(\alpha^p)^{-1}\},\\ &\varphi(\delta_o,\delta,r) < 1, \end{aligned} \tag{3.2}$$

then $R(\tilde{\alpha}^{p-1}) = N(\alpha^p)$, $R(\tilde{\alpha}^p)$ is closed, and

$$\max\{\gamma(\tilde{\alpha}^{p-1})^{-1},\gamma(\tilde{\alpha}^p)^{-1}\} \leqq (1-\varphi(\delta_o,\delta,r))^{-1}\rho_1(\delta_o,\delta,r), \tag{3.3}$$

with φ and ρ_1 as in Remark 2.4.1°.

2° We take some positive numbers δ_o,δ and r that satisfy (3.2) for both steps p-1 and p+1 (notice that (3.2) is written for the step p). From (3.3) and Lemma 2.9 we infer that

$$\begin{aligned} &\hat{\delta}(R_o(\alpha^{p-1}),R_o(\tilde{\alpha}^{p-1})) \leqq \max\{(1-\varphi(\delta_o,\delta,r))^{-1}\rho_1(\delta_o,\delta,r),\\ &\qquad (1+\gamma(\alpha^{p-1})^{-2})^{1/2}\}\hat{\delta}_o(\alpha^{p-1},\tilde{\alpha}^{p-1}),\\ &\hat{\delta}(N_o(\alpha^p),N_o(\tilde{\alpha}^p)) \leqq \max\{(1-\varphi(\delta_o,r,\delta)^{-1}\rho_1(\delta_o,\delta,r),\\ &\qquad (1+\gamma(\alpha^p)^{-2})^{1/2}\}\hat{\delta}_o(\alpha^p,\tilde{\alpha}^p). \end{aligned} \tag{3.4}$$

Let η_{p1} and η_{p2} be the right sides of the inequalities (3.4), and let $\eta_p := \max\{\eta_{p1},\eta_{p2}\}$. If $m_p := \dim H^p(\alpha)$ is finite and if

$$\eta_p(1+m_p)(1+\sqrt{2}(1+m_p)(1+\eta_p)) < 1, \tag{3.5}$$

which is possible if $\hat{\delta}_o(\alpha^{p-1},\tilde{\alpha}^{p-1})$ and $\hat{\delta}_o(\alpha^p,\tilde{\alpha}^p)$ are sufficiently small, then by Corollary 2.5,

$$\dim H^p(\tilde{\alpha}) = \dim N_o(\tilde{\alpha}^p)/R_o(\tilde{\alpha}^{p-1}) = \dim N_o(\alpha^p)/(R_o(\alpha^{p-1}) = \dim H^p(\alpha).$$

The case $m_p = \infty$ follows from Lemma 2.16. Indeed, if

$$2\eta_p(1 + 2(1 + \eta_p)) < 1, \tag{3.6}$$

then according to Lemma 2.16, we cannot have $\dim H^p(\tilde{\alpha}) < \infty$ (otherwise we would have $\dim H^p(\alpha) < \infty$).

The following statements are concerned with the "global" consequences of the results from the second section. For every pair $\alpha, \beta \in \partial(\mathfrak{X})$ we define its "gap" by the equation

$$\hat{\delta}_o(\alpha, \beta) = \sup_{p \in \mathbb{Z}} \hat{\delta}_o(\alpha^p, \beta^p). \tag{3.7}$$

One always has $\hat{\delta}_o(\alpha, \beta) \leq 1$. Moreover, this function induces a metric topology on $\partial(\mathfrak{X})$.

3.5. THEOREM. <u>Let $\alpha = (\alpha^p)_{p \in \mathbb{Z}} \in \partial(\mathfrak{X})$ be a semi-Fredholm complex that is not Fredholm. Then there exists an $\varepsilon > 0$ such that if $\tilde{\alpha} = (\tilde{\alpha}^p)_{p \in \mathbb{Z}} \in \partial(\mathfrak{X})$ and $\hat{\delta}_o(\alpha, \tilde{\alpha}) < \varepsilon$, then $\tilde{\alpha}$ is also semi-Fredholm. Moreover, for each $p \in \mathbb{Z}$ we have $\dim H^p(\tilde{\alpha}) = 0$ if $\dim H^p(\alpha) = 0$, $\dim H^p(\tilde{\alpha}) = \infty$ if $\dim H^p(\alpha) = \infty$, and $\dim H^p(\tilde{\alpha}) < \infty$ if $\dim H^p(\alpha) < \infty$. In particular, $\operatorname{ind} \tilde{\alpha} = \operatorname{ind} \alpha$.</u>

<u>In addition, for every non-negative index q there exists a positive number $\varepsilon_q \leq \varepsilon$ such that if $\hat{\delta}(\alpha, \tilde{\alpha}) \leq \varepsilon_q$, then $\dim H^p(\tilde{\alpha}) \leq \dim H^p(\alpha)$ when $|p| \leq q$.</u>

<u>Proof</u>. With no loss of generality we may suppose that the function $k \to \dim H^{2k}(\alpha)$ is finite and has finite support. Therefore, these exists a non-negative integer k_o such that $H^{2k}(\alpha) = 0$ if $|k| > k_o$.

Let us consider some positive numbers $\delta_o \geqq \hat{\delta}_o(\alpha,\tilde{\alpha})$, $\delta \geqq (1+\gamma(\alpha)^{-1})^{1/2}\,\hat{\delta}_o(\alpha,\tilde{\alpha})$ and $r \geqq \gamma(\alpha)^{-1}$. If $\hat{\delta}_o(\alpha,\tilde{\alpha})$ is sufficiently small, then we may assume that $\varphi(\delta_o,\delta,r) < 1$, with φ as in Remark 2.4.1^o. Hence, if $H^p(\alpha) = 0$ for a certain p, then $H^p(\tilde{\alpha}) = 0$, by Proposition 3.4.1^o. In particular, H $H^{2k}(\tilde{\alpha}) = 0$ if $|k| > k_o$. We also note that the right side of (3.3) does not depend on $p = 2k$ if $|k| > k_o$. Thus

$$\inf_{|k|>k_o} \{\gamma(\tilde{\alpha}^{2k-1}), \gamma(\tilde{\alpha}^{2k})\} > 0.$$

If η_p is the function from (3.5) and if $\eta := \sup\{\eta_{2k}; |k| > k_o\}$, then, for $\hat{\delta}_o(\alpha,\tilde{\alpha})$ sufficiently small, we have $2\eta(2\eta+3) < 1$. In particular, (3.6) is fulfilled for $|k| > k_o$, and hence $\dim H^{2k+1}(\alpha) = \infty$ (or $\dim H^{2k+1}(\alpha) < \infty$) implies $\dim H^{2k+1}(\tilde{\alpha}) = \infty$ (or $\dim H^{2k+1}(\tilde{\alpha}) < \infty$).

Now, we have to discuss only a finite number of cases. Since $(\alpha^{2k-1}, \alpha^{2k})$ is a semi-Fredholm pair, then $(\tilde{\alpha}^{2k-1}, \tilde{\alpha}^{2k})$ is also semi-Fredholm if $\hat{\delta}_o(\alpha,\tilde{\alpha})$ is sufficiently small. In particular, $\gamma(\tilde{\alpha}^{2k-1}) > 0$ and $\gamma(\tilde{\alpha}^{2k}) > 0$ if $|k| \leqq k_o$, so that $\gamma(\tilde{\alpha}) > 0$.

If $\dim H^{2k+1}(\alpha) = \infty$ and either $\dim H^{2k}(\alpha)$ or $\dim H^{2k+2}(\alpha)$ is non-zero, then Proposition 3.4.2^o does not apply. In this case we can use Proposition 2.18. Indeed, assuming with no loss of generality that $\overline{D(\alpha^p)} = X^p/X_o^p$ for all p, we have

$$\alpha_o^{2k} \in \Phi_+(X^{2k}/R_o(\alpha^{2k-1}), X^{2k+1}/X_o^{2k+1}),$$

where α_o^{2k} is induced by α^{2k}. We also have

$$\alpha^{2k+1} \in \Phi_-(X^{2k+1}/X_o^{2k+1}, N_o(\alpha^{2k+2})/X_o^{2k+2}).$$

Since $\delta(R_o(\alpha^{2k-1}), R_o(\tilde{\alpha}^{2k-1}))$ and $\delta(N_o(\tilde{\alpha}^{2k+2}), N_o(\alpha^{2k+2}))$ can be made as small as we want (Lemma 2.9), and $\delta_o(\alpha_o^{2k}, \tilde{\alpha}_o^{2k}) = \delta_o(\alpha^{2k}, \tilde{\alpha}^{2k})$ (where $\tilde{\alpha}_o^{2k}$ is obtained from $\tilde{\alpha}^{2k}$ as α_o^{2k} from α^{2k}), we derive, from Proposition 2.18, that $\dim H^{2k+1}(\tilde{\alpha}) = \infty$ if $\hat{\delta}(\alpha, \tilde{\alpha})$ is sufficiently small.

In particular we have $\operatorname{ind} \alpha = \operatorname{ind} \tilde{\alpha} = -\infty$ if $\dim H^{2k+1}(\alpha) = \infty$ for some $k \in \mathbb{Z}$. Suppose now that $\dim H^{2k+1}(\alpha) < \infty$ for all $k \in \mathbb{Z}$. As α is not Fredholm, the function $k \to \dim H^{2k+1}(\alpha)$ cannot have finite support. Assume now that $H^{2k+1}(\tilde{\alpha}) = 0$ for some $k \in \mathbb{Z}$ with $|k| > k_o$. We can chose δ_o and δ so small that we have (with η_p as in the proof of Proposition 3.4.2°)

$$\eta_p(1 + \sqrt{2}\,(1 + \eta_p)) < 1 \qquad (3.\tilde{5})$$

for all $p \in \mathbb{Z}$, $|p| > k_o$. From (3.4) and $(3.\tilde{5})$ we see that Corollary 2.5 can be applied with $X = N(\tilde{\alpha}^{2k+1})$, $Y = R(\tilde{\alpha}^{2k})$, $\tilde{X} = N(\alpha^{2k+1})$, $\tilde{Y} = R(\alpha^{2k})$, and $m = \dim H^{2k+1}(\tilde{\alpha}) = 0$. We conclude that also $H^{2k+1}(\alpha) = 0$. Hence, $k \mapsto H^{2k+1}(\tilde{\alpha})$ cannot have finite support and we conclude that $\operatorname{ind} \alpha = \operatorname{ind} \tilde{\alpha} = -\infty$.

The last assertion follows from Proposition 3.4.1° (the cases of infinite dimension are already settled, by the previous argument). Note that the proof of Proposition 3.4.1°, which is based on Corollary 2.5, involves the dimension of a quotient space, and the set of all these dimensions is, in general, non-bounded. Therefore, the condition from Corollary 2.5 can be realized, in general, only for a finite number of indices. Hence, we also have the following:

3.6. COROLLARY. <u>Assume that the complex α from Theorem 3.5. has finite length. Then, if q is big enough we may take $\varepsilon := \varepsilon_q$, so that $\dim H^p(\tilde{\alpha}) \leqq \dim H^p(\alpha)$ for all $p \in \mathbb{Z}$.</u>

3.7. THEOREM. Let $\alpha = (\alpha^p)_{p\in\mathbb{Z}} \in \partial(\mathfrak{X})$ be a Fredholm complex. Then there exists an $\varepsilon > 0$ such that if $\tilde{\alpha} = (\tilde{\alpha}^p)_{p\in\mathbb{Z}} \in \partial(\mathfrak{X})$ and $\hat{\delta}_o(\alpha,\tilde{\alpha}) < \varepsilon$, then the complex $\tilde{\alpha}$ is also Fredholm, $\dim H^p(\tilde{\alpha}) \leqq \dim H^p(\alpha)$ for all $p \in \mathbb{Z}$ and $\operatorname{ind} \tilde{\alpha} = \operatorname{ind} \alpha$.

Proof. Since α is a Fredholm complex, there exists a non-negative index p_o such that $H^p(\alpha) = 0$ if $|p| \geqq p_o$. Therefore, as in the proof of Theorem 3.5, if $\hat{\delta}_o(\alpha,\tilde{\alpha})$ is sufficiently small, then $H^p(\tilde{\alpha}) = 0$ if $|p| \geqq p_o$. According to the same argument, we have $\gamma(\tilde{\alpha}) > 0$ and $\dim H^p(\tilde{\alpha}) \leqq \dim H^p(\alpha)$ for all $p \in \mathbb{Z}$, provided that $\hat{\delta}_o(\alpha,\tilde{\alpha})$ is sufficiently small (we note that in the present case no infinite dimensions occur and all pairs (α^{p-1},α^p) are semi-Fredholm.

The only thing to be proved is that $\operatorname{ind}\tilde{\alpha} = \operatorname{ind}\alpha$ if $\hat{\delta}_o(\alpha,\tilde{\alpha})$ is sufficiently small. With no loss of generality we may assume that $\dim H^p(\alpha) = 0$ if $p < 0$. Set

$$n(\alpha) := \min\{n \geqq 0;\ H^p(\alpha) = 0 \quad \forall\ p \geqq n\}.$$

We prove our assertion by induction with respect to $n := n(\alpha)$.

If $n = 0$, then the assertion follows from the first part of the proof. Assume that the assertion is true for a certain $n \geqq 0$, an let α have the property that $n(\alpha) = n+1$. Then we can write $R(\alpha^n) + M = N(\alpha^{n+1})$, where M is a complement of $R(\alpha^n)$ in $N(\alpha^{n+1})$ with $\dim M = \dim H^n(\alpha) < \infty$. Set $\mathfrak{Y} = \mathfrak{X} \oplus M$. We shall consider a certain complex $\beta = (\beta^p)_{p\in\mathbb{Z}} \in \partial(\mathfrak{Y})$ that extends the complex α. Namely, by identifying $\mathfrak{X}$ with $\mathfrak{X} \oplus 0$, we set $\beta^p = \alpha^p$ if $p \neq n-1$ and $\beta^{n-1}(\xi \oplus v) = \alpha^{n-1}(\xi) + v$ for all $\xi \in D(\alpha^{n-1})$ and $v \in M$. Then β is a Fredholm complex with the property that $n(\beta) = n$ (since $R(\beta^{n-1}) = N(\beta^n)$). Note also that $N(\beta^{n-1}) = N(\alpha^{n-1})$. The complex $\tilde{\alpha}$ will be extended in a similar way. Namely, we define $\tilde{\beta}^p = \tilde{\alpha}^p$ if $p \neq n-1$, whereas $\tilde{\beta}^{n-1}$ is obtained from $\tilde{\alpha}^{n-1}$ as in

Lemma 2.19 (with α^{n-1} for S, $\tilde{\alpha}^{n-1}$ for $\tilde{S}$ and the canonical inclusion $M \subset N(\alpha^{n+1})$ for A). Since $R(\alpha^{n-1}) \subset N(\alpha^n)$, $R(\tilde{\alpha}^{n-1}) \subset N(\tilde{\alpha}^n)$ and, by (3.4),

$$\hat{\delta}(N_o(\alpha^n), N_o(\tilde{\alpha}^n)) \leq c\,\hat{\delta}_o(\alpha,\tilde{\alpha}),$$

where $c \geq 0$ does not depend on $\tilde{\alpha}$, it follows that $\hat{\delta}_o(\beta^{n-1},\tilde{\beta}^{n-1})$ can be made as small as one wants, on account of the estimates from Lemma 2.19. Therefore $\hat{\delta}(\beta,\tilde{\beta})$ is as small as we desire if $\hat{\delta}(\alpha,\tilde{\alpha})$ is sufficiently small. By the induction hypothesis, we know that $\tilde{\beta}$ is Fredholm and that $\operatorname{ind}\tilde{\beta} = \operatorname{ind}\beta$. Since we have

$$\operatorname{ind}\beta = \operatorname{ind}\alpha + (-1)^{n-1}\dim M,$$
$$\operatorname{ind}\tilde{\beta} = \operatorname{ind}\tilde{\alpha} + (-1)^{n-1}\dim M,$$

which follows by Proposition 2.9 from [19] (or directly from (2.1)), we obtain that $\operatorname{ind}\tilde{\alpha} = \operatorname{ind}\alpha$ if $\hat{\delta}(\alpha,\tilde{\alpha})$ is sufficiently small, and the proof of the theorem is complete.

4. SOME APPLICATIONS

In this section we shall present various consequences of the results from the previous sections, as well as related observations.

a) We start with some aspects concerning the geometry of the metric space $\mathcal{S}(\mathfrak{X})$, where $\mathfrak{X}$ is a fixed Banach space.

4.1. DEFINITION. Let $\{X^1, Y^1, X^2, Y^2\} \subset \mathcal{S}(\mathfrak{X})$ be a quadruplet with the property that $X^1 + Y^1 \subset X^2 \cap Y^2$.

We say that this quadruplet is semi-Fredholm if $\dim (X^2 \cap Y^2)/(X^1 + Y^1) < \infty$ and $X^2 + Y^2$ is closed.

Let us mention that such quadruplets have been presented in [12] under the name of <u>Fredholm links.</u> The next result is a slight extension of a statement in [12]. We are not aware of any published proof of this result.

4.2. PROPOSITION. <u>Let</u> $\{X^1, Y^1, X^2, Y^2\} \subset \mathcal{S}(\mathfrak{X})$ <u>be a semi-Fredholm quadruplet</u>. <u>If</u> $\{\tilde{X}^1, \tilde{Y}^1, \tilde{X}^2, \tilde{Y}^2\} \subset \mathcal{S}(\mathfrak{X})$ <u>is such that</u> $\tilde{X}^1 + \tilde{Y}^1 \subset \tilde{X}^2 \cap \tilde{Y}^2$ <u>and if the numbers</u> $\delta(X^1, \tilde{X}^1)$, $\delta(Y^1, \tilde{Y}^1)$, $\delta(\tilde{X}^2, X^2)$, $\delta(\tilde{Y}^2, Y^2)$ <u>are sufficiently small, then the quadruplet</u> $\{\tilde{X}^1, \tilde{Y}^1, \tilde{X}^2, \tilde{Y}^2\}$ <u>is also semi-Fredholm and</u>

$$\dim \tilde{X}^2 \cap \tilde{Y}^2/(\tilde{X}^1 + \tilde{Y}^1) \leq \dim (X^2 \cap Y^2)/(X^1 + Y^1). \tag{4.1}$$

<u>Proof</u>. Consider the operators $S^p \in \mathcal{L}(X^p \oplus Y^p, \mathfrak{X} \oplus \mathfrak{X})$ given by the equation $S^p(x \oplus y) = (x+y) \oplus (-x-y)$ for all $x \in X^p$ and $y \in Y^p$, $p = 1,2$. Note that

$$R(S^1) = \{v \oplus (-v); v \in X^1 + Y^1\} \subset \{v \oplus (-v); v \in X^2 \cap Y^2\} = N(S^2).$$

Moreover, $R(S^1)$ is isomorphic to $X^1 + Y^1$, $N(S^2)$ is isomorphic to $X^2 \cap Y^2$ and $R(S^2)$ is isomorphic to $X^2 + Y^2$. Therefore the pair (S^1, S^2) is semi-Fredholm (Definition 2.11). Let $\tilde{S}^k$ be defined in a similar way for the spaces $(\tilde{X}^k, \tilde{Y}^k)$, $k = 1,2$. It will be enough to prove that the pair $(\tilde{S}^1, \tilde{S}^2)$, and this fact will follow from Proposition 2.10. Since

$$\|x \oplus y \oplus (x+y) \oplus (-x-y) - \tilde{x} \oplus \tilde{y} \oplus (\tilde{x}+\tilde{y}) \oplus (-\tilde{x}-\tilde{y})\|^2 \leq$$
$$\leq 5(\|x - \tilde{x}\|^2 + \|y - \tilde{y}\|^2)$$

for all $x \in X^p$, $y \in Y^p$, $\tilde{x} \in \tilde{X}^p$, $\tilde{y} \in \tilde{Y}^p$, $p = 1,2$, we derive that

$$\delta(S^p, \tilde{S}^p)^2 \leq 5(\delta(X^p, \tilde{X}^p)^2 + \delta(Y^p, \tilde{Y}^p)^2), \tag{4.2}$$

and a similar relation with changed order. In particular, if $\delta(X^1, \tilde{X}^1)$, $\delta(Y^1, \tilde{Y}^1)$, $\delta(\tilde{X}^2, X^2)$ and $\delta(\tilde{Y}^2, Y^2)$ are sufficiently small, then $\delta(S^1, \tilde{S}^1)$ and $\delta(\tilde{S}^2, S^2)$ are as small as we want, and the conclusion of the proposition follows via Proposition 2.10.

4.3. Remarks. 1° Proposition 4.2 suggests a concept of a semi-Fredholm complex of subspaces. First of all, a double family of subspaces $(X,Y) = (X^p,Y^p)_{p\in\mathbb{Z}} \subset \mathcal{S}(\mathfrak{X})$ is said to be a complex of subspaces if $X^{p-1} + Y^{p-1} \subset X^p \cap Y^p$ for all $p \in \mathbb{Z}$. Using an idea from the proof of Proposition 2.2, we can reduce the study of complexes of subspaces to the study of complexes (of operators). Namely, if $S^p \in \mathcal{L}(X^p \oplus Y^p, \mathfrak{X} \oplus \mathfrak{X})$ is defined as in Proposition 2.2 (for an arbitrary p), then $R(S^p) \subset N(S^{p+1})$, $R(S^p)$ is isomorphic to $X^p + Y^p$ and $N(S^{p+1})$ is isomorphic to $X^{p+1} \cap Y^{p+1}$. Moreover, from (4.2),

$$\hat{\delta}(S^p,\tilde{S}^p)^2 \leq 5\ (\hat{\delta}(X^p,\tilde{X}^p)^2 + \hat{\delta}(Y^p,\tilde{Y}^p)^2),$$

where $(\tilde{X},\tilde{Y}) = (\tilde{X}^p,\tilde{Y}^p)_{p\in\mathbb{Z}} \subset \mathcal{S}(\mathfrak{X})$ is another complex of subspaces and $(\tilde{S}^p)_{p\in\mathbb{Z}}$ are the corresponding operators. Therefore, the whole information concerning the complex of subspaces (X,Y) is transmitted to the complex $S = (S^p)_{p\in\mathbb{Z}} \in \partial(\mathfrak{X}\oplus\mathfrak{X})$. In particular, a Fredholm theory for complexes of subspaces can be obtained from the Fredholm theory of complexes (of operators), as developed in the preceding section.

2° Conversely, every complex $\alpha = (\alpha^p)_{p\in\mathbb{Z}} \in \partial(\mathfrak{X})$ can be associated with a complex of subspaces. Namely, if $\alpha^p \in \mathcal{C}(X^p/X_o^p, X^{p+1}/X_o^{p+1})$, then we define the spaces

$$Y^p = \ldots \oplus \mathfrak{X} \oplus \mathfrak{X} \oplus G_o(\alpha^p) \oplus X_o^{p+2} \oplus X_o^{p+3} \oplus \ldots,$$

$$Z^p = \ldots \oplus \mathfrak{X} \oplus \mathfrak{X} \oplus \mathfrak{X} \oplus X_o^{p+1} \oplus X_o^{p+2} \oplus \ldots,$$

which are subspaces of the space $\bigoplus_{p\in\mathbb{Z}} \mathfrak{X}$. Since

$$G_o(\alpha^p) \cap (\mathfrak{X} \oplus X_o^{p+1}) = N_o(\alpha^p) \oplus X_o^{p+1}$$

and $\mathfrak{X} \oplus X_o^{p+1} + G_o(\alpha^p) = \mathfrak{X} \oplus R_o(\alpha^p)$, it is easily seen that

$Y^p + Z^p \subset Y^{p+1} \cap Z^{p+1}$ and that the quotient space $(Y^{p+1} \cap Z^{p+1})/(Y^p + Z^p)$ is isomorphic to $H^{p+1}(\alpha)$. However, the metric relations do not seem to be as good as in the previous case. Indeed, if $\tilde{\alpha} \in \partial(\mathfrak{X})$ is another complex and we construct the spaces $\tilde{Y}^p$ and $\tilde{Z}^p$ for $\tilde{\alpha}$ in a similar way, then $\hat{\delta}(Y^p, \tilde{Y}^p)$ (or $\hat{\delta}(Z^p, \tilde{Z}^p)$) should be expressed in terms of $\hat{\delta}(\alpha^p, \tilde{\alpha}^p)$ but also of $\hat{\delta}(X_o^{p+k}, \tilde{X}_o^{p+k})$ for $k \geq 2$. As we have seen in the preceding section, our main results (see Theorems 3.5 and 3.7) do not require the use of the numbers $\hat{\delta}(X_o^p, \tilde{X}_o^p)$.

3^o One can consider semi-Fredholm pairs of subspaces of a Banach space $\mathfrak{X}$, in the sense of [8], Chapt. IV. Such a pair $(X,Y) \subset \mathcal{S}(\mathfrak{X})$ can be regarded as a complex of subspaces with $X^o = X$, $Y^o = Y$, $X^p = Y^p = 0$ if $p < 0$ and $X^p = Y^p = \mathfrak{X}$ if $p \geq 1$. Then the associated complex (of operators) is semi-Fredholm and we can apply our results from the previous section. In particular, we obtain stability results when both variables are perturbed in the gap topology. Although mentioned in [8], such results are not proved there.

b) We shall show that the family of semi-Fredholm (Fredholm) complexes in a Banach space $\mathfrak{X}$ is invariant under a class of natural transformations, which also preserve the index, provided that these transformations are "close" to the identity in a certain sense to be specified.

Let $\partial_o(\mathfrak{X})$ be the family of all complexes $\alpha = (\alpha^p)_{p \in \mathbb{Z}}$ in $\mathfrak{X}$ such that $D(\alpha^p) \in \mathcal{S}(\mathfrak{X})$ for all p. In particular, $\alpha^p \in \mathcal{L}(D(\alpha^p), D(\alpha^{p+1}))$. (We work with the family $\partial_o(\mathfrak{X})$ only for the sake of simplicity; most of the assertions are valid in general, with suitable modifications.) For every pair $\alpha, \beta \in \partial_o(\mathfrak{X})$ we define the set $\mathrm{Hom}(\alpha,\beta)$ as the collection of maps $\lambda = (\lambda^p)_{p\in\mathbb{Z}}$ such that $\lambda^p \in \mathcal{L}(D(\alpha^p), D(\beta^p))$ and $\beta^p\lambda^p = \lambda^{p+1}\alpha^p$ for all $p \in \mathbb{Z}$. For every $\lambda \in \mathrm{Hom}(\alpha,\beta)$ we define the quantity

$$\nu(\lambda) := \sup_{p\in\mathbb{Z}} \sup_{\substack{x\in D(\alpha^p)\\ \|x\|\leq 1}} \|x - \lambda^p x\|, \tag{4.3}$$

which is not necessarily a finite number. Note that we have

$$\sup_{p\in\mathbb{Z}} \delta(D(\alpha^p), D(\beta^p)) \leqq \nu(\lambda). \tag{4.4}$$

We can even give estimates of the same type for $\delta(\alpha^p, \beta^p)$. Indeed, if $x \oplus \alpha^p x \in G(\alpha^p)$, then $\lambda^p x \oplus \lambda^{p+1}\alpha^p x = \lambda^p x \oplus \beta^p\lambda^p x \in G(\beta^p)$ and one has

$$\|x \oplus \alpha^p x - \lambda^p x \oplus \beta^p \lambda^p x\|^2 = \|x - \lambda^p x\|^2 + \|\alpha^p x - \lambda^{p+1}\alpha^p x\|^2 \leqq$$
$$\leqq \nu(\lambda)^2 \quad \|x \oplus \alpha^p x\|^2 .$$

Hence

$$\sup_{p\in\mathbb{Z}} \delta(\alpha^p, \beta^p) \leqq \nu(\lambda). \tag{4.5}$$

4.4. PROPOSITION. _Let_ $\alpha = (\alpha^p)_{p\in\mathbb{Z}} \in \partial_o(\mathfrak{X})$ _be a semi-Fredholm complex and let_ $\varepsilon > 0$ _be given by Theorem 3.5 (or Theorem 3.7). Assume that for a certain_ $\beta = (\beta^p)_{p\in\mathbb{Z}} \in \partial_o(\mathfrak{X})$ _there are_ $\lambda \in \mathrm{Hom}(\alpha,\beta)$ _and_ $\mu \in \mathrm{Hom}(\beta,\alpha)$ _such that_ $\nu(\lambda) < \varepsilon$ _and_ $\nu(\mu) < \varepsilon$. _Then the complex_ β _is also semi-Fredholm and_ $\mathrm{ind}\,\beta = \mathrm{ind}\,\alpha$.

Proof. Indeed, from (4.5) it follows that

$$\hat{\delta}_o(\alpha,\beta) = \hat{\delta}(\alpha,\beta) \leqq \max\{\nu(\lambda), \nu(\mu)\} < \varepsilon.$$

Consequently, we can apply Theorem 3.5 (or Theorem 3.7), which implies the conclusion.

Of course, the other consequences from Theorem 3.5 (or Theorem 3.7) are also true for the complex β.

4.5. COROLLARY. <u>Let</u> $S \in \mathcal{L}(X^1, X^2)$ <u>and let</u> $T \in \mathcal{L}(Y^1, Y^2)$, <u>where</u> $X^1, X^2, Y^1, Y^2 \in \mathcal{L}(\mathfrak{X})$. <u>Let also</u> $A^p \in \mathcal{L}(X^p, Y^p)$ <u>and</u> $B^p \in \mathcal{L}(Y^p, X^p)$ $(p = 1,2)$ <u>be such that</u> $TA^1 = A^2 S$ <u>and</u> $SB^2 = B^1 T$. <u>If</u> $S \in \Phi_{\pm}(X^1, X^2)$ <u>and if the numbers</u> $\|1 - A^p\|, \|1 - B^p\|$ $(p = 1,2)$ <u>are sufficiently small, then</u> $T \in \Phi_{\pm}(Y^1, Y^2)$ <u>and</u> $\text{ind}\, T = \text{ind}\, S$.

<u>Proof</u>. The result is a simple consequence of Proposition 4.4. We only note that $\|1 - A^p\|$ (or $\|1 - B^p\|$) is computed in $\mathcal{L}(X^p, \mathfrak{X})$ (or $\mathcal{L}(Y^p, \mathfrak{X})$).

c) Given a Banach space $\mathfrak{X}$, one can consider collections of operators $(\alpha^p)_{p \in \mathbb{Z}}$ in $\mathfrak{X}$ that are no longer complexes but "almost" complexes, in a sense which will be specified (i.e. modulo compact operators).

Let $\ell^\infty(\mathfrak{X})$ (or $\tau(\mathfrak{X})$) denote the Banach space of all bounded (or totally bounded) sequences of elements of $\mathfrak{X}$. Let also $\kappa(\mathfrak{X}) := \ell^\infty(\mathfrak{X})/\tau(\mathfrak{X})$. If $\mathfrak{Y}$ is another Banach space, then for every operator $S \in \mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ there exists an operator $\kappa(S) \in \mathcal{L}(\kappa(\mathfrak{X}), \kappa(\mathfrak{Y}))$ that is induced by the action of S on components. In fact, the assignments

$$\mathfrak{X} \to \kappa(\mathfrak{X}), \quad S \to \kappa(S)$$

define a functor which is mentioned and studied in [3] (see also [5]). Since $\kappa(S) = 0$ if and only if S is compact, the action of $\kappa(S)$ is determined by the action of S modulo compact operators.

4.6. DEFINITION. Let $\alpha = (\alpha^p)_{p \in \mathbb{Z}}$ be a collection of closed operators in $\mathfrak{X}$ with $X^p := D(\alpha^p) \in \mathcal{S}(\mathfrak{X})$ (i.e. each α^p is bounded), $R(\alpha^p) \subset X^{p+1}$ for all p and $D(\alpha^p) = 0$ for all but a finite family of indices. We say that α is an essential complex if $\alpha^{p+1}\alpha^p \in \mathcal{L}(X^p, X^{p+2})$ is compact for all $p \in \mathbb{Z}$. Then $\kappa(\alpha) := (\kappa(\alpha^p))_{p\in\mathbb{Z}} \in \partial(\ell^\infty(\mathfrak{X}))$, where $\kappa(\alpha^p) \in \mathcal{L}(\kappa(X^p), \kappa(X^{p+1}))$. The family of all essential complexes in $\mathfrak{X}$ will be denoted by $\partial_e(\mathfrak{X})$. An essential complex α is said to be essentially Fredholm if $H^p(\kappa(\alpha)) = C$ for all $p \in \mathbb{Z}$.

Let us remark that the study of essential complexes requires the general setting (i.e. operators between quotient spaces) from the preceding section.

As in the case of complexes (see eq. (3.7)), the topology of $\partial_e(\mathfrak{X})$ is induced by the function

$$\hat{\delta}(\alpha,\beta) = \max_{p\in\mathbb{Z}} \hat{\delta}(\alpha^p,\beta^p) \qquad , \alpha,\beta \in \partial_e(\mathfrak{X}).$$

We shall show that the set of essentially Fredholm complexes is open in $\partial_e(\mathfrak{X})$ with respect to this topology. We need some auxiliary results.

4.7. LEMMA. <u>Let</u> $X, Y \in \mathcal{S}(\mathfrak{X})$. <u>Then one has the estimates</u>

$$\delta(\ell^\infty(X), \ell^\infty(Y)) \leqq \delta(X,Y),$$
$$\delta(\tau(X), \tau(Y)) \leqq \delta(X,Y).$$

<u>Proof</u>. The first estimate is obvious, so that we shall deal only with the second one.

Let $\delta > \delta(X,Y)$, and let $\{\delta_n\}_n$ and $\{\varepsilon_n\}_n$ be two sequences of positive numbers such that $\delta > \delta_n > \delta(X,Y)$, $\delta_n \to \delta$ as $n \to \infty$, $7\varepsilon_n + \delta_n < \delta_{n+1}$ for all n and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. We can also assume that $\delta \leqq 2$. Let $0 \neq \xi = \{x_k\}_k \in \tau(X)$, and let A be the sequence ξ regarded as a subset of X. Since A is totally bounded, for every integer $n \geqq 1$ there exists a finite subset $A_n \subset A$ such that $d(x, A_n) < \varepsilon_n \|\xi\|$

for all $x \in A$. With no loss of generality we may assume that $A_n = \{x_1, x_2, \ldots, x_{k_n}\}$. We want to construct similar sets in Y. Let $B_1 = \{y_1, \ldots, y_{k_1}\} \subset Y$ be such that $\|x_j - y_j\| < \delta_1 \|x_j\| \leqq \delta_1 \|\xi\|$ for $j = 1, \ldots, k_1$. Assume that the sets $B_1, \ldots, B_n$ have been constructed such that $d(y, B_k) < 6\varepsilon_k \|\xi\|$ for all $y \in B_{k+1}$ and $k = 1, \ldots, n-1$, and for every $u \in A_k$ one can find a vector $v \in B_k$ such that $\|u - v\| < \delta_k \|\xi\|$ for each $k = 1, \ldots, n$. Let us obtain the set B_{n+1}. Let $x_j \in A_{n+1}$ be with $k_n + 1 \leqq j \leqq k_{n+1}$. Then there exists a vector $u_j \in A_n$ such that $\|x_j - u_j\| < \varepsilon_n \|\xi\|$. By the induction hypothesis, there exists $v_j \in B_n$ such that $\|u_j - v_j\| < \delta_n \|\xi\|$. Let also $w_j \in Y$ be such that

$$\|x_j - u_j - w_j\| < \delta_n \|x_j - u_j\| \leqq \varepsilon_n \delta_n \|\xi\|.$$

Then define $y_j := v_j + w_j$, and note that

$$d(y_j, B_n) \leqq \|w_j\| \leqq (1+\delta_n)\|x_j - u_j\| < \varepsilon_n \delta_n (1+\delta_n)\|\xi\| \leqq 6\varepsilon_n \|\xi\|.$$

We also have

$$\|x_j - y_j\| \leqq \|x_j - u_j\| + \|u_j - v_j\| + \|w_j\| \leqq$$

$$\leqq (\varepsilon_n + \delta_n + \varepsilon_n \delta_n (1+\delta_n))\|\xi\| \leqq (7\varepsilon_n + \delta_n)\|\xi\| < \delta_{n+1}\|\xi\|.$$

The sequence $\eta = \{y_k\}_k$ is a totally bounded set. Indeed, if $\varepsilon > 0$ is given, then there is an index n such that $6\|\xi\| \sum_{k \geqq n} \varepsilon_k > \varepsilon$. It is easily seen that $d(y, B_n) < \varepsilon$ for every vector y in the sequence η, by the properties of the sequence $\{B_k\}_k$. We also have

$$\|\xi - \eta\| = \sup_k \|x_k - y_k\| \leqq \delta \|\xi\|,$$

whence we derive the desired estimate.

4.8. LEMMA. Let $S \in \mathcal{L}(X,Y)$ and let $\tilde{S} \in \mathcal{L}(\tilde{X},\tilde{Y})$, where $X,\tilde{X},Y,\tilde{Y} \in \mathcal{S}(\mathfrak{X})$. Then we have the estimate

$$\delta_0(\kappa(S),\kappa(\tilde{S})) \leqq 5(1+\|S\|^2)^{1/2}\max\{\delta(S,\tilde{S}),\delta(Y,\tilde{Y})\}.$$

Proof. Let $\xi \oplus \eta \in G_0(\kappa(S))$. Then $\eta - S\xi \in \kappa(Y)$, where $\xi = \{x_k\}_k \in \ell^\infty(X)$ and $S\xi := \{Sx_k\}_k$. Fix a number $\delta > \delta(S,\tilde{S})$. Then for every index k there exists a vector $\tilde{x}_k \oplus \tilde{S}\tilde{x}_k \in G(\tilde{S})$ such that

$$\|x_k \oplus Sx_k - \tilde{x}_k \oplus \tilde{S}\tilde{x}_k\|^2 < \delta^2\|x_k \oplus Sx_k\|^2 .$$

Fix another number $\delta_1 > \delta(Y,\tilde{Y})$. By Lemma 4.7, we can find a vector $\tilde{\xi} \in \tau(\tilde{Y})$ such that

$$\|\eta - S\xi - \tilde{\xi}\| < \delta_1\|\eta - S\xi\| .$$

If $\tilde{\xi} := \{\tilde{x}_k\}_k$, we set $\tilde{\eta} := \tilde{S}\tilde{\xi} + \tilde{\xi}$. Note that we have

$$\|\xi \oplus \eta - \tilde{\xi} \oplus \tilde{\eta}\|^2 = \sup_k\|x_k - \tilde{x}_k\|^2 + \|\eta - \tilde{S}\tilde{\xi} - \tilde{\xi}\|^2 \leqq$$

$$\leqq \delta^2 \sup_k (\|x_k\|^2 + \|Sx_k\|^2) + (\delta_1\|\eta - S\xi\| + \|S\xi - \tilde{S}\tilde{\xi}\|)^2 \leqq$$

$$\leqq \delta^2(1+\|S\|^2)\|\xi\|^2 + 2[\delta_1^2(1+\|S\|^2)\|\xi\oplus\eta\|^2 + \delta^2(1+\|S\|^2)\|\xi\|^2] \leqq$$

$$\leqq 5(1+\|S\|^2)\max\{\delta^2,\delta_1^2\}\|\xi\oplus\eta\|^2,$$

from which we infer easily the desired estimate.

4.9. LEMMA. Let S and $\tilde{S}$ be as in Lemma 4.8. Then $\delta(X,\tilde{X}) \leqq (1+\|S\|^2)^{1/2}\delta(S,\tilde{S})$. Moreover, if $(1+\|S\|)\delta(\tilde{S},S) < 1$, then

$$\|\tilde{S}\| \leqq \frac{(1+\|S\|)\delta(\tilde{S},S)+\|S\|}{1-(1+\|S\|)\delta(\tilde{S},S)}.$$

Proof. Let $\delta > \delta(S,\tilde{S})$. Then for every $x \in X$ one can find $\tilde{x} \in \tilde{X}$ such that

$$\|x \oplus Sx - \tilde{x} \oplus \tilde{S}\tilde{x}\|^2 < \delta^2\|x \oplus Sx\|^2 .$$

In particular,

$$\|x - \tilde{x}\|^2 < \delta^2(1+\|S\|^2)\|x\|^2,$$

which implies readily the first assertion.

Now, let $\delta > \delta(\tilde{S},S)$ be such that $(1+\|S\|)\delta < 1$. Let also $\tilde{x} \in \tilde{X}$ be such that $\|\tilde{x}\| \leq 1$. Then we choose $x \oplus Sx \in G(S)$ such that

$$\|\tilde{x} \oplus \tilde{S}\tilde{x} - x \oplus Sx\| < \delta \|\tilde{x} \oplus \tilde{S}\tilde{x}\|.$$

From this estimate we derive that

$$\|x\| \leq \|\tilde{x}\| + \delta(\|\tilde{x}\| + \|\tilde{S}\tilde{x}\|) \leq 1 + \delta + \delta\|\tilde{S}\tilde{x}\|.$$

Similarly,

$$\|\tilde{S}\tilde{x}\| \leq \|Sx\| + \delta(\|\tilde{x}\| + \|\tilde{S}\tilde{x}\|) \leq \delta + (1+\delta)\|S\| + \delta(1+\|S\|)\|\tilde{S}\tilde{x}\|.$$

Hence

$$\|\tilde{S}\tilde{x}\| \leq (1-\delta(1+\|S\|))^{-1}(\delta + (1+\delta)\|S\|),$$

which implies the desired estimate.

4.10. PROPOSITION. <u>Let $\alpha = (\alpha^p)_{p\in\mathbb{Z}} \in \partial_e(\mathfrak{X})$. If α is essentially Fredholm, then there exists an $\varepsilon > 0$ such that if $\beta = (\beta^p)_{p\in\mathbb{Z}} \in \partial_e(\mathfrak{X})$ and if $\hat{\delta}(\alpha,\beta) < \varepsilon$, then β is also essentially Fredholm.</u>

<u>Proof.</u> We shall apply Theorem 3.7 to the complex $\kappa(\alpha)$. Let $\|\alpha\| := \max_p \|\alpha^p\| < \infty$, and similarly $\|\beta\| := \max_p \|\beta_p\| < \infty$ (since $\alpha^p \neq 0$ and $\beta^p \neq 0$ only for a finite number of indices). If $(1+\|\alpha\|)\hat{\delta}(\alpha,\beta) < 1$, then, by Lemma 4.9,

$$\|\beta\| \leq \frac{(1+\|\alpha\|)\hat{\delta}(\alpha,\beta) + \|\alpha\|}{1-(1+\|\alpha\|)\hat{\delta}(\alpha,\beta)}.$$

This shows, in particular, that $\|\beta\|$ is bounded if $\hat{\delta}(\alpha,\beta)$ is sufficiently small.

Now, let $X^p := D(\alpha^p)$ and let $Y^p := D(\beta^p)$. According to Lemma 4.8, we have for every $p \in \mathbb{Z}$

$$\hat{\delta}_o(\kappa(\alpha^p),\kappa(\beta^p)) \leq \sqrt{5}(1+\max\{\|\alpha\|,\|\beta\|\})\max\{\hat{\delta}(\alpha,\beta), \hat{\delta}(X^{p+1},Y^{p+1})\}.$$

On the other hand,

$$\hat{\delta}(X^p, Y^p) \leqq (1 + \max\{\|\alpha\|, \|\beta\|\})\hat{\delta}(\alpha,\beta)$$

for all p, by Lemma 4.9.

Therefore, $\hat{\delta}(\kappa(\alpha),\kappa(\beta))$ can be made as small as we want if $\hat{\delta}(\alpha,\beta)$ is sufficiently small. By Theorem 3.7, the complex $\kappa(\beta)$ is Fredholm and $H^p(\kappa(\beta)) = C$ for all p when $\hat{\delta}(\alpha,\beta)$ is sufficiently small. The proof of the proposition is complete.

4.11. <u>Remarks</u>. 1^o A particular case of Proposition 4.10 occurs in [5].

2^o One can define a concept of an essentially semi-Fredholm complex and obtain similar statements. Also complexes of infinite length can be taken into consideration.

3^o We end this discussion with the following question: Is there any way to assign an index to every essentially Fredholm complex?

d) The results of the previous section can be applied to obtain statements concerning the semicontinuity of the joint spectrum (in the sense of [17]) of several commuting operators.

Let $\mathfrak{X}$ be a Banach space and let $a = (a_1,\ldots,a_n) \subset \mathcal{L}(\mathfrak{X})$ be a commuting system. Set

$$\mathrm{Lat}(a) := \{X \in \mathcal{G}(\mathfrak{X});\ a_j X \subset X,\ j=1,\ldots,n\}.$$

For every $X \in \mathrm{Lat}(a)$ let $\sigma(a,X)$ denote the joint spectrum of a, when acting in X. This set is defined in the following way: Let $\sigma = (\sigma_1,\ldots,\sigma_n)$ be a fixed system of indeterminates. Then $\Lambda[\sigma]$ (or $\Lambda^p[\sigma]$) is the exterior algebra over $\mathbb{C}$ generated by $\sigma_1,\ldots,\sigma_n$ (or the space of homogeneous exterior forms of degree p in $\sigma_1,\ldots,\sigma_n$, $0 \leqq p \leqq n$). Then a point $z = (z_1,\ldots,z_n) \notin \sigma(a,X)$

iff the complex $(\delta^p_{z-a})_{p\in\mathbb{Z}}$ is exact, where

$\delta^p_{z-a} := \delta_{z-a}|\Lambda^p[\sigma,X]$, $\Lambda^p[\sigma,X] := X\otimes\Lambda^p[\sigma]$ and

$$\delta_{z-a}\xi = (a_1\otimes\sigma_1+\ldots+a_n\otimes\sigma_n)\wedge\xi$$

for all $\xi\in\Lambda[\sigma,\mathfrak{X}] := \mathfrak{X}\otimes\Lambda[\sigma]$. Here one puts $\delta^p_{z-a} = 0$ if $p<0$ or $p\geq n$ (see [17] or [19] for details).

If $X_o, X\in \mathrm{Lat}(a)$ and $X_o\subset X$, then the system a also acts in X/X_o (the induced action) so that one can define the set $\sigma(a,X/X_o)$ in a similar way. We shall show that the joint spectrum $\sigma(a,X/X_o)$ is semicontinuous in all of its arguments.

Let $b=(b_1,\ldots,b_n)\subset\mathcal{L}(\mathfrak{X})$ be another commuting system. We define a distance between the systems a and b by the formula

$$\|a-b\| := \max_{1\leq j\leq n}\|a_j-b_j\|.$$

4.12. PROPOSITION. <u>Let</u> $a=(a_1,\ldots,a_n)\subset\mathcal{L}(\mathfrak{X})$ <u>be a commuting system and let</u> $X_o, X\in \mathrm{Lat}\ a$ <u>be such that</u> $X_o\subset X$. <u>Then for every open set</u> $U\supset\sigma(a,X/X_o)$, <u>there exists a positive number</u> δ_U <u>such that if</u> $b=(b_1,\ldots,b_n)\subset\mathcal{L}(\mathfrak{X})$ <u>is a commuting system,</u> $Y_o, Y\in \mathrm{Lat}(b)$, $Y_o\subset Y$, $\|a-b\|<\delta_U$, $\hat{\delta}(X_o,Y_o)<\delta_U$ <u>and</u> $\hat{\delta}(X,Y)<\delta_U$, <u>then</u> $\sigma(b,Y/Y_o)\subset U$.

<u>Proof.</u> We note first the inclusion

$$\sigma(a,X/X_o)\subset\prod_{j=1}^{n}\{z_j\in\mathbb{C};\ |z_j|\leq\|a_j\|\},$$

and a similar inclusion for $\sigma(b,Y/Y_o)$ [17]. On account of these inclusions and since $\|b_j\|$ is as close of $\|a_j\|$ as we want if $\|a-b\|$ is sufficiently small, it will be enough to consider the case

$$U\subset B := \{z\in\mathbb{C}^n;\ \|z\|\leq R\},$$

with $R>0$ sufficiently large.

For every $z \in B \setminus U$ the complex $(\delta^p_{z-a})_{p \in \mathbb{Z}}$ is exact (note that here δ^p_{z-a} acts on $\Lambda^p[\sigma, X/X_o]$, which is naturally isomorphic to $\Lambda^p[\sigma,X]/\Lambda^p[\sigma,X_o]$), and hence Fredholm. Let $\varepsilon_z > 0$ be the positive number given by Theorem 3.7 for this Fredholm complex. If $w \in \mathbb{C}^n$ and if

$$\sup_{p \in \mathbb{Z}} \hat{\delta}_o(\delta^p_{z-a}, \delta^p_{w-b}) < \varepsilon_z, \tag{4.6}$$

then the complex $(\delta^p_{w-b})_{p \in \mathbb{Z}}$ is also exact, by virtue of Theorem 3.7. If δ^p_{z-a} and δ^p_{w-b} are regarded as operators on $\Lambda^p[\sigma, \mathfrak{X}]$ (which is isomorphic to a direct sum of $\binom{n}{p}$ copies of $\mathfrak{X}$ and is given the topology induced by this isomorphism), then we have

$$\| \delta^p_{z-a} - \delta^p_{w-b} \| \leq n(\|a-b\| + \|z-w\|),$$

since $\|\sigma_j \wedge \xi\| \leq \|\xi\|$ for all j and ξ. We also have

$$\hat{\delta}(\Lambda^p[\sigma,X], \Lambda^p[\sigma,Y]) \leq \hat{\delta}(X,Y),$$

and a similar formula for X_o and Y_o. Therefore the fulfillment of (4.6) will follow from the following:

4.13. LEMMA. *Let* $S, T \in \mathcal{L}(\mathfrak{X})$, *let* $X_o, X \in \mathrm{Lat}(S)$, *let* $Y_o, Y \in \mathrm{Lat}(T)$ *and let* S_o *and* T_o *be the operators induced by* S *and* T *in* X/X_o *and* Y/Y_o, *respectively. Then we have the estimate*

$$\delta_o(S_o, T_o) \leq 2(1+\|S\|)(\delta(X,Y)+\delta(X_o,Y_o))+4\|S-T\|.$$

Proof of Lemma 4.13. We take $\delta > \delta(X,Y)$ and $\delta_o > \delta(X_o,Y_o)$. If $x \oplus u \in G_o(S_o)$, then $u - Sx \in X_o$. We choose the vectors $y \in Y$ and $w \in Y_o$ such that $\|x-y\| < \delta\|x\|$ and $\|u - Sx - w\| < \delta_o\|u - Sx\|$. Set $v := Ty + w$. Then $y \oplus v \in G_o(T_o)$ and we have

$$\|x\oplus u - y\oplus v\| \leq \|x-y\| + \|u-v\| \leq \delta\|x\| + \|u-Sx-w\| + \|Sx-Ty\| \leq$$
$$\leq \delta\|x\| + \delta_o(\|u\| + \|S\|\,\|x\|) + \delta\|S\|\,\|x\| + (1+\delta)\|S\|\,\|x\| \leq$$
$$\leq 2(\delta(1+\|S\|) + \delta_o(1+\|S\|) + (1+\delta)\|S-T\|)\|x\oplus u\|,$$

which implies the desired estimate.

Returning to the proof of Proposition 4.12, if $\|a-b\|$, $\|z-w\|$, $\hat{\delta}(X,Y)$ and $\hat{\delta}(X_o,Y_o)$ are sufficiently small, then (4.6) is fulfilled and we can apply Theorem 3.7 and obtain that the complex $(\delta^p_{w-b})_{p\in\mathbb{Z}}$ is exact if w is in a neighbourhood of z. Since $B\setminus U$ is compact, it can be covered with a finite number of such neighbourhoods, and (4.6) must be fulfilled only for a finite set of points z. Consequently, if $\|a-b\|$, $\hat{\delta}(X,Y)$ and $\hat{\delta}(X_o,Y_o)$ are sufficiently small, then the complex $(\delta^p_{w-b})_{p\in\mathbb{Z}}$ is exact for every $w\in B\setminus U$, so that $\sigma(b,Y/Y_o)\subset U$.

Using the functor considered in the previous example, we can give the following:

4.14. DEFINITION [5]. Let $a=(a_1,\dots,a_n)\subset\mathcal{L}(\mathfrak{X})$ be an essentially commuting system (i.e. $\kappa(a)$ is a commuting system in $\mathcal{L}(\kappa(\mathfrak{X}))$ and let $X\in \mathrm{Lat}(a)$. We define the essential joint spectrum of a in X by the formula

$$\sigma_{ess}(a,X) := \sigma(\kappa(a), \kappa(X)),$$

where $\kappa(a) := (\kappa(a_1),\dots,\kappa(a_n))$.

A version of Proposition 4.12 can be also stated for the essential joint spectrum.

4.15. PROPOSITION. <u>Let</u> $a=(a_1,\dots,a_n)\subset\mathcal{L}(\mathfrak{X})$ <u>be an essentially commuting system, and let</u> $X\in\mathrm{Lat}(a)$. <u>Then for every open set</u> $U\supset\sigma_{ess}(a,X)$ <u>there is a positive number</u> δ_U <u>such that if</u> $b=(b_1,\dots,b_n)\subset\mathcal{L}(\mathfrak{X})$ <u>is an essentially commuting system,</u> $Y\in\mathrm{Lat}(b)$, $\|a-b\|<\delta_U$ <u>and</u> $\hat{\delta}(X,Y)<\delta_U$, <u>then</u> $U\supset\sigma_{ess}(b,Y)$.

Proof. Since $U \supset \sigma(\kappa(a),\kappa(X))$, we may try to apply Proposition 4.12 to the system of operators $\dot{a} = (a_1,\dots,a_n)$ when acting in $\ell^\infty(\mathfrak{X})$ (on components), and to the invariant subspaces $X^\infty := \ell^\infty(X)$ and $X_o^\infty := \tau(X)$. Let also $Y^\infty := \ell^\infty(Y)$ and $Y_o^\infty := \tau(Y)$. Since $\|a-b\|_\infty \leq \|a-b\|$ ($\|a-b\|_\infty$ is computed in $\ell^\infty(\mathfrak{X})$), $\hat{\delta}(X^\infty,Y^\infty) \leq \hat{\delta}(X,Y)$ and $\hat{\delta}(X_o^\infty,Y_o^\infty) \leq \hat{\delta}(X,Y)$ (both estimates follow by Lemma 4.7), on account of Proposition 4.12 we deduce that if $\|a-b\|$ and $\hat{\delta}(X,Y)$ are sufficiently small, then

$$U \supset \sigma(\kappa(b),\kappa(Y)) = \sigma_{ess}(b,Y).$$

e) Complexes of infinite length occur in the cohomology theory of Banach algebras. We shall use the notations of B. E. Johnson's survey article in [21]. Let $\mathfrak{A}$ and $\mathfrak{X}$ be two Banach spaces and consider the situation $(A,Q,X,P,\rho,\mu_L,\mu_R)$ where A resp. X is a complemented closed subspace of $\mathfrak{A}$ resp. $\mathfrak{X}$, $Q \in \mathcal{L}(\mathfrak{A})$ resp. $P \in \mathcal{L}(\mathfrak{X})$ is a projection with $R(Q) = A$ resp. $R(P) = X$, $\rho: A \times A \to A$ is a continuous associative multiplication on A which turns A into a Banach algebra, and $\mu_L: A \times X \to X$ resp. $\mu_R: X \times A \to X$ are left resp. right A-module multiplications such that X becomes a two-sided Banach A-module. We also shall consider a perturbed situation $(\tilde{A},\tilde{Q},\tilde{X},\tilde{P},\tilde{\rho},\tilde{\mu}_L,\tilde{\mu}_R)$. We define the continuous bilinear mappings $B: \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$, $M_L: \mathfrak{A} \times \mathfrak{X} \to \mathfrak{X}$, and $M_R: \mathfrak{X} \times \mathfrak{A} \to \mathfrak{X}$ by $B(a,b) := \rho(Qa,Qb)$, $M_L(a,x) := \mu_L(Qa,Px)$, and $M_R(x,a) := \mu_R(Px,Qa)$ for $a,b \in \mathfrak{A}$, $x \in \mathfrak{X}$. The corresponding mappings in the perturbed situation will be denoted by $\tilde{B},\tilde{M}_L,\tilde{M}_R$.

4.16. PROPOSITION. <u>Suppose that the Hochschild cohomology groups $\mathcal{H}^p(A,X)$ are finite dimensional for $0 \leq p \leq n$ $(n \in \mathbb{N}_o)$ and that the coboundary operator $S^{n+1}: \mathcal{L}^n(A,X) \to \mathcal{L}^{n+1}(A,X)$ has closed</u>

range. Then there exists an $\varepsilon=\varepsilon(n)>0$ such that for all perturbed situations with

$$\|P-\tilde{P}\|<\varepsilon, \|Q-\tilde{Q}\|<\varepsilon, \|B-\tilde{B}\|<\varepsilon, \|M_L-\tilde{M}_L\|<\varepsilon, \|M_R-\tilde{M}_R\|<\varepsilon \qquad (4.7)$$

the coboundary operator $\tilde{S}^{n+1}:\mathcal{L}^n(\tilde{A},\tilde{X})\to\mathcal{L}^{n+1}(\tilde{A},\tilde{X})$ of the perturbed situation has closed range and such that for $0\le p\le n$,

$$\dim\mathcal{H}^p(\tilde{A},X)\le\dim\mathcal{H}^p(A,X).$$

Proof. First let us notice that each $T\in\mathcal{L}^p(A,X)$ has a canonical extension to a p-linear mapping from $\mathfrak{A}^p$ to $\mathfrak{X}$ given by

$$(a_1,\dots,a_p)\to T(Qa_1,\dots,Qa_p)=: T_e(a_1,\dots,a_p)$$

for $a_1,\dots,a_p\in\mathfrak{A}$. In this way we may consider $\mathcal{L}^p(A,X)$ as a closed linear subspace of $\mathcal{L}^p(\mathfrak{A},\mathfrak{X})$. The same can be done in the perturbed situation. For all $p\in\mathbb{N}$, $T\in\mathcal{L}^p(A,X)$ we then have $S^p(T)_e=S^p_e(T_e)$, where $S^p_e:\mathcal{L}^{p-1}(\mathfrak{A},\mathfrak{X})\to\mathcal{L}^p(\mathfrak{A},\mathfrak{X})$ is defined by

$$D(S^p_e)=\{T_e\,|\,T\in\mathcal{L}^p(\mathfrak{A},X)\} \quad\text{and}$$

$$\begin{aligned}(S^p_e(T_e))(a_1,\dots,a_p) = {}& M_L(a_1,T(Qa_2,\dots,Qa_p))-T(B(a_1,a_2),Qa_3,\dots,Qa_p)\\ &+T(Qa_1,B(a_2,a_3),Qa_4,\dots,Qa_p)+\dots\\ &+(-1)^{p-1}T(Qa_1,\dots,Qa_{p-2},B(a_{p-1},a_p))+\\ &+(-1)^p M_R(T(Qa_1,\dots,Qa_{p-1}),a_p)\end{aligned}$$

for $T\in\mathcal{L}^p(A,X)$ and $a_1,\dots,a_p\in\mathfrak{A}$. Thus, instead of considering the complexes $(S^p)_{p\in\mathbb{N}}$ and $(\tilde{S}^p)_{p\in\mathbb{N}}$, we now work with $(S^p_e)_{p\in\mathbb{N}}$ and $(\tilde{S}^p_e)_{p\in\mathbb{N}}$. Then $G(S^p_e)$ and $G(\tilde{S}^p_e)$ are both contained in $\mathcal{L}^{p-1}(\mathfrak{A},\mathfrak{X})\times\mathcal{L}^p(\mathfrak{A},\mathfrak{X})$. The result will now follow from Proposition 2.10, if we show that $\hat{\delta}(S^p_e,\tilde{S}^p_e)$ $(p=1,\dots,n+1)$ is as small as we desire if ε is small enough.

Fix an arbitrary $T_e\in D(S^p_e)$. We define $\tilde{T}_e\in\mathcal{L}^{p-1}(\mathfrak{A},\mathfrak{X})$ by $\tilde{T}_e(a_1,\dots,a_{p-1}):=\tilde{P}\,T_e(\tilde{Q}a_1,\dots,\tilde{Q}a_{p-1})$ for $a_1,\dots,a_{p-1}\in\mathfrak{A}$. Clearly, $\tilde{T}_e\in D(\tilde{S}^p_e)$ and for $a_1,\dots,a_{p-1}\in\mathfrak{A}$ we have

$$\|(T_e-\tilde{T}_e)(a_1,\ldots,a_{p-1})\| \leq$$

$$\leq \|(P-\tilde{P})T_e(a_1,\ldots,a_{p-1})\| +$$

$$+ \|\tilde{P}\sum_{j=1}^{p-1} T_e(a_1,\ldots,a_{j-1},(Q-\tilde{Q})a_j,\tilde{Q}a_{j+1},\ldots,\tilde{Q}a_{p-1})\|$$

$$\leq \varepsilon\|T_e\|\cdot\|a_1\|\cdot\ldots\cdot\|a_{p-1}\|(1+\|\tilde{P}\|\sum_{j=0}^{p-2}\|\tilde{Q}\|^j) .$$

This shows that for $\varepsilon<1$ we have

$$\|T_e-\tilde{T}_e\| \leq \varepsilon C_p\|T_e\| \tag{4.8}$$

where $C_p=1+(\|P\|+1)\sum_{j=0}^{p-2}(\|Q\|+1)^j$ does not depend on T or ε.

Moreover,

$$(S_e^p(T_e)-\tilde{S}_e^p(\tilde{T}_e))(a_1,\ldots,a_p) =$$

$$= M_L(a_1,T_e(a_2,\ldots,a_p))-\tilde{M}_L(a_1,\tilde{T}_e(a_2,\ldots,a_p)) -$$

$$- (T_e(B(a_1,a_2),a_3,\ldots,a_p)-\tilde{T}_e(\tilde{B}(a_1,a_2),a_3,\ldots,a_p))+\ldots+$$

$$+ (-1)^{p-1}(T_e(a_1,\ldots,a_{p-2},B(a_{p-1},a_p))-\tilde{T}_e(a_1,\ldots,a_{p-2},\tilde{B}(a_{p-1},a_p)))+$$

$$+ (-1)^p(M_R(T_e(a_1,\ldots,a_{p-1}),a_p)-\tilde{M}_R(\tilde{T}_e(a_1,\ldots,a_{p-1}),a_p) .$$

From this, (4.7), and (4.8) we see easily that there is some constant $C'_p>0$ not depending on T_e or ε, such that

$$\|S_e^p(T_e)-\tilde{S}_e^p(\tilde{T}_e)\| \leq \varepsilon\|T_e\|\cdot C'_p$$

This and (4.8) now imply that $\delta(S_e^p,\tilde{S}_e^p) \leq \varepsilon\sqrt{C_p^2+C_p'^2}$. In the same way one obtains $\delta(\tilde{S}_e^p,S_e^p) \leq \varepsilon C''_p$ for some constant C''_p.

(f) We end this section with a second simple example involving complexes of infinite length.

For $p\in\mathbb{N}$ we denote by $\mathfrak{S}_p$ the group of all permutations of $\{1,\ldots,p\}$. If K is a set and $k=(k_1,\ldots,k_p)\in K^p$ then we define for $\pi\in\mathfrak{S}_p$: $\pi^*(k):=(k_{\pi(1)},\ldots,k_{\pi(p)})$. Let now Ω be a compact Hausdorff space and denote by $\Lambda^p(\Omega,\mathfrak{X})$ the space of all continuous functions $f:\Omega^p\to\mathfrak{X}$ that are antisymmetric, that is, such that

$$f(\pi^*(\omega)) = f(\omega_{\pi(1)},\dots,\omega_{\pi(p)}) = \operatorname{sgn}\pi \cdot f(\omega)$$

for all $\omega=(\omega_1,\dots,\omega_p)\in\Omega^p$ and all $\pi\in\mathfrak{S}_p$. It is clear that $\Lambda^p[\Omega,\mathfrak{X}]$ is a Banach space with respect to the norm $\|f\| := \sup\{\|f(\omega)\| : \omega\in\Omega^p\}$. We also define $\Lambda^0[\Omega,\mathfrak{X}] := \mathfrak{X}$ and $\Lambda^p[\Omega,\mathfrak{X}] := \{0\}$ for $p<0$. Let now $a:\Omega\to\mathcal{L}(\mathfrak{X})$ be a continuous function with commuting range (i.e. whose range is contained in a commutative subalgebra of $\mathcal{L}(\mathfrak{X})$). We define continuous linear operators

$$\alpha^p(a):\Lambda^p[\Omega,\mathfrak{X}]\to\Lambda^{p+1}[\Omega,\mathfrak{X}]$$

by $\alpha^p(a)=0$ for $p<0$, $(\alpha^0(a)x)(\omega) := a(\omega)x$ for $x\in\mathfrak{X}, \omega\in\Omega$, and

$$(\alpha^p(a)f)(\omega) := \frac{1}{p+1}\sum_{k=1}^{p+1}(-1)^{k+1}a(\omega_k)f(\omega_1,\dots,\hat{\omega}_k,\dots,\omega_{p+1})$$

for $f\in\Lambda^p[\Omega,\mathfrak{X}]$ and $\omega=(\omega_1,\dots,\omega_{p+1})\in\Omega^{p+1}$. One can easily see that $\alpha^{p+1}(a)\alpha^p(a)=0$ for all $p\in\mathbb{Z}$, so that $(\Lambda^p[\Omega,\mathfrak{X}],\alpha^p(a))_{p\in\mathbb{Z}}$ is a complex of Banach spaces (in general of infinite length).

Let us denote by $\operatorname{Lat} a$ the family of all $X\in\mathcal{S}(\mathfrak{X})$ such that $a(\omega)X\subset X$ for all $\omega\in\Omega$. For $X\in\operatorname{Lat} a$, we have $\alpha^p(a)\Lambda^p[\Omega,X]\subset\Lambda^{p+1}[\Omega,X]$, so that we may also consider the complex $(\Lambda^p[\Omega,X],\alpha^p(a,X))_{p\in\mathbb{Z}}$ where $\alpha^p(a,X) := \alpha^p(a)|\Lambda^p[\Omega,X]$.

<u>DEFINITION</u>. The function a is said to be of semi-Fredholm type on $X\in\operatorname{Lat} a$ if the associated complex $(\Lambda^p[\Omega,X],\alpha^p(a,X))_{p\in\mathbb{Z}}$ is semi-Fredholm. In this case we define the index of a on X, say $\operatorname{ind}_X a$, as the index of $(\Lambda^p[\Omega,X],\alpha^p(a,X))_{p\in\mathbb{Z}}$.

4.17. PROPOSITION. <u>Let $a:\Omega\to\mathcal{L}(\mathfrak{X})$ be of semi-Fredholm type on $X\in\operatorname{Lat} a$. If $\tilde{a}:\Omega\to\mathcal{L}(\mathfrak{X})$ is also a continuous function with commuting range and if the numbers</u>

$\|a-\tilde{a}\| = \sup_{\Omega\in\Omega}\|a(\omega)-\tilde{a}(\omega)\|$ <u>and</u> $\hat{\delta}(X,\tilde{X})$

are sufficiently small, where $\tilde{X} \in \text{Lat}\,\tilde{a}$, then $\tilde{a}$ is of semi-Fredholm type on $\tilde{X}$ and $\text{ind}_{\tilde{X}}\tilde{a} = \text{ind}_X a$.

PROOF. Clearly $\hat{\delta}(\alpha^p(a,X),\alpha^p(\tilde{a},\tilde{X})) = 0$ for $p<0$ and a direct computation shows

$$\hat{\delta}(\alpha^0(a,X),\alpha^0(\tilde{a},\tilde{X})) \leq (1+\hat{\delta}(X,\tilde{X}))(\|a-\tilde{a}\|+(1+\|a\|)\hat{\delta}(X,\tilde{X})).$$

Fix now $p \geq 1$. For $f \in \Lambda^p[\Omega,X]$, $\tilde{f} \in \Lambda^p[\Omega,\tilde{X}]$ we have

$$\|\alpha^p(a,X)f-\alpha^p(\tilde{a},\tilde{X})\tilde{f}\| \leq \|a\|\cdot\|f-\tilde{f}\|+\|a-\tilde{a}\|\cdot\|\tilde{f}\| \leq$$
$$\leq \|a\|\cdot\|f-\tilde{f}\|+\|a-\tilde{a}\|(\|f\|+\|f-\tilde{f}\|).$$

From this we obtain

$$\hat{\delta}(\alpha^p(a,X),\alpha^p(\tilde{a},\tilde{X})) \leq (1+\|a\|)\hat{\delta}_p+\|a-\tilde{a}\|(1+\hat{\delta}_p), \tag{4.9}$$

where $\hat{\delta}_p := \hat{\delta}(\Lambda^p[\Omega,X],\Lambda^p[\Omega,\tilde{X}])$. We shall now prove

$$\hat{\delta}_p \leq \hat{\delta}(X,\tilde{X}) \quad \text{for all } p \in \mathbb{N}. \tag{4.10}$$

Fix an arbitrary $\delta > \hat{\delta}(X,\tilde{X})$. Then there are $\delta',\rho > 0$ such that $\hat{\delta}(X,\tilde{X}) < \delta' < \delta'+\rho < \delta$. If $f \in \Lambda^p[\Omega,X]$, then by the continuity of f and the compactness of Ω we find a finite open covering $\{U_1,\ldots,U_n\}$ of Ω such that for all $i=(i_1,\ldots,i_p) \in \mathbb{N}_n^p$ (with $\mathbb{N}_n := \{1,\ldots,n\}$) and all $\omega,\omega' \in U_{i_1}\times\ldots\times U_{i_p}$ we have

$$\|f(\omega)-f(\omega')\| \leq \rho\|f\|.$$

For each $i \in \mathbb{N}_n^p$ we fix now an arbitrary point $\omega^i = (\omega_1^i,\ldots,\omega_p^i) \in U_{i_1}\times\ldots\times U_{i_p}$. As $\{U_1,\ldots,U_n\}$ is an open covering of Ω there exist continuous functions $\varphi_j:\Omega \to [0,1]$ with $\text{supp}\,\varphi_j \subset U_j$ $(j=1,\ldots,n)$ and $\varphi_1+\ldots+\varphi_n \equiv 1$ on Ω. By the choice of δ', there exist for each $i \in \mathbb{N}_n^p$ elements $\tilde{x}^i \in X$ such that $\|\tilde{x}^i - f(\omega^i)\| \leq \delta'\|f(\omega^i)\| \leq \delta'\|f\|$. We define now a continuous function $\tilde{f}:\Omega^p \to \tilde{X}$ by

$$\tilde{f}(\omega) := \frac{1}{p!}\sum_{\pi\in\mathfrak{S}_p} \text{sgn}\,\pi \sum_{i\in\mathbb{N}_n^p} \varphi_{i_1}(\omega_1)\cdot\ldots\cdot\varphi_{i_p}(\omega_p)\tilde{x}^{\pi^*(i)}$$

for $\omega=(\omega_1,\dots,\omega_p)\in\Omega^p$. Let us show that $\tilde{f}\in\Lambda^p[\Omega,\tilde{X}]$.
For $\sigma\in\mathfrak{S}_p$ we have

$$\tilde{f}(\sigma^*(\omega))=\frac{1}{p!}\sum_{\pi\in\mathfrak{S}_p}\operatorname{sgn}\pi\sum_{i\in\mathbb{N}_n^p}\varphi_{i_1}(\omega_{\sigma(1)})\cdot\ldots\cdot\varphi_{i_p}(\omega_{\sigma(p)})\tilde{x}^{\pi^*(i)}$$

$$=\frac{1}{p!}\sum_{\pi\in\mathfrak{S}_p}\operatorname{sgn}\pi\sum_{i\in\mathbb{N}_n^p}\varphi_{i_{\sigma^{-1}(1)}}(\omega_1)\cdot\ldots\cdot\varphi_{i_{\sigma^{-1}(p)}}(\omega_p)\tilde{x}^{(\pi\sigma)^*((\sigma^{-1})^*(i))}$$

$$=\frac{1}{p!}\sum_{\pi\in\mathfrak{S}_p}\operatorname{sgn}(\pi\sigma)\operatorname{sgn}(\sigma^{-1})\sum_{j\in\mathbb{N}_n^p}\varphi_{j_1}(\omega_1)\cdot\ldots\cdot\varphi_{j_p}(\omega_p)\tilde{x}^{(\pi\sigma)^*(j)}$$

$$=\operatorname{sgn}\sigma\cdot\frac{1}{p!}\sum_{\tau\in\mathfrak{S}_p}\operatorname{sgn}\tau\sum_{j\in\mathbb{N}_n^p}\varphi_{j_1}(\omega_1)\cdot\ldots\cdot\varphi_{j_p}(\omega_p)\tilde{x}^{\tau^*(j)}=$$

$$=\operatorname{sgn}\sigma\cdot\tilde{f}(\omega),$$

where we used $\operatorname{sgn}\sigma=\operatorname{sgn}\sigma^{-1}$ and the fact that the mappings $\pi\to\pi\sigma$ from $\mathfrak{S}_p$ to $\mathfrak{S}_p$ and $\sigma^{-1*}:\mathbb{N}_n^p\to\mathbb{N}_n^p$ are bijective. Hence, we have $\tilde{f}\in\Lambda^p[\Omega,\tilde{X}]$.

Fix now $\omega\in\Omega^p$. As f is antisymmetric and
$$\sum_{j\in\mathbb{N}_n^p}\varphi_{j_1}(\omega_1)\cdot\ldots\cdot\varphi_{j_p}(\omega_p)=1$$
we obtain

$$f(\omega)=\sum_{j\in\mathbb{N}_n^p}\varphi_{j_1}(\omega_1)\cdot\ldots\cdot\varphi_{j_p}(\omega_p)f(\omega)=$$

$$=\sum_{j\in\mathbb{N}_n^p}\varphi_{j_1}(\omega_1)\cdot\ldots\cdot\varphi_{j_p}(\omega_p)\frac{1}{p!}\sum_{\pi\in\mathfrak{S}_p}\operatorname{sgn}\pi\, f(\pi^*(\omega))=$$

$$=\frac{1}{p!}\sum_{\pi\in\mathfrak{S}_p}\operatorname{sgn}\pi\sum_{j\in\mathbb{N}_n^p}\varphi_{j_1}(\omega_1)\cdot\ldots\cdot\varphi_{j_p}(\omega_p)f(\pi^*(\omega)),$$

so that

$$\tilde{f}(\omega)-f(\omega)=\frac{1}{p!}\sum_{\pi\in\mathfrak{S}_p}\operatorname{sgn}\pi\sum_{j\in\mathbb{N}_n^p}\varphi_{j_1}(\omega_1)\cdot\ldots\cdot\varphi_{j_p}(\omega_p)(\tilde{x}^{\pi^*(j)}-f(\pi^*(\omega))) \quad (4.11)$$

For $\omega=(\omega_1,\ldots,\omega_p)\in\operatorname{supp}\varphi_{j_1}\times\ldots\times\operatorname{supp}\varphi_{j_p}\subset U_{j_1}\times\ldots\times U_{j_p}$ and $\pi\in\mathfrak{S}_p$ we have

$$\pi^*(\omega)=(\omega_{\pi(1)},\ldots,\omega_{\pi(p)})\in U_{j_{\pi(1)}}\times\ldots\times U_{j_{\pi(p)}}.$$

On the other hand, also

$$\omega^{\pi^*(j)}\in U_{j_{\pi(1)}}\times\ldots\times U_{j_{\pi(p)}}$$

so that

$$\|x^{\pi^*(j)}-f(\pi^*(\omega))\|\le\|x^{\pi^*(j)}-f(\omega^{\pi^*(j)})\|+\|f(\omega^{\pi^*(j)})-f(\pi^*(\omega))\|\le$$

$$\le\delta'\|f\|+\rho\|f\|\le\delta\|f\|.$$

Hence we obtain from (4.11),

$$\|\tilde{f}(\omega)-f(\omega)\|\le\frac{1}{p!}\sum_{\pi\in\mathfrak{S}_p}|\operatorname{sgn}\pi|\sum_{j\in\mathbb{N}_n^p}\varphi_{j_1}(\omega_1)\cdot\ldots\cdot\varphi_{j_p}(\omega_p)\delta\|f\|=\delta\|f\|$$

and therefore,

$\|\tilde{f}-f\|\le\delta\|f\|$. As δ was arbitrary with $\delta>\hat{\delta}(X,\tilde{X})$, we see that

$$\delta(\Lambda^p[\Omega,X],\Lambda^p[\Omega,\tilde{X}])\le\hat{\delta}(X,\tilde{X}).$$

In the same way one obtains

$$\delta(\Lambda^p[\Omega,\tilde{X}],\Lambda^p[\Omega,X])\le\hat{\delta}(X,\tilde{X})$$

and (4.10) is proved.

Now, because of (4.9) and (4.10) we see that

$$\hat{\delta}(\alpha(a,X),\alpha(\tilde{a},\tilde{X}))=\sup_{p\in\mathbb{Z}}\hat{\delta}(\alpha^p(a,X),\alpha^p(\tilde{a},\tilde{X}))$$
$$\le(1+\|a\|)\hat{\delta}(X,\tilde{X})+(1+\delta(X,\tilde{X}))\|a-\tilde{a}\|,$$

since this is smaller than a given $\varepsilon>0$ for $\hat{\delta}(X,\tilde{X})$ and $\|a-\tilde{a}\|$ small enough, we obtain the statement of the proposition now from Theorem 3.5 resp. 3.7.

REFERENCES

1. S. BANACH, Théories des opérations linéaires, Monografie Matematyczne, Warszawa-Lwów, 1932.

2. R. G. BARTLE and L. M. GRAVES, Mappings between function spaces, Trans. Amer. Math. Soc. 72 (1952), 400-413.

3. J. J. BUONI, R. HARTE, and T. WICKSTEAD, Upper and lower Fredholm spectra, Proc. Amer. Math. Soc. 66 (1977), 309-314.

4. R. E. CURTO, Fredholm and invertible tuples of bounded linear operators, Thesis, State University of New York at Stony Brook, 1978.

5. A. S. FAINSTEIN, A stability of Fredholm complexes of Banach spaces with respect to perturbations which are small in q-norm, (in Russian) Izv. Akad. Nauk Az. SSR, Ser. Fiz.-Tekh. Mat. Nauk 1980, No. 1 (1980), 3-8.

6. G. B. FOLLAND and J. J. KOHN, The Neumann problem for the Cauchy-Riemann complex, Princeton University Press, Princeton, New Jersey, 1972.

7. G. HENKIN, Uniform estimates for solutions of the $\overline{\partial}$-problem in a Weil domain, (in Russian) Uspekhi Mat. Nauk 26 (1971), 211-217.

8. T. KATO, Perturbation theory for linear operators, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

9. N. KERZMAN, Hölder and L^p-estimates for solutions of $\overline{\partial}u = f$ in strongly pseudoconvex domains. Comm. Pure Appl. Math. 24 (1971), 301-379.

10. I. LIEB, Beschränktheitsaussagen für den d"-Operator, Nachr. der Akad. der Wissensch. Göttingen, Math.-Phys. Kl. (1970), 1-7.

11. G. NEUBAUER, Über den Index abgeschlossener Operatoren in Banachräumen. Math. Ann. 160 (1965), 93-130.

12. G. NEUBAUER, Fredholm links and perturbation theorems, Studia Math. 38 (1970), 459.

13. N. ØVRELID, Integral representation formulas and L^p-estimates for the $\overline{\partial}$-equation Math. Scand. 29 (1971), 137-160.

14. R. S. PALAIS et al., Seminar on the Atiyah-Singer index theorem. Princeton University Press, Princeton, New Jersey, 1965.

15. A. PIETSCH, Operator ideals, North-Holland, Amsterdam-New York-Oxford, 1980.

16. M. A. ŠUBIN, Pseudo-differential operators and spectral theory (in Russian), Nauka, Moskow 1978.

17. J. L. TAYLOR, A joint spectrum for several commuting operators, J. Functional Analysis 6 (1970), 172-191.

18. D. TIMOTIN, Remark on the Bartle-Graves Theorem, Preprint.

19. F.-H. VASILESCU, Stability of the index of a complex of Banach spaces, J. Operator Theory 2 (1979), 247-275.

20. F.-H. VASILESCU, Analytic functional calculus and spectral decompositions, D. Reidel Publishing Company, Dordrecht and Editura Academiei, Bucureşti 1982.

21. J. H. WILLIAMSON (Ed.), Algebras in analysis, Academic Press, London-New York-San Francisco 1975.

22. M. G. ZAIDENBERG, S. G. KREĬN, P. A. KUČMENT, A. A. PANKOV, Banach bundles and linear operators, Russian Math. Surveys 30 (1975), 115-175.