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Alexandru Buium

0. Introduction.

Throughout this paper, the ground field is the complex field $\mathbb{C}$. By curve we will always mean a Cohen-Macaulay quasi-projective scheme over $\mathbb{C}$ of pure dimension one.

It is well known [1,7] that if C is a reduced projective curve embedded in a smooth projective 3-fold V such that C is locally embeddable in smooth surfaces, then C globally lies on a smooth surface contained in V.

Note that the above turns out to be false if one does not suppose C is reduced, even if the support of C is assumed to be smooth connected (see [3]). So nilpotents on C may impose singularities for the surfaces containing C. The aim of this paper is to get a picture of these singularities. We will restrit ourselves to the case when the support of C is smooth, connected. We prove:

THEOREM. Let C be a projective curve embedded in a smooth projective 3-fold. Suppose $D=C_{red}$ is smooth, connected and C has a 2-dimensional Zariski tangent space at each closed point. Let H be a very ample divisor on V and let $|mH-C|$ be the linear system of surfaces in $|mH|$ passing through C. Let n be the multiplicity of the local ring of C at some (or equivalently, any) closed point of C, and put $2t=(D_1^2)_Z-(D_2^2)_Z$ where $Z=\mathbb{P}(\check{N}_{D/V})$, $\mathcal{O}_Z(D_1)=\mathcal{O}_Z(1)$ and D_2 is the section of $Z \longrightarrow D$ defined by the injection of vector bundles $T_C/T_D \subset T_V/T_D=N_{D/V}$.

Then there exists an integer m_o such that for any $m \geqslant m_o$ there is a Zariski open subset $U \neq \phi$ of $|mH-C|$ with the property that for any $S \in U$,

1) S has only A_{n-1} singularities.
2) The number of these singularities is $m(H.D)_V+t$.
3) C is a Cartier divisor on S.

Let us make some remarks:

(0.1) Since in our Theorem, $m(H.D)_V+t>0$ for $m \gg 0$, it follows that S is always singular for $m \gg 0$. So the Theorem appears as an analogus of a result of Severi [9] (see also [6,10]) which says that hypersurfaces F of sufficiently high degree containing a smooth subvariety X of dimension $k \geqslant N/2$ in $\mathbb{P}^N$ must have singularities, the singular locus of F having dimension $2k-N$ and a degree which is computable in terms of $\deg(F)$ and some invariants of X.

(0.2) The Theorem is also analogus toa result of Bloch [4] saying that if X is a smooth subvariety of a smooth variety

$Y \subset \mathbb{P}^N$ and $\dim(Y)=2\dim(X)$ then the generic hypersurface section of Y containing X has only ordinary double points.

(0.3) One could prove the above theorem in the following way: First prove a local analogus of the theorem (which seems in fact to be classical cf. [6 p.17]) and then globalize it with the same method used in proving the Bertini type theorems in [1]. This method is probably easier then the one we adopted in our paper. The reason we chose a longer way is that this way gives a quite nice picture of the simultaneous desingularization of the generic surfaces S passing through C. We think that in fact the real interest for the theorem lies in this picture which will be implicitely described in the proof.

The rest of this paper is devoted to the proof of the theorem.

I am endebted to C.Bănică and F.Catanese for several valuable discussions and fruitful suggestions.

1. The main construction.

Start with the following

(1.1) LEMMA. Let $\delta \subset F \subset W$ be closed imersions of smooth connected quasi-projective varieties, where δ is a curve, F is a surface and W is a 3-fold. Let $W_1 \longrightarrow W$ be the blowing up of W along δ, Z the exceptinal locus, F_1 the proper transform of F and δ_1 the curve cut by F_1 on Z. Define γ to be the Cartier divisor $(k+1)\delta$ on F and γ_1 to be the Cartier divisor $k\delta_1$ on F_1 regarded as schemes, where $k \geq 1$ is some integer. Let $G \subset W$ be an irreducible normal surface distinct from F and G_1 its proper transform on W_1. The following are equivalent:

1) G contains γ.
2) G_1 contains γ_1.

Proof. Take a generic $p \in \delta$, denote by p_1 the corresponding point on δ_1, take a germ R of smooth surface in W passing through p and transversal to δ and denote by R_1 its strict transform on W_1. It turns out that the local intersection numbers $(F \frown R.G \frown R)_{R,p}$ and $(F_1 \frown R_1.G_1 \frown R_1)_{R_1,p_1}$ differ precisely by 1 which leads to the conclusion of our lemma. We omit details.

Now suppose we are in the hypothesis of the Theorem. We perform the following construction. Cover C with Zariski open sets $W_i \subset V$ such that there exist smooth surfaces $F_i \subset W_i$ having the property that $C \cap W_i = n(D \cap W_i)$ as Cartier divisors on F_i. Let $f_1 : V_1 \longrightarrow V_o = V$ be the blowing up of V_o along $\delta_o = D$. Let Z_1 be the exceptional locus of f_1, F_{1i} the strict transforms of F_i via $f_1^{-1}(W_i) = W_{1i} \longrightarrow W_i$ and $\delta_{1i} = F_{1i} \cap Z_1$. By Lemma (1.1) δ_{1i} stick together giving a curve δ_1. Let $f_2 : V_2 \longrightarrow V_1$ be the blowing up of V_1 along δ_1, Z_2 the exceptional locus of f_2, F_{2i} the strict transforms of F_{1i} via $f_2^{-1}(W_{1i}) = W_{2i} \longrightarrow W_{1i}$ and $\delta_{2i} = F_{2i} \cap Z_2$. Again by Lemma (1.1), δ_{2i} stick together giving a curve δ_2. Now since F_{1i} are transversal to Z_1 it follows that δ_2 does not meet the proper transform Z_{12} of Z_1 on V_2. Iterating the above procedure we obtain a sequence of morphisms

$$V_n \xrightarrow{f_n} V_{n-1} \longrightarrow \cdots \longrightarrow V_1 \xrightarrow{f_1} V_o$$

and curves δ_k on V_k, $0 \leqslant k \leqslant n-1$, such that f_{k+1} is the blowing up of V_k along δ_k. Denote by Z_{k+1} the exceptional locus of f_{k+1} and by Z_{js} the strict transform of Z_j on V_s for $j < s$. It follows that $\delta_s \cap Z_{js} = \phi$ for $j < s$; in particular Z_{jn} are isomorphic to Z_j for any $j = 1,2,\ldots,n-1$. Note also that Lemma (1.1) implies the independence of the above construction from the choice of W_i and F_i.

2. The linear system $|T_m|$.

In the notations of §1 put $\psi_k = f_{k+1}f_{k+2}\dots f_n$ $(\psi_n = id)$, and consider the linear system on V_n :

$$|T_m| = |\psi_o^{*}mH - \psi_1^{*}Z_1 - \dots - \psi_{n-1}^{*}Z_{n-1} - Z_n|$$

where m is any integer. The aim of this § is to prove the following:

(2.1) <u>LEMMA</u>. The linear system $|T_m|$ is base point free for $m \gg 0$.

We will prove first the following "uniformity" lemma concerning linear systems on surfaces:

(2.2) <u>LEMMA</u>. There exists a function $m_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ having the following property. Let Y be a smooth projective surface and consider any sequence of morphisms

$$X = Y_n \xrightarrow{g_n} Y_{n-1} \longrightarrow \dots \longrightarrow Y_1 \xrightarrow{g_1} Y_o = Y$$

where g_1 is the blowing up of Y_o at d arbitrary distinct points $p_{o1},\dots,p_{od} \in Y_o$ and for $k \geq 1$, g_{k+1} is the blowing up of Y_k at a set of points $\{p_{k1},\dots,p_{kd}\}$ such that for each $j=1,\dots,d$, p_{kj} lies on the exceptional line $E_{kj} = f_k^{-1}(p_{k-1,j})$ but p_{kj} does not lie on any strict transform of a $E_{k_o j}$ with $k_o < k$. Let M be a very ample divisor on Y and $s \geqslant 1$ such that the linear system $|sM-K_Y|$ is base point free, where K_Y is a canonical divisor on Y. Then the linear system

$$|\tau_m| = |\varphi_o^* mM - \sum_j \varphi_1^* E_{1j} - \ldots - \sum_j \varphi_{n-1}^* E_{n-1,j} - \sum_j E_{nj}|$$

is base point free and

$$H^1(X, \mathcal{O}_X(\tau_m)) = 0$$

for $m \geqslant m_1(d,n) + s$, where $\varphi_k = g_{k+1} g_{k+2} \ldots g_n$.

<u>Proof</u>. Using the same technique as in [5] we may easily reduce ourselves to proving that for any blowing up $\sigma : \tilde{X} \longrightarrow X$ at $x \in X$, with exceptional locus E, the linear systems

$$P = |\varphi_o^*(m-s)M - R|$$
$$P' = |\sigma^* \varphi_o^*(m-s)M - R'|$$

(where $R = 2\sum_j \varphi_1^* E_{1j} + \ldots + 2\sum_j E_{nj}$ and $R' = \sigma^* R + 2E$) have selfintersections > 0 and contain 1-connected divisors. Note that selfintersections can be made > 0 provided $m-s$ is bigger than an obvious bound depending only on n and d.

Now choose: for each $j=1,\ldots,d$ an irreducible reduced divisor $M_j \in |M|$ passing through p_{oj}, an irreducible reduced $M_y \in |M|$ passing through $y = \varphi_o(x)$ and an arbitrary irreducible reduced $M \in |M|$. Define

$$\tilde{A} = 2n \sum_j M_j + (m-s-2nd)M$$
$$\tilde{B} = 2n \sum_j M_j + 2M_y + (m-s-2nd-2)M$$

Then:

$$A = \varphi_o^* \widetilde{A} - R \in P$$

$$B = \sigma^* \varphi_o^* \widetilde{B} - R' \in P'$$

because $\sigma^* \varphi_o^* M_y - E$ and $\varphi_o^* M_j - \varphi_k^* E_{kj}$ are effective for any $k=1,\dots,n$.

We will prove that B is 1-connected; in the same way one may prove that A is 1-connected.

Suppose $B = B_1 + B_2$ with $B_s > 0$, $s=1,2$. Writing

$$B_s = \sigma^* \varphi_o^* \widetilde{B}_s + \sum_k \sum_j a_{skj} \sigma^* \varphi_k^* E_{kj} + a_s E$$

where a_{skj}, a_s are integers, we get $\widetilde{B}_s \geqslant 0$, $\widetilde{B}_1 + \widetilde{B}_2 = \widetilde{B}$, $a_1 + a_2 = -2$, $a_{1kj} + a_{2kj} = -2$. We may suppose $\widetilde{B}_2 \neq 0$ and put $a_{kj} = a_{1kj}$, $a = a_1$. We get

$$B_1 B_2 = \widetilde{B}_1 \widetilde{B}_2 + \sum_k \sum_j (a_{kj}^2 + 2a_{kj}) + a^2 + 2a$$

If both $\widetilde{B}_1$ and $\widetilde{B}_2$ are non-zero, we have $\widetilde{B}_s \sim b_s M$ with $b_s > 0$ hence $\widetilde{B}_1 \widetilde{B}_2 > b_1 b_2 \geqslant b_1 + b_2 - 1 = m-s-1$. Consequently $B_1 B_2 \geqslant 1$ provided $m-s \geqslant 2nd+2$.

Now suppose $\widetilde{B}_1 = 0$. Let F_{kj} be the strict transform of E_{kj} on X. Because of the conditions imposed to the centers of the blowing ups, we have

$$\varphi_k^* E_{kj} = F_{kj} + F_{k+1,j} + \dots + F_{n,j}$$

hence

$$\sum_k \sum_j \varphi_k^* E_{kj} = \sum_j (a_{1j} F_{1j} + (a_{1j} + a_{2j}) F_{2j} + \dots$$

$$\dots + (a_{1j} + \dots + a_{nj}) F_{nj})$$

Let G_{kj} be the strict transform of F_{kj} on $\tilde{X}$.

Three cases may occur:

Case 1 : $x \notin F_{kj}$ for any k and any j

Case 2 : $x \in F_{ij_o}$ but $x \notin F_{kj_o}$ for $k \neq i$

Case 3 : $x \in F_{i-1,j_o} \cap F_{ij_o}$.

We have accordingly:

$$B_1 = \sum_j (a_{1j}G_{1j} + \ldots + (a_{1j}+\ldots+a_{nj})G_{nj}) + (a+c)E$$

where

$$c = \begin{cases} 0 \text{ in Case 1.} \\ a_{1j_o} + \ldots + a_{ij_o} \text{ in Case 2.} \\ 2a_{1j_o} + \ldots + 2a_{i-1,j_o} + a_{ij_o} \text{ in Case 3.} \end{cases}$$

Since $B_1 > 0$ we get $\sum_k \sum_j a_{kj} \geqslant 0$. In Case 1, we also get $a \geqslant 0$ and since at least one of the numbers a, a_{kj} is non-zero we get $B_1 B_2 \geqslant 1$. Suppose now we are in one of the cases 2 or 3. If all the numbers a_{kj} are zero, we must have $a \geqslant 1$ hence $B_1 B_2 \geqslant 3$. If there is precisely one non-zero number between the numbers a_{kj} then $B_1 B_2 \geqslant 3 + a^2 + 2a \geqslant 2$. If there are at least two non-zero numbers between the numbers a_{kj} then $B_1 B_2 \geqslant 2 + a^2 + 2a \geqslant 1$ and we are done.

Let us prove now Lemma (2.1).

Take an arbitrary $x \in V_n$ and look for a member of $|T_m|$ not passing through x, where m is any integer bigger that some bound not depending on x.

Put $y=\psi_o(x)$. By Bertini-type results, there exists asmooth connected surface $Y \in |H|$ passing through y and meeting D in $d=(H.D)_V$ distinct points $p_{o1},\dots,p_{od}$. Let Y_k be the strict transform of $Y=Y_o$ on V_k. Then $x \in X=Y_n$, X is linearly equivalent with $\psi_o^* H$ on V_n and the induced sequence

$$Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_o$$

is precisely of the form described in Lemma (2.1). Furthermore, if M is the trace of H on X, we have $\mathcal{O}_X(\mathcal{T}_m)=\mathcal{O}_X \otimes \mathcal{O}_{V_n}(T_m)$. Choose an integer s such that the linear system $|(s-1)H - K_V|$ is base point free. By adjunction, $|sM - K_Y|$ is base point free, so by Lemma (2.1), $|\mathcal{T}_m|$ is base point free and $H^1(\mathcal{O}_X(\mathcal{T}_m))=0$ for $m \geqslant m_1(d,n)+s$; in particular there exists a section $u \in H^o(\mathcal{O}_X(\mathcal{T}_m))$ with $u(x) \neq 0$. Consider the exact sequence:

$$H^o(\mathcal{O}_{V_n}(T_m)) \xrightarrow{\alpha_m} H^o(\mathcal{O}_X(\mathcal{T}_m)) \longrightarrow$$

$$\longrightarrow H^1(\mathcal{O}_{V_n}(T_{m-1})) \xrightarrow{\beta_m} H^1(\mathcal{O}_{V_n}(T_m)) \longrightarrow H^1(\mathcal{O}_X(\mathcal{T}_m))$$

Since β_m is surjective for $m \geqslant m_1(d,n)+s$ and since $\{\dim H^1(\mathcal{O}_{V_n}(T_m))\}_m$ is a decreasing sequence of natural numbers, there exists m_2, depending only on V,H,C, such that β_m is an isomorphism for $m \geqslant m_2$, so α_m is surjective for $m \geqslant m_2$ hence u lifts to $H^o(\mathcal{O}_{V_n}(T_m))$ and we are done.

3. Conclusion of the proof.

Keeping in mind the notations from §1 we will conclude the proof of the Theorem.

It follows from [7] that there exists a non-empty Zariski open subset U_o of $|mH-C|$ whose members are normal (connected) surfaces, provided $m \gg 0$. Take $S \in U_o$ and let S_k be the strict transform of S on V_k for $k=1,\dots,n$. From Lemma (1.1) we deduce that for $k \leq n-1$, S_k passes through δ_k; consequently, if R_n denotes $\psi_1^* Z_1 + \dots + Z_n$, then the divisor $\psi_o^* S - R_n$ is effective. Consider the diagram of algebraic morphisms:

$$\begin{array}{ccccc} |mH| & \xrightarrow{\psi_o^*} & |\psi_o^* mH| & \xrightarrow{\psi_{o*}} & |mH| \\ \cup & & \uparrow{\scriptstyle +R} & & \\ U_o & \xrightarrow{h} & |\psi_o^* mH - R_n| = |T_m| & & \end{array}$$

where $h(S) = \psi_o^* S - R_n$. Since h has a retract, it is injective, in particular $\dim |mH-C| \leq \dim |\psi_o^* mH - R_n|$. We will prove that there exists a non-empty Zariski open subset U' of $|\psi_o^* mH - R_n|$ such that for any $S_n \in U'$, $S_o = \psi_{o*} S_n$ contains C, has only A_{n-1} singularities and the number of the singularities is that given in the Theorem. Suppose we already have proved this. In particular $\psi_{o*}(U') \subset U_o$. Since h is a retract for $\varepsilon = \psi_{o*} : U' \longrightarrow U_o$ it follows that ε is injective and dominant, hence the image of ε contains a non-empty Zariski open set U which will be good for our Theorem.

Let us prove now the existence of U'. Take $S_n \in$ $\in |\psi_{o}^{*}mH-R_n|$ and put $S_k = \psi_{k*}S_n$. Let Γ_{ks} be an arbitrary fibre of Z_{ks} $(k < s)$ and Γ_k an arbitrary fibre of Z_k. We have:

$$(S_n \cdot \Gamma_{kn})_{V_n} = -(Z_k \cdot \Gamma_k)_{V_k} - (Z_{k+1} \cdot \Gamma_{k,k+1})_{V_{k+1}} = 1 - 1 = 0.$$

$$(S_k \cdot \Gamma_k) = -(Z_k \cdot \Gamma_k) = 1.$$

By Lemma (2.1) and Bertini's theorem, there exists a non-empty Zariski open subset U_1' of $|T_m|$ consisting of smooth members. Let η be an arbitrary section of Z_{1n}. Let $\phi \neq U_2' \subset |T_m|$ be a Zariski open subset all of whose members cut η in distinct points. The number of these points may be calculated as follows: put $\eta_1 = \psi_{1*}\eta$, $\eta_2 = \psi_{2*}\eta$, $\delta_{12} = Z_{12} \cap Z_2$ and then

$$(S_n \cdot \eta)_{V_n} = m(H.D)_V - (Z_1 \cdot \eta_1)_{V_1} - (Z_2 \cdot \eta_2)_{V_2} =$$

$$= m(H.D)_V + (\mathcal{O}_{Z_1}(1) \cdot \eta_1)_{Z_1} - (\delta_{12} \cdot \eta_2)_{Z_{12}}$$

Now $(\delta_{12} \cdot \eta_2)_{Z_{12}} = (\delta_1 \cdot \eta_1)_{Z_1}$ so finally $(S_n \cdot \eta) =$ $= m(H.D)_V + t$ where t is the number defined in the Theorem.

Write for any fibre Γ_n an exact sequence

$$0 \longrightarrow H^o(V_n, \mathcal{J}_{\Gamma_n}(T_m)) \longrightarrow H^o(V_n, \mathcal{O}_{V_n}(T_m)) \xrightarrow{\alpha} H^o(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$$

where $\mathcal{J}_{\Gamma_n}$ is the sheaf of ideals of $\Gamma_n \simeq \mathbb{P}^1$ on V_n. Note that α is surjective because $\operatorname{Im}\alpha$ defines a base point-free linear subsystem of $|\mathcal{O}_{\mathbb{P}^1}(1)|$ which must be automatically complete.

Hence $\dim|\mathcal{J}_{\Gamma_n}(T_m)| = \dim|T_m| - 2$, consequently the dimension of the set of divisors in $|T_m|$ containing at least a fibre Γ_{n} is $\leq \dim T_m - 1$. Let $U_3' \neq \emptyset$ be a Zariski open subset of $|T_m|$, all of whose members don't contain any Γ_n. Finally put $U' = U_1' \cap U_2' \cap U_3'$. By the intersection relations computed above it follows that for $S_n \in U'$, the (set-theoretic) intersection between S_n and $Z_{1n} \cup Z_{2n} \cup \ldots \cup Z_{n-1,n} \cup Z_n$ has precisely $\nu = m(H.D)_V+t$ connected components, each connected component being a curve of the form $\Gamma_{1n} \cup \Gamma_{2n} \cup \ldots \cup \Gamma_{n-1,n}$ with intersection numbers $(\Gamma_{in}.\Gamma_{i+1,n})_{S_n} = 1$ for all $i=1,\ldots,n-2$ and all other intersection numbers $(\Gamma_{in}.\Gamma_{jn})_{S_n}$ equal to zero. Denote the above connected components by $\Gamma^{(1)},\ldots,\Gamma^{(\nu)}$. We claim that $S_{n-1} \longrightarrow S_o$ is a desingularisation of S_o with exceptional locus $f_{n*}(\Gamma^{(1)} \cup \ldots \cup \Gamma^{(\nu)})$. This is a consequence of the following :

(3.1) <u>LEMMA</u>. Let V be a smooth 3-fold, D a smooth curve which is a closed subscheme in V, $f:V_1 \longrightarrow V$ the blowing up of V along D, Z the exceptional locus of f, Γ a fibre of the ruled surface $Z \longrightarrow D$ and S a divisor on V_1 such that $(S.\Gamma)_{V_1} = 1$, $S \cap \Gamma$ consists of one point p and S is smooth at p. Then $f|_S : S \longrightarrow V$ is a closed immersion around p (in the complex topology).

<u>Proof</u>. The local injectivity of f is obvious. Now suppose there is a non-zero tangent vector $v \in T_pS$ which is carried by the tangent map $T_pf : T_pV_1 \longrightarrow T_pV$ into zero. It fiollows that $v \in T_p\Gamma$ which contradicts the relation $(S.\Gamma)_{V_1} = 1$ and we are done.

Returning to the proof of the Theorem, note that since S_o is normal, Lemma (1.1) implies that S_o contains C. To prove that S_o has only A_{n-1} singularities, it is sufficient to show that each irreducible component of each $f_{n*}\Gamma^{(i)}$ has selfintersection -2. Since S_o is Gorenstein, it is sufficient by [2] to prove that $g^*K_{S_o} = K_{S_{n-1}}$ where $g : S_{n-1} \longrightarrow S_o$ is the restriction of $f : V_{n-1} \longrightarrow V_o$. Now if $q : S_{n-1} \hookrightarrow V_{n-1}$ and $r : S_o \hookrightarrow V_o$ denote the natural inclusions and if we put $R_{n-1} = f_{n*}R_n$ we get

$$g^*K_{S_o} = g^*r^*(K_{V_o} + S_o) = q^*f^*(K_{V_o} + S_o) =$$

$$= q^*(K_{V_{n-1}} - R_{n-1} + S_{n-1} + R_{n-1}) =$$

$$= q^*(K_{V_{n-1}} + S_{n-1}) = K_{S_{n-1}} .$$

In the end we will show that C is a Cartier divisor on S_o. Note first that the restriction of C to the regular locus of S_o is equal as a Cartier divisor to n times its support. So we will be finished if we prove the following:

(3.2) LEMMA. Let $(A,\underline{m})$ be an A_{n-1} surface singularity, $P \subset A$ a prime of height one and $I \subset A$ an ideal of height one with $\mathrm{prof}(A/I)=1$. Suppose the sheaves of ideals $\widetilde{I}$ and $(P^n)^{\sim}$ are equal on the punctured spectrum $\mathrm{Spec}(A) \setminus \underline{m}$. Then I is principal.

Proof. Since I has no embedded components, it must be primary so it is equal to the symbolic power $P^{(n)}$. On the other hand by [8] the divisor class group of the completion of $(A,\underline{m})$ has cardinal n, and since $\mathrm{Cl}(A) \subset \mathrm{Cl}(\hat{A})$, $P^{(n)}$ must be principal.

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