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by

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HYPERPLANE SECTIONS AND DEFORMATIONS

Lucian Bădescu*)

Introduction

This paper is a continuation of [3] and [4]. Here we especially determine all normal projective varieties X containing a certain given projective variety Y as an ample Cartier divisor. In many cases we shall be dealing with, the variety X turns out to be a cone over Y if X is assumed to be singular. Some situations of this kind were already encountered in [3] and [4]. The paper is divided in four Sections. The first one deals with the cases in which Y is either an elliptic curve (see Theorem 1), or a smooth projective curve of genus $\geqslant 2$ (see Theorem 2). Since in [3] and [4] we classified all smooth projective 3-folds containing a geometrically ruled surface as an ample divisor, it is natural also to see what is going on in the singular case. And indeed, Section 2 takes up this problem, giving a complete answer if Y is the surface F_e, with $e \geqslant 0$ (see Theorems 3 and 4), and a partial one if Y is a P^1-bundle over a smooth non-rational curve (see Theorem 6). The higher dimensional case (i.e. Y is a P^n-bundle over a smooth curve, with $n \geqslant 2$) is much easier to handle. In Section 3 we improve a result of T. Fujita concerning the Grassmann variety (see Theorem 7). Finally, the last Section relates the results obtained previously with the deformation theory.

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Notations and terminology

Throughout this paper we shall fix an algebraically closed base field k. In general the terminology and notations are standard, with the following precisations.

Unless otherwise stated, all schemes we shall be dealing with will be algebraic schemes over k. The term "algebraic variety" means an irreducible and

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reduced algebraic scheme over k.

By a polarized variety we understand a pair (Y,L) consisting of a projective variety Y and an ample line bundle L over Y. The graded k-algebra $S = S(Y,L)$ associated to the polarized variety (Y,L) is the algebra $\bigoplus_{n=0}^{\infty} H^o(Y,L^n)$ graded in the natural way. Consider the polynomial algebra $S[z]$ over S in one variable z graded by the condition that $\deg(sz^n) = \deg(s) + n$ for every homogeneous element $s \in S$. Then the variety $C(Y,L) = \mathrm{Proj}(S[z])$ will be referred as the projective cone associated to the polarized variety (Y,L).

Let Y be an effective Cartier divisor on the variety X. We shall denote by $O_X(Y)$ the invertible sheaf (or line bundle) associated to the divisor Y, and by $N_{Y,X} = O_X(Y) \otimes O_Y$ the normal bundle of Y in X. A global equation of the divisor Y on X is a section $\sigma \in H^o(X,O_X(Y))$ whose associated divisor in Y.

If Z is an arbitrary algebraic scheme over k, we shall denote by ω_Z the Grothendieck dualizing sheaf of Z. If D is an effective Cartier divisor on Z, then one has the adjunction formula $\omega_D = \omega_Z \otimes O_Z(D) \otimes O_D$. This gives in particular the genus'formula of a curve over a smooth projective surface.

If $D_1,\ldots,D_d$ are Cartier divisors on a proper d-dimensional scheme Z over k, then $D_1.D_2\ldots D_d$ will denote the intersection number of the divisors $D_1,\ldots,D_d$. If in particular, $D_i = D$ for every $i = 1,\ldots,d$, then $D_1.D_2\ldots D_d$ will be also denoted by $D^{\cdot d}$, or simply by D^d.

If F is a Coherent sheaf on the scheme Z and D a Cartier divisor on Z, then F(D) will denote the sheaf $F \otimes O_Z(D)$. If moreover Z is proper over k we shall denote by $h^i(Z,F)$ the dimension over k of the vector space $H^i(Z,F)$.

If E is a vector bundle over Z, $\check{E}$ stands for the dual of E. If E is of rank one, we shall also write E^{-1} instead of $\check{E}$.

If S is a graded k-algebra and r a natural number, then $S^{(r)}$ is the graded k-algebra such that $(S^{(r)})_n = S_{nr}$, where S_m denotes the homogeneous part of degree m of S.

§1. Surfaces containing a given curve as an ample Cartier divisor

Let us begin by recalling two well-known results:

Theorem A. Let X be a normal projective surface containing $Y = P^1$ as an ample Cartier divisor. Then (up to an isomorphism) one has one of the following three possibilities:

a) X is P^2 and Y is either a straight line or a conic in P^2;

b) X is the geometrically ruled surface $F_e = P(O_{P^1} \oplus O_{P^1}(-e))$ $(e \geqslant o)$ and Y is a section of the canonical projection $p: F_e \longrightarrow P^1$;

c) X is the projective cone over P^1 with respect to the s-fold Veronese embedding $v_s: P^1 \longrightarrow P^s$ $(s \geqslant 2)$ and Y is the intersection of X with the hyperplane at infinity of P^{s+1}.

Theorem A is classical. A modern reference for it is [18].

Theorem B (Char(k) = o). Let X be a smooth projective surface containing the elliptic curve Y as an ample divisor. Then (up to isomorphism) one has one of the following two possibilities:

a) X is a Del Pezzo surface and -Y is a canonical divisor on X;

b) X is a geometrically ruled surface $p: P(E) \longrightarrow Y$ over Y and the inclusion $Y \subset X$ is a section of p (with E a rank two vector bundle over Y).

Theorem B is also classical. A modern reference for it is [17]. In connection with theorem B, it is natural to classify also all normal (singular) projective surfaces containing a given elliptic curve as an ample Cartier divisor. As far as we know such a classification is not explicitly contained in any paper, although it turns out to be closely related to the classification of all surfaces of degree d in P^d which are not contained in any hyperplane of P^d (see [31]). The result is the following:

Theorem 1 (Char(k) = o). Let X be a normal (singular) projective surface containing the elliptic curve Y as an ample Cartier divisor. Then one has one of the following two possibilities:

a) X is a surface with only rational double points as singularities and -Y is a canonical divisor on X. These surfaces are classified in [9], [11], [24] (see Theorem C below).

b) X is the projective cone over the polarized curve $(Y, N_{Y,X})$, and Y is embedded in X as the infinite section (i.e. X is an elliptic cone over Y).

Proof. Let $f:\tilde{X} \longrightarrow X$ be the minimal desingularization of X, i.e. the exceptional fibres of f do not contain any exceptional curve of the first kind. Since Y does not meet the singular locus of X, Y is also contained in $\tilde{X}$ and and the normal bundles of Y in X and of Y in $\tilde{X}$ are the same. In particular, $(Y^2)_{\tilde{X}} > o$ and $Y.E \geqslant o$ for every integral curve E on $\tilde{X}$. The exact sequence

$$o \longrightarrow \omega_{\tilde{X}} \longrightarrow \omega_{\tilde{X}} \otimes O_{\tilde{X}}(Y) \longrightarrow \omega_Y = O_Y \longrightarrow o$$

yields the exact sequence of cohomology

$$o \longrightarrow H^o(\omega_{\tilde{X}}) \longrightarrow H^o(\omega_{\tilde{X}} \otimes O_{\tilde{X}}(Y)) \xrightarrow{\ \varepsilon\ } H^o(O_Y) = k \longrightarrow H^1(\omega_{\tilde{X}}) \longrightarrow$$
$$\longrightarrow H^1(\omega_{\tilde{X}} \otimes O_{\tilde{X}}(Y)).$$

By duality and the Kodaira-Ramanujam vanishing theorem, $H^1(\omega_{\tilde{X}} \otimes O_{\tilde{X}}(Y)) = o$, which implies that $q = h^1(O_{\tilde{X}}) = h^1(\omega_{\tilde{X}}) \leqslant 1$.

If q = o the map ε is surjective, and therefore there is a section $s \in H^o(\tilde{X}, \omega_{\tilde{X}} \otimes O_{\tilde{X}}(Y))$ whose restriction to Y is 1; in other words, there is a canonical divisor K on $\tilde{X}$ of the form K = D-Y, with $D \geqslant o$ and $Supp(D) \cap Y = \phi$. Since Y is ample on X, the support of D (if $D \neq o$) is contained in the exceptional fibres of f. On the other hand, since f is minimal, $o \leqslant K.E = D.E - Y.E = D.E$ for every irreducible component E of the exceptional fibres of f. Then a standard argument (see [2], propos. 2) shows that $D \leqslant o$. Recalling that $D \geqslant o$, we get D = o, and in particular, $\omega_{\tilde{X}}/\tilde{X}-Y \cong O_{\tilde{X}}/\tilde{X}-Y$. This implies that ω_X is invertible and isomorphic to $O_X(-Y)$, and that $f^*(\omega_X) = \omega_{\tilde{X}}$. By [2] X has only rational double points as singularities. Therefore q = o leads to case a).

Now assume q = 1. First we show that $\tilde{X}$ is ruled.

Claim. Assuming that $\tilde{X}$ is not ruled, then $\omega_{\tilde{X}}.Z \geqslant o$ for every integral curve Z on $\tilde{X}$ such that $Z^2 > o$ and $p_a(Z) \geqslant 1$.

Proof of the claim. We shall proceed by induction on the number n of quadratic transformations in order to reach $\tilde{X}$ from its (unique) minimal model (see [8]). If n = o, i.e. if $\tilde{X}$ is itself the minimal model, then the classification of surfaces (loc. cit.) shows that $\omega_{\tilde{X}}.Z \geqslant o$ for every curve Z on $\tilde{X}$. Thus we can assume n > o, and let E be an exceptional curve of the first kind on $\tilde{X}$. Let $\sigma : \tilde{X} \longrightarrow \bar{X}$ be the morphism contracting E to a smooth point $x \in \bar{X}$, and set $Z' = \sigma(Z)$. We have $Z'^2 = Z^2 + m^2$, where $m \geqslant o$ is the multiplicity of the point x on Z' ($m = o$ if $x \notin Supp(Z')$). Therefore $Z'^2 \geqslant Z^2 > o$. Moreover,

$p_a(Z') \geq p_a(Z) \geq 1$. Using the inductive hypothesis we infer that $\omega_{\bar{X}}.Z' \geq o$. But $o \leq \omega_{\bar{X}}.Z' = \sigma^*(\omega_{\bar{X}}).\sigma^*(Z') = \sigma^*(\omega_{\bar{X}}).Z + m\sigma^*(\omega_{\bar{X}}).E = \sigma^*(\omega_{\bar{X}}).Z$. Therefore $\sigma^*(\omega_{\bar{X}}).Z \geq o$. Since $\omega_{\tilde{X}} = \sigma^*(\omega_{\bar{X}}) \otimes O_{\tilde{X}}(E)$, we have $\omega_{\tilde{X}}.Z = \sigma^*(\omega_{\bar{X}}).Z + E.Z \geq$ $\geq E.Z \geq o$, the last inequality coming from the fact that E and Z are different integral curves. The claim is proved.

Returning to the proof of theorem 1 and using the claim (in the assumption that $\tilde{X}$ is not ruled), one gets $\omega_{\tilde{X}}.Y \geq o$. Then the genus formula yields the desired contradiction:

$$1 = p_a(Y) = 1/2.Y^2 + 1/2.\omega_{\tilde{X}}.Y + 1 \geq 1/2.Y^2 + 1 > 1.$$

This proves that $\tilde{X}$ is ruled if q = 1. Now we can apply corollary 2.4 of [22] in order to deduce that the inclusion $Y \subset \tilde{X}$ is equivalent to a section of a geometrically ruled surface $p: P(E) \longrightarrow Y$.

The point is to show that the inclusion $Y \subset \tilde{X}$ actually coincides to a section of a geometrically ruled surface $\tilde{X} = P(E) \longrightarrow Y$. But this is standard as following. Let $g: \tilde{X} \longrightarrow Y$ be the rulling morphism of $\tilde{X}$. We have to check that all the fibres of g are irreducible, knowing that Y.F = 1 for a general fibre F of g. Since $\tilde{X}$ is the minimal desingularization of X, there are no exceptional curves of the first kind not meeting Y. Consider a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\sigma} & P(E) \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle g'} & \\ & Y & \end{array}$$

where σ is a birational morphism (such a diagram always exists because the minimal models of surfaces birationally equivalent to $P^1 \times Y$ are the geometrically ruled surfaces $g': P(E) \longrightarrow Y$ over Y). Since Y is a section of g, $\sigma(Y)$ is also a section of g', and in particular, $\sigma(Y)$ is smooth. Let m be the maximal number of irreducible components of fibres of g, and let F be a fibre of g having exactly m components. An easy induction on m shows that if $m > 1$, there is an exceptional curve E' of the first kind contained in F such that $E' \cap Y = \emptyset$. In this way, the assumption that $m > 1$ contradicts the minimality of the desingularization $f: \tilde{X} \longrightarrow X$. Therefore m = 1, or else $\tilde{X}$ is geometrically ruled over Y and the inclusion $Y \subset \tilde{X}$ is a section of $g: \tilde{X} \longrightarrow Y$.

Write $\tilde{X} = P(E)$, with E a normalized vector bundle of rank 2 over Y (see [21], page 373). Set e = -deg(E), and let C_o be the minimal section of P(E). Then $C_o^2 = -e$ and $O_{P(E)}(C_o) = O_{P(E)}(1)$. Assume that the vector bundle E is indecomposable. Then by loc. cit., Theorem 2.15 (page 377) and Propositions

2.2o and 2.21 (page 382) it is easy to deduce that every integral curve on $\widetilde{X}$ has non-negative self-intersection, contradicting the fact that $\widetilde{X}$ carries at least an integral curve with negative self-intersection. Consequently E is decomposable, i.e. $E = O_Y \oplus L$, with $-e = \deg(L) \leqslant o$. Again if $e = o$ every integral curve on $\widetilde{X}$ has non-negative self-intersection. Therefore $e > o$. Then C_o is the only curve on $\widetilde{X}$ with negative self-intersection, and consequently X is obtained from $\widetilde{X}$ by contracting C_o to a point. Since C_o can also be regarded as the zero section of the line bundle $V(L^{-1})$ and $\deg(L^{-1}) = e > o$, [19] II §8 shows that X is isomorphic to $\mathrm{Proj}(S[z])$, where $S = \bigoplus_{m=0}^{\infty} H^o(X, L^{-m})$ (with the natural gradation), z is a variable over S, and if $s \in S$ is a homogeneous element, then $\deg(sz^m) = \deg(s) + m$. In other words, X is the projective cone of the polarized curve (Y, L^{-1}). Moreover, $Y \approx C_o + eF$, and therefore the inclusion $Y \subset X$ can be viewed as the infinite section of $\mathrm{Proj}(S[z])$ (corresponding to the equation $z = o$). Q.E.D.

Remarks. 1. One could also prove the ruledness of $\widetilde{X}$ and the fact that the inclusion $Y \subset \widetilde{X}$ is equivalent to a section of a geometrically ruled surface, using the same kind of arguments as in [17], Proposition 1.11.

2. Let (Y,L) be a polarized variety, with Y a normal projective variety of dimension $\geqslant 1$, and let $X = C(Y,L)$ be the cone of (Y,L). Embed Y in X as the infinite section, so that Y becomes an ample Cartier divisor on X. Then:

i) $H^1(O_X) = o$.

ii) If $H^1(Y,L) = o$ then $H^1(O_X(Y)) = o$.

iii) If $H^i(Y,L^n) = o$ for every $n \geqslant 1$ and $i = 1,2$, and $H^2(O_Y) = o$, then $H^2(O_X) = o$.

The proofs of these statements are standard and left to the reader. Applying them to the elliptic cone X (if Y is an elliptic curve), one gets that $H^i(O_X) = o$ for $i = 1,2$ and $H^1(O_X(Y)) = o$.

3. In both cases of Theorem 1 the surface X is Gorenstein and ω_X^{-1} is ample. The surfaces with these properties have been studied and classified in [9], [11] and [24]. The main results about them can be summarized as follows:

Theorem C. ([9], [11], [24]) If X is a normal (singular) projective surface with ω_X^{-1} ample and if we put $d = \omega_X . \omega_X$, then:

a) The anti-canonical linear system $|\omega_X^{-1}|$ contains a smooth elliptic curve Y.

b) If $Y \in |\omega_X^{-1}|$ is as above and $\sigma \in H^o(X,\omega_X^{-1})$ is a global equation of Y, then there is a natural identification of graded k-algebras $\bigoplus_{m=o}^{\infty} H^o(Y,N_{Y,X}^{-m}) = S/\sigma S$, where $S = \bigoplus_{m=o}^{\infty} H^o(X,\omega_X^{-m})$ is the anti-canonical ring of X. In particular ω_X^{-1} is very ample if $d \geqslant 3$.

c) $\mathrm{Dim}_k H^o(X,\omega_X^{-m}) = 1/2.dm(m+1) + 1$ for every $m \geqslant o$. In particular, if $d \geqslant 3$, the anti-canonical linear system of X yields an embedding of X in P^d as a subvariety of degree d.

d) (Classification) If X is an elliptic cone, all positive values of d are possible. If X is not an elliptic cone (i.e. if the minimal desingularization of X has irregularity zero), then $1 \leqslant d \leqslant 8$. If $d = 8$, X is isomorphic to the quadric cone of P^3. If $1 \leqslant d \leqslant 7$, there exists a set S of 9-d points of P^2 in almost general position (but not in general position) such that X is obtained by contracting all the -2-curves of the surface V(S) obtained from P^2 by blowing up the points of S.

Recall that a subset $S = \{P_1,\ldots,P_r\}$ $(r \leqslant 8)$ of points of P^2 is said to be in general position if no three of them are on a line, no six of them are on a conic, and if $r = 8$, there is no cubic passing through all points and having a singularity in one of them. Then one can prove that S is a set of points in general position iff the anti-canonical class of the surface V(S) is ample (see [11]). In the case of points in almost general position, infinitely near points of P^2 are allowed in S (see loc. cit. for the precise definition). Anyhow, the points of S are in almost general position iff the canonical class K of V(S) has the property that $K^2 > o$ and $K.C \leqslant o$ for every integral curve on V(S). Moreover, there are only finitely many curves C with $K.C = o$, and all of these are smooth rational curves with self-intersection -2 (loc. cit.).

Therefore we see that - a posteriori - there is no difference between the class of normal (singular) Gorenstein projective surfaces with ample anti-canonical class and the class of normal (singular) projective surfaces supporting elliptic curves as ample Cartier divisors. However, Theorem 1 cannot be directly deduced from Theorem C because -a priori - we do not know that such a surface X is Gorenstein or that the elliptic curve Y belongs necessarily to the anti-canonical linear system.

Via part c) of Theorem C, we also see that if $d = Y^2 \geqslant 3$, then Y is a very ample Cartier divisor on X, giving rise to an embedding of X in P^d as a surface of degree d. Therefore, if $d \geqslant 3$, the classification of these surfaces also comes from [31], Theorem 8.

Theorem 2 (Char(k) = o). Let Y be a smooth projective curve of genus $g \geq 2$, and X a normal projective surface containing Y as an ample Cartier divisor. Assume that $Y^2 \geq 4g+5$. Then (up to isomorphism) one has one of the following two possibilities:

a) X is a geometrically ruled surface $p:P(E) \longrightarrow Y$ over Y and Y is contained in X as a section of p;

b) X is the projective cone $C(Y,N_{Y,X})$ and Y is embedded in X as the infinite section.

Proof. According to the proof of Theorem 1, let $f:\widetilde{X} \longrightarrow X$ be the minimal desingularization of X. We have $(Y^2)_{\widetilde{X}} \geq 4g+5$ by hypothesis. Since $\widetilde{X}$ is smooth we can use Theorem 4.1 of [22] in order to deduce that X is a ruled surface and the embedding $Y \subset \widetilde{X}$ is equivalent to a section of a geometrically ruled surface $p:P(E) \longrightarrow Y$. Using the fact that $f:\widetilde{X} \longrightarrow X$ is minimal one shows (exactly as in the proof of Theorem 1) that $\widetilde{X}$ is itself geometrically ruled, i.e. $\widetilde{X} = P(E)$, with E a vector bundle of rank 2 over Y, and Y is embedded in $\widetilde{X}$ as a section of p. If $\widetilde{X} = X$ (i.e. if X is smooth) we get directly situation a). If not, $\widetilde{X}$ carries at least an irreducible and reduced curve with negative self-intersection and not meeting Y. Then, exactly as in the proof of Theorem 1, one can assume that E is necessarily of the form $E = O_Y \oplus L$, with $\deg(L) = -e < o$. From this point one easily gets that we are in situation b). Q.E.D.

Remark. Let X be $P^1 \times P^1$ and Y a smooth member of the linear system $|O(2,g+1)|$. Then Y has genus g and $Y^2 = 4g+4$. This shows that Theorem 2 is sharp and this is because Theorem 4.1 of [22] (the key point in the proof) is so.

§2. Normal 3-folds whose hyperplane sections are geometrically ruled surfaces

In [3] and [4] the following result is proved:

Theorem D (Char(k) = o). Let B be a smooth projective curve and E a vector bundle over B of rank $r \geq 2$. Denote by Y the projective bundle P(E), and assume that the smooth projective (r+1)-dimensional variety X contains Y as an ample divisor. Then:

a) If $Y \not\cong P^1 \times P^1$, there exists an exact sequence of vector bundles over B

$$o \longrightarrow O_B \longrightarrow F \xrightarrow{\varphi} E' = E \otimes L \longrightarrow o,$$

with F ample and $L \in Pic(B)$, such that X is isomorphic to P(F) and $Y \cong P(E')$ is embedded in X via surjection φ.

b) If $Y = P^1 \times P^1$, one has one of the following possibilities:

i) X is isomorphic to P^3 and Y is a quadric in X;

ii) X is isomorphic to a hyperquadric of P^4 and Y is a hyperplane section;

iii) There is an exact sequence of vector bundles over P^1 of the form

$$o \longrightarrow O_{P^1} \longrightarrow F = O(a) \oplus O(b) \oplus O(c) \xrightarrow{\varphi} E' = O(s) \oplus O(s) \longrightarrow o,$$

with $a > o$, $b > o$, $c > o$ and $s > o$ (satisfying the equation $a+b+c = 2s$), such that X is isomorphic to P(F) and $Y \cong P(E')$ is embedded in X via surjection φ.

The aim of this section is to determine also the normal singular projective (r+1)-dimensional varieties X supporting Y = P(E) as an ample Cartier divisor, where E is a vector bundle of rank $r \geq 2$ over P^1. As in the proof of Theorem D, the really difficult case is the one where Y is a surface. We have to treat separately the case $Y = P^1 \times P^1$.

First of all let us recall the following general and useful lemma (see [37], [16], *lemma 2.2*):

Lemma 1. Let X be a normal projective variety and Y an ample Cartier divisor on X. Assume that Y is smooth and of dimension ≥ 2. Let U = Reg(X) be the smooth locus of X, and F a coherent O_X-module such that F/U is locally free and $depth_x(F_x) \geq 2$ for every $x \in X-U$. If $H^p(Y, F \otimes N^{-m}_{Y,X}) = o$ for every p = o,1 and $m \geq o$, then $H^p(X,F) = o$ for p = o,1.

Now one can state:

Theorem 3 (Char(k) = o). Let X be a normal projective 3-fold with singularities containing $Y = P^1 \times P^1$ as an ample Cartier divisor. Then X is isomorphic to the projective cone $C(Y, N_{Y,X})$ and the inclusion $Y \subset X$ is the infinite section.

Proof. Set $N_{Y,X} = O(a,b)$, with $a > o$ and $b > o$. Consider first the case where a = 1 or b = 1. Then, using the explicit computation of the cohomology of $P^1 \times P^1$ together with lemma 1, we deduce that $H^1(X, O_X(mY)) = o$ for every $m \in \mathbb{Z}$. Therefore, for every $m \geq o$ we get the exact sequence:

$$o \longrightarrow H^o(O_X((m-1)Y)) \xrightarrow{\sigma} H^o(O_X(mY)) \longrightarrow H^o(O(ma,mb)) \longrightarrow o.$$

Thus we have a natural isomorphism of graded k-algebras $S/\sigma S = \bigoplus_{m=0}^{\infty} H^o(O(ma,mb))$, where $S = \bigoplus_{m=0}^{\infty} H^o(O_X(mY))$ and $\sigma \in S_1$ is a global equation of Y. Since the graded k-algebra $\bigoplus_{m=0}^{\infty} H^0(O(ma,mb))$ is Cohen-Macaulay and generated by its homogeneous part of degree one, we infer that S is Cohen-Macaulay and generated by S_1. In particular, we get that Y is very ample and X = Proj(S) is locally Cohen-Macaulay. Let $X \hookrightarrow P^N = |Y|$ be the closed immersion given by $|Y|$ and H a hyperplane in P^N such that $X \cap H = Y$. Let H' be another hyperplane not passing through the singularities of X (which are finitely many) and such that $C = X \cap H \cap H' = Y \cap H'$ is a smooth curve. Since $Y = P^1 \times P^1$ and a = 1 or b = 1, C is rational. Let then $x \in X$ be a singular point of X and H" the hyperplane passing through $H \cap H'$ and the point x (note that since $x \notin H'$, $x \notin H \cap H'$), and denote by S the surface $X \cap H''$. Since X is Cohen-Macaulay, S is also Cohen-Macaulay. Moreover, S supports the smooth curve C as an ample Cartier divisor, and hence the singularities of S are isolated. This proves that S is normal, having a singularity in x. Since S contains the smooth rational curve C as an ample Cartier divisor, we can apply Theorem A to deduce that S is the cone over C in P^N. Now, varying the smooth curve C in the linear system $|N_{Y,X}| = |O(a,b)|$, we infer that for every point $y \in Y$ the line in P^N passing through x and y is contained in X. This shows that X is the cone over Y in P^N, and concludes the proof of Theorem 3 in case a = 1 or b = 1.

Assume now $a \geq 2$ and $b \geq 2$. If moreover $a \neq b$, then part b) of Theorem 5 in [4] shows that X is isomorphic to $C(Y, N_{Y,X})$. The extra-hypothesis "$a \neq b$" was necessary in that theorem only to deduce that the restriction map α: Pic(U) $\longrightarrow$ Pic(Y) is an isomorphism, where U = Reg(X). Therefore it will be sufficient to prove that α is an isomorphism even in case $a = b \geq 2$.

Assume the contrary, i.e. $a = b \geq 2$ and α not an isomorphism. Then according to the proof of Theorem 5 in [4], α is injective and Coker(α) is torsion-free. Since Pic(Y) = $\mathbb{Z} \times \mathbb{Z}$, Pic(U) = $\mathbb{Z}L$, with L ample on U. Write $L \otimes O_Y = O(s,t)$, and since Coker(α) has no torsion, s and t are relatively prime positive integers. Write $O_X(Y)/U = L^r$ and $\omega_X/U = L^d$. Then the adjunction formula yields $s(d+r) = t(d+r) = -2$, and thus $s = t = 1$ and $r = a = b \geq 2$. Therefore we get the exact sequence

$$o \longrightarrow L^{m-r} \xrightarrow{\sigma} L^m \longrightarrow O(m,m) \longrightarrow o,$$

with $m \geq o$ and σ a global equation of Y. Applying lemma 1 to the sheaf $F = j_*(L^m)$, where $j: U \hookrightarrow X$ is the canonical inclusion, we get that $H^p(X,F) = o$ for p = o,1 and every $m \geq o$. In particular, the above exact sequence yields for every $m \geq o$ the exact sequence:

$$(1) \quad o \longrightarrow H^o(U,L^{m-r}) \xrightarrow{\ \sigma\ } H^o(U,L^m) \longrightarrow H^o(O(m,m)) \longrightarrow o.$$

Denote by S the graded k-algebra $\bigoplus_{m=o}^{\infty} H^o(U,L^m) = \bigoplus_{m=o}^{\infty} H^o(X,j_*(L^m))$. Then $\sigma \in S_r$ and (1) shows that $S/\sigma S = \bigoplus_{m=o}^{\infty} H^o(O(m,m)) \cong k[T_o,T_1,T_2,T_3]/(T_oT_1-T_2T_3)$, where $k[T_o,T_1,T_2,T_3]$ is the polynomial k-algebra in four variables (graded in the usual way). Since $S^{(r)} = \bigoplus_{m=o}^{\infty} H^o(X,j_*(L^{mr})) = \bigoplus_{m=o}^{\infty} H^o(X,O_X(mY))$ and Y is ample on X, we have $X = \mathrm{Proj}(S^{(r)}) = \mathrm{Proj}(S)$.

Now, using the elementary arguments from the proof of Theorem 3.6 in [28] one deduces that $S = k[T_o,T_1,T_2,T_3,T_4]/(Q(T_o,\dots,T_4))$, where $\deg(T_4) = r$, $T_4 \bmod (Q) = \sigma$ and $Q(T_o,T_1,T_2,T_3,o) = T_oT_1 - T_2T_3$. In other words, $X = \mathrm{Proj}(S)$ is the hyperquadric $Q = o$ in the weighted projective space $P(1,1,1,1,r) = \mathrm{Proj}(k[T_o,T_1,T_2,T_3,T_4])$. If $r \geqslant 3$ then necessarily $Q(T_o,\dots,T_4) = T_1T_2 - T_3T_4$, and hence X is the projective cone $C(Y,N_{Y,X})$. Therefore the case $r \geqslant 3$ implies that the map α is an isomorphism (since X is a cone over Y), contrary to the hypothesis that $\mathrm{Pic}(U) = \mathbb{Z}L$.

If $r = 2$ we have two possibilities: either T_4 does not occur in Q (and one gets the same contradiction as in case $r \geqslant 3$), or T_4 does occur, and then $Q = aT_4 + T_1T_2 - T_3T_4$ (with $a \in k-\{o\}$). In the latter case $S = k[T_o,\dots,T_4]/(Q) \cong k[T_o,\dots,T_3]$. Consequently, X is isomorphic to P^3, a contradiction because X was supposed to have singularities. This shows that if $a \geqslant 2$, $b \geqslant 2$, the map α is always bijective, and -via the proof of Theorem 5, b) in [4]- Theorem 3 is completely proved. Q.E.D.

Remarks. 1. Let Y be a smooth hypersurface in P^m $(m \geqslant 4)$ of degree $r \geqslant 2$, and let X be a normal projective variety supporting Y as an ample Cartier divisor. Since $\mathrm{Pic}(Y) = \mathbb{Z}O_Y(1)$, $N_{Y,X} = O_Y(s)$ for some $s > o$. If $s > r$ then Theorem 4 of [4] shows that X is necessarily the cone $C(Y,O_Y(s))$. If instead $s = r$, the arguments of the proof of Theorem 3 (concerning the case $a = b = r = 2$) can be used to prove that X is either P^m or $C(Y,O_Y(r))$.

2. Theorem 3 could also be proved in case $a = 1$ or $b = 1$ by observing that (if e.g. $a = 1$) the subvariety X of P^N (embedded via the complete linear system $|Y|$) satisfies the limiting condition $\deg(X) = \mathrm{codim}(X,P^N) + 1$, and using a well-known and classical result classifying the non-degenerate subvarieties of P^N satisfying the above equality.

Now consider the case $Y = F_e = P(O_{P^1} \oplus O_{P^1}(-e))$, with $e \geqslant 1$. Denote by $p: Y \longrightarrow P^1$ the canonical projection, and let X be a normal (singular) projective 3-fold containing Y as an ample Cartier divisor. First we prove a key lemma:

<u>Lemma 2 (Char(k) = o). In the above notations and assumptions, the restriction map α: Pic(U) $\longrightarrow$ Pic(Y) is an isomorphism, where U = Reg(X).</u>

<u>Proof.</u> Exactly as in the proof of Theorem 3, if α would not be bijective then $\mathrm{Pic}(U) = \mathbb{Z}L$, with $L \in \mathrm{Pic}(U)$ ample. Denote by $O_Y(1)$ the tautological invertible sheaf of $Y = F_e$ relative to p, and by $O(1)$ the tautological sheaf of P^1. Then $O_Y(1)$ and $p^*O(1)$ form a base for Pic(Y), and hence we can write $L \otimes O_Y \cong O_Y(b) \otimes p^*O(a)$ for some (uniquely determined) integers a and b. Since $L \otimes O_Y$ is ample on F_e one has $a > be > o$ (see [21], page 380, Corollary 2.18).

Since L generates Pic(U) we can also write $O_X(Y)/U \cong L^r$ and $\omega_X/U \cong L^d$. The adjunction formula yields $b(d+r) = -2$ and $a(d+r) = -2-e$. If $e \geqslant 2$, then we easily get a contradiction from these equations, and therefore lemma 2 is proved in this case.

The case e = 1 is more subtle. From now on (till the end of the proof of lemma2) we shall assume $Y = F_1$. Since a and b are relatively prime integers (recall that Coker(α) has no torsion), the above equations yield a = 3 and b = 2, i.e. $L \otimes O_Y \cong O_Y(2) \otimes p^*O(3) = \omega_Y^{-1}$. Exactly as in the proof of Theorem 3 one deduces (using lemma 1) that the following sequence is exact:

$$o \longrightarrow H^o(U,L^{m-r}) \xrightarrow{\ \sigma\ } H^o(U,L^m) \longrightarrow H^o(Y,\omega_Y^{-m}) \longrightarrow o \quad (m \geqslant o).$$

Therefore, if we denote by S the graded k-algebra $\bigoplus_{m=o}^{\infty} H^o(U,L^m)$, we have $\sigma \in S_r$ and $S/\sigma S = \bigoplus H^o(Y,\omega_Y^{-m})$. The latter ring is generated by its homogeneous part of degree one and $|\omega_Y^{-1}|$ yields an embedding of Y in P^8 as a subvariety of degree 8. Moreover, using [29], Theorem 8 and its Corollary, we infer that Y is given in P^8 by n hyperquadrics $f_1,\ldots,f_n$, or else

$$\bigoplus_{m=o}^{\infty} H^o(Y,\omega_Y^{-m}) \cong k[T_o,\ldots,T_8]/(f_1,\ldots,f_n).$$

Now we proceed again as in the proof of Theorem 3.6 of [28]. Let $\xi_i = T_i \bmod (f_1,\ldots,f_n)$. Since $S/\sigma S \cong k[T_o,\ldots,T_8]/(f_1,\ldots,f_n)$ we can find (for every $i = o,1,\ldots,8$) some $\eta_i \in S_1$ such that $\eta_i \bmod (\sigma S) = \xi_i$. We can then construct n hyperquadrics $F_1,\ldots,F_n$ from $k[T_o,\ldots,T_8,T_9]$ such that $\deg(T_9) = r$, $F_i(T_o,\ldots,T_8,o) = f_i(T_o,\ldots,T_8)$ $(i = 1,\ldots,n)$ and $S = k[T_o,\ldots,T_9]/(F_1,\ldots,F_n)$.

If $r \geqslant 3$ then the variable T_9 cannot occur in F_i, i.e. $F_i(T_o,\ldots,T_9) = f_i(T_o,\ldots,T_8)$, $i = 1,\ldots,n$. This shows that $X = C(Y,\omega_Y^{-r})$, contradicting the assumption that α is not an isomorphism.

If $r = 2$ and T_9 does not occur in any F_i, one gets in the same way a contradiction. If $r = 2$ but T_9 does occur in at least one F_i, then T_9 occurs precisely in one F_i (say in F_1), provided that the system of generators $(f_1,\dots,f_n)$ is supposed to be minimal. In other words we have:

$$F_1(T_o,\dots,T_9) = aT_9 + f_1(T_o,\dots,T_8)\text{ , with } a\in k-\{o\}\text{, and}$$
$$F_i(T_o,\dots,T_9) = f_i(T_o,\dots,T_8) \quad \text{for } i = 2,\dots,n.$$

Then we get

$$S = k[T_o,\dots,T_9]/(F_1,\dots,F_n) \cong k[T_o,\dots,T_8]/(F_2,\dots,F_n) = k[T_o,\dots,T_8]/(f_2,\dots,f_n).$$

Therefore $X = \mathrm{Proj}(S) = \mathrm{Proj}(k[T_o,\dots,T_8]/(f_2,\dots,f_n)) \subseteq P^8$. Then the intersection of the hypersurface $f_1 = o$ with X is (transversal and equals) Y. Therefore $\deg(X) = 1/2.\deg(Y) = 4$. But X is a non-degenerate subvariety of P^8 of dimension 3 and degree 4, which contradicts the well-known inequality $\deg(X) \geqslant \mathrm{codim}(X,P^8) + 1$.

The last case to consider is the one where $r = 1$. Since $S/\sigma S = \bigoplus_{m=o}^{\infty} H^o(\omega_Y^{-m})$, the latter algebra is generated by its homogeneous part of degree one and the degree of σ is 1, $L \cong O_X(Y)$ is very ample and yields an embedding of X in P^9 as a subvariety of degree 8. Moreover, $\omega_X^{-1} \cong L^2$, i.e. X is a singular Fano 3-fold of index 2 in the terminology of Iskovskih [26]. If X would be smooth then Iskovskih proved that such a 3-fold cannot exist (see loc. cit., page 504). We shall mimic the proof of Iskovskih in order to show that such a singular 3-fold also cannot exist. The method (classically called the sweeping method) consists in the following. Let H_o be a hyperplane of P^9 such that $X\cap H_o = Y$ and H_1 another hyperplane such that $X\cap H_1 \cong F_1$ and the curve $C = X\cap H_o\cap H_1$ is smooth. Then C is necessarily elliptic. Consider the pencil (H_λ) of hyperplanes containing $H_o\cap H_1$. We get a rational map $X \dashrightarrow P^1$ which is not defined precisely along the curve C. Let P be the divisor on X which is the closure of the subvariety of X swept out by the lines E_λ of $X\cap H_\lambda \cong F_1$ (E_λ is the only one curve of $X\cap H_\lambda$ with negative self-intersection). For every λ we have $(C.E)_{X\cap H_\lambda} = (C.E_o)_Y = Y.E_o = 1$ (E_o is the unique curve of Y with negative self-intersection). We get that P is a ruled surface, which cannot contain the curve C because C is elliptic and $(C.E_o) = 1$. This easily implies that $P\cap Y = E_o$, and therefore there is a line bundle $M\in \mathrm{Pic}(U)$, with $M = O_U(P/U)$, such that $M\otimes O_Y = O_Y(1)$. But since

Pic(U) was supposed to be generated by L, we get obviously a contradiction because M cannot be a multiple of L (otherwise $\mathcal{O}_Y(1)$ would be a multiple of ω_Y^{-1}). Lemma 2 is completely proved. Q.E.D.

Remark. The proof of lemma 2 extends the arguments of [5].

Theorem 4 (Char(k) = o). Let X be a normal (singular) projective 3-fold containing the surface $Y = F_e$, with $e \geq 1$, as an ample Cartier divisor. Then X is isomorphic to the cone $C(Y,N_{Y,X})$ and Y is embedded in X as the infinite section of $C(Y,N_{Y,X})$.

Proof. Write $N_{Y,X} = O_Y(t) \otimes p^*O(s)$, with $s > te > o$ (see the proof of lemma 2). If t = 1 the proof works exactly as the proof of Theorem 3, case a = 1 or b = 1 (using Theorem A). Therefore we can assume $t \geq 2$. By lemma 2 there are $L,M \in Pic(U)$ such that $L \otimes O_Y \cong p^*O(1)$ and $M \otimes O_Y \cong O_Y(1)$. If F is a coherent sheaf on U we shall denote by F' the sheaf $j_*(F)$, where $j:U \hookrightarrow X$ is the canonical inclusion. Then using lemma 1 one can show that $H^p(L' \otimes O_X(mY)) = o$ for every $m \in \mathbb{Z}$ and p = o,1. This implies in particular that the restriction map $H^o(U,L) = H^o(X,L') \longrightarrow H^o(Y,p^*O(1))$ (whose kernel and cokernel are respectively $H^o(X,L' \otimes O_X(-Y))$ and $H^1(X,L' \otimes O_X(-Y))$) is an isomorphism. From this we infer that the linear system $|L|$ on U yields a rational mapping $q:U \dashrightarrow P^1$, and there is no loss of generality in assuming it is defined on U (either simply restricting U to a smaller neighbourhood of Y in X, or using the arguments of the proof of Theorem 1 in [4]). Moreover, the map q has the properties that $q/Y = p$ and $q^*O(1) \cong L$.

Consider now the coherent O_X-module $F = (\underset{\sim}{Hom}_U(q^*(O \oplus O(-e)),M))' = (M \oplus (M \otimes L^e))'$, whose restriction to Y is $F_Y = \underset{\sim}{Hom}_Y(p^*(O \oplus O(-e)),O_Y(1)) = O_Y(1) \oplus (O_Y(1) \otimes p^*O(e))$. We have the exact sequence

$$o \longrightarrow G = ((M^{1-t} \otimes L^{-s}) \oplus (M^{1-t} \otimes L^{e-s}))' \longrightarrow F \longrightarrow F_Y \longrightarrow o$$

(indeed, since $N_{Y,X} = O_Y(t) \otimes p^*O(s)$ and the fact that α is bijective, we have $O_X(Y) \cong M^t \otimes L^s$). Now we claim that

(2) $\qquad H^1(X,G) = o.$

Let us assume for the moment (2) proved. From the above exact sequence it follows that the restriction map $Hom(q^*(O \oplus O(-e)),M) \longrightarrow Hom(p^*(O \oplus O(-e)),O_Y(1))$ is surjective. Since p is the projection of $P(O \oplus O(-e))$, the definition of the projective bundle yields a canonical surjective homomorphism

$$\varphi \in Hom(p^*(O \oplus O(-e)),O_Y(1)).$$

By surjectivity of the above restriction map we infer that there is a map $\varphi' \in \mathrm{Hom}(q^*(O\oplus O(-e)),M)$ such that $\varphi'/Y = \varphi$. Since φ is surjective, Nakayama's lemma shows that we can assume that φ' is also surjective. Then by the definition of the P^1-bundle $P(O\oplus O(-e))$, there exists a unique morphism $\pi: U \longrightarrow Y = F_e$ such that $\pi^*(O_Y(1)) \cong M$ and $\pi/Y = \mathrm{id}$. Then the conclusion of Theorem 4 comes from the following general lemma:

Lemma 3. Let Y be a smooth projective variety of dimension $\geqslant 2$, and X a normal projective variety X containing Y as an ample Cartier divisor. Assume that the following two conditions are fulfilled:

i) $H^1(Y, N_{Y,X}^{-m}) = o$ for every $m > o$ (automatically fulfilled if $\mathrm{char}(k) = o$ by Kodaira Vanishing Theorem).

ii) There exists a rational mapping $\pi: X \longrightarrow Y$ defined in a neighbourhood of Y in X, such that $\pi/Y = \mathrm{id}$.

Then X is isomorphic to the projective cone $C(Y, N_{Y,X})$ and Y is the infinite section.

Proof of lemma 3. Let U be a (Zariski) open neighbourhood of Y in X such that $U \subseteq X_{reg}$ and π is defined in U, and $i: Y \hookrightarrow X$ the canonical inclusion. Since $\pi \circ i = \mathrm{id}$, the composition of the natural maps

$$\mathrm{Pic}(Y) \xrightarrow{\ \pi^*\ } \mathrm{Pic}(U) \xrightarrow{\ i^*\ } \mathrm{Pic}(Y)$$

is also identity. We claim that $O_X(Y)/U \cong \pi^*(N_{Y,X})$. Indeed, since both line bundles are mapped by i^* into $N_{Y,X}$, it will be sufficient to know that i^* is injective. But the injectivity of i^* follows from i) and [20], exposé XI, Theorem 3.12.

Now, since X–U is finite and X normal, $H^o(U, O_X(mY)) = H^o(X, O_X(mY))$ for every $m \geqslant o$. Set $S' = \bigoplus_{m=o}^{\infty} H^o(X, O_X(mY))$ and $S = \bigoplus_{m=o}^{\infty} H^o(Y, N^m_{Y,X})$. Consider the natural homomorphism of graded k-algebras $\pi^*: S \longrightarrow S'$, so that we get a homomorphism $\lambda: S[T] \longrightarrow S'$ by $\lambda/S = \pi^*$ and $\lambda(T) = \sigma$, where $\sigma \in S'_1$ is a global equation of Y, and T a variable over S (such that the gradation of $S[T]$ is given by $\deg(sT^m) = \deg(s) + m$, where $s \in S$ is a homogeneous element). The point is to show that λ is actually an isomorphism. In order to do this, it will be sufficient to check the surjectivity of λ, because $S[T]$ and S' are both integral domains of the same dimension. The surjectivity of λ is a consequence of the isomorphism $S'/\sigma S' \cong S$. To establish the latter isomorphism, one has just to look at the following commutative diagram ($m \geqslant o$):

$$\begin{array}{ccc} H^o(X,O_X(mY)) = H^o(U,O_X(mY)) & = & H^o(U,\pi^* N^m_{Y,X}) \\ & \nearrow^{\pi^*} & \downarrow i^* \\ H^o(Y,N^m_{Y,X}) & = = = = & H^o(Y,N^m_{Y,X}) \end{array}$$

(from which we deduce that the restriction map $S' \longrightarrow S$ (whose kernel is $\sigma S'$) is surjective). Q.E.D.

Returning to the proof of Theorem 4, it remains to prove (2). But (2) is equivalent to the following two equalities:

$$H^1(X,(M^{1-t}\otimes L^{-s})') = H^1(X,(M^{1-t}\otimes L^{e-s})) = o.$$

Using lemma 1 these equalities follow from:

$$(2')\quad H^1(Y,O_Y(1-t(m+1))\otimes p^*O(-s(m+1))) = o \quad \text{and}$$

$$(2'')\quad H^1(Y,O_Y(1-t(m+1))\otimes p^*O(e-s(m+1))) = o \quad \text{for every } m\geqslant o.$$

In order to prove equation (2') consider the Leray spectral sequence for the morphism p (with $n = m+1 \geqslant 1$):

$$E_2^{ij} = H^i(P^1,O(-ns)\otimes R^j p_* O_Y(1-nt)) \Longrightarrow H^{i+j}(Y,O_Y(1-nt)\otimes p^*O(-ns)),$$

whose exact sequence in low degrees $o \longrightarrow E_2^{1,o} \longrightarrow H^1 \longrightarrow E_2^{o,1}$ shows that (2') follows from $E_2^{1,o} = E_2^{o,1} = o$. Now we have:

$$E_2^{1,o} = H^1(P^1,O(-ns)\otimes p_* O_Y(1-nt)) = o$$

since for every $n\geqslant 1$ and $t\geqslant 2$ we have $1-nt<o$, and hence $p_*O_Y(1-nt) = o$ since p is the projection of a P^1-bundle.

On the other hand, by the relative duality we have

$$R^1p_*O_Y(1-nt) \cong \underset{\sim\sim}{Hom}_{P^1}(O(-e)\otimes p_*O_Y(nt-3),\, O_{P^1}) \cong O(e)\otimes(S^{nt-3}(O\oplus O(-e)))^{\vee} \cong$$

$$\cong \bigoplus_{i=o}^{nt-3} O((i+1)e) \qquad (\text{if } n = 1,\ S^{nt-3}(O\oplus O(-e)) = o).$$

In these formulae we used the fact that $\omega_{Y,P^1} = O_Y(-2)\otimes p^*O(-e)$ and the notation $S^m(E)$ for the m^{th} symmetric power of the vector bundle E. Therefore we get

$$E_2^{o,1} = \bigoplus_{i=o}^{nt-3} H^o(P^1,O(-ns+(i+1)e)).$$

But $-ns+(i+1)e \leqslant -ns+(nt-2)e = -n(s-te)-2e < o$ for every $o\leqslant i\leqslant nt-3$. Therefore $E_2^{o,1} = o$.

In a completely similar way one proceeds for (2") using the inequalities $-ns+e+(i+1)e\leqslant -ns+(nt-2)e+e = -n(s-te)-e<o$ for $o\leqslant i\leqslant nt-3$.

Theorem 4 is completely proved. Q.E.D.

The case where Y is of the form $P(E)\longrightarrow P^1$, with E a vector bundle of rank $\geqslant 3$ over P^1 is completely analogous and in fact much easier to prove.

The result is:

Theorem 5. If $Y = P(E)$, with E a vector bundle of rank $r \geqslant 3$ over P^1, is an ample Cartier divisor on the normal (singular) projective variety X, then X is isomorphic to the projective cone $C(Y, N_{Y,X})$ and Y is embedded in X as the infinite section.

Theorem D also regards the situation when Y is a P^r-bundle ($r \geqslant 1$) over a curve of positive genus and X is smooth. As far as the case where Y is a P^r-bundle over such a curve and X has singularities is concerned, we have the following partial answer:

Theorem 6 (Char(k) = o). Let B be a smooth projective curve of genus $g \geqslant 1$, E a vector bundle of rank $r \geqslant 2$ over B, and $p: Y = P(E) \longrightarrow B$ the canonical projection of the projective bundle $P(E)$. Assume that the normal (singular) variety X contains Y as an ample Cartier divisor. Write $N_{Y,X} \cong O_{P(E)}(t) \otimes p^*(L)$, with $t > o$ and $L \in \mathrm{Pic}(B)$, and assume $t \geqslant 2$. Then X is isomorphic to the projective cone $C(Y, N_{Y,X})$ and Y is the infinite section of X in any of the following two cases:

a) $r \geqslant 3$, or

b) $r = 2$ and E is decomposable.

Proof. If $U = \mathrm{Reg}(X)$ the Lefschetz theorem and the Albanese mapping (see [4], Theorem 3) yield a commutative diagram

$$\begin{array}{ccc} Y & \hookrightarrow & U \\ & {}_{p}\searrow \quad \swarrow_{q} & \\ & B & \end{array}$$

and the existence of a $M \in \mathrm{Pic}(U)$ such that $M \otimes O_Y \cong O_Y(1)$ and $O_X(Y)/U \cong$ $\cong M^t \otimes q^*(L)$. Consider the following exact sequence

$$o \longrightarrow (q^*(\check{E} \otimes L^{-1}) \otimes M^{1-t})' \longrightarrow (\underset{U}{\underline{\mathrm{Hom}}}(q^*(E), M))' = (q^*(\check{E}) \otimes M)' \longrightarrow$$
$$\longrightarrow \underset{Y}{\underline{\mathrm{Hom}}}(p^*(E), O_Y(1)) \longrightarrow o,$$

where, as in the proof of Theorem 4, we denote by F' the coherent O_X-module $j_*(F)$, where F is a coherent O_U-module and $j: U \hookrightarrow X$ is the canonical inclusion. If we show that

(3) $\quad H^1(X, (q^*(\check{E} \otimes L^{-1}) \otimes M^{1-t})') = o,$

then (exactly as in the proof of Theorem 4) one gets a surjective homomorphism $\varphi' \in \mathrm{Hom}_U(q^*(E), M)$ whose restriction to Y is the canonical surjection

$\varphi \in \mathrm{Hom}(p^*(E), O_{P(E)}(1))$, and therefore a morphism $\pi : U \longrightarrow Y$ such that $\pi/Y =$ $=$ id. Then again using lemma 3 one gets the conclusion.

It remains therefore to prove (3). By lemma 1 (3) will follow from

$$H^1(Y, p^*(\check{E} \otimes L^{-1-s}) \otimes O_Y(1-t-st)) = o \quad \text{for every } s \geqslant o,$$

or else (replacing s by s+1):

$$(4) \quad H^1(Y, p^*(\check{E} \otimes L^{-s}) \otimes O_Y(1-ts)) = o \quad \text{for every } s \geqslant 1.$$

To prove (2) consider the Leray spectral sequence

$$E_2^{ij} = H^i(B, \check{E} \otimes L^{-s} \otimes R^j p_* O_Y(1-ts)) \Longrightarrow H^{i+j}(Y, p^*(\check{E} \otimes L^{-s}) \otimes O_Y(1-ts)),$$

and its associated exact sequence in low degrees:

$$(5) \quad o \longrightarrow E_2^{1o} \longrightarrow H^1(Y, p^*(\check{E} \otimes L^{-s}) \otimes O_Y(1-ts)) \longrightarrow E_2^{o1}.$$

If $r \geqslant 3$ then $R^j p_* O_Y(1-ts) = o$ for every $s \geqslant 1$, $t \geqslant 2$ and $j = o, 1$, and therefore in this case (4) follows from (5) because $E_2^{1o} = E_2^{o1} = o$.

Now consider the case $r = 2$. Then again $E_2^{1o} = o$, and it remains to check that E_2^{o1} also vanishes if E is decomposable.

Since $\omega_{Y/B} = O_Y(-2) \otimes p^*(\det(E))$, the relative duality with respect to p gives:

$$R^1 p_* O_Y(1-st) \cong \underset{\sim\sim}{\mathrm{Hom}}_B(p_*(O_Y(st-3) \otimes p^*(\det(E))), O_B) \cong$$
$$\cong \underset{\sim\sim}{\mathrm{Hom}}_B(\det(E) \otimes S^{st-3}(E), O_B) \cong \det(E)^{-1} \otimes S^{st-3}(E)^{\vee}.$$

Therefore we also have $E_2^{o1} = o$ if we show that:

$$(6) \quad H^o(B, \check{E} \otimes L^{-s} \otimes \det(E)^{-1} \otimes S^{st-3}(E)^{\vee}) = o \quad \text{for every } s \geqslant 1.$$

In order to do it, we can assume E normalized, i.e. $E = O_B \oplus L'$, where $L' = \det(E)$ and $\deg(L') = -e \leqslant o$. If we set $F = \check{E} \otimes L^{-s} \otimes \det(E)^{-1} \otimes S^{st-3}(E)^{\vee}$, we have

$$F = (O_B \oplus L'^{-1}) \otimes L^{-s} \otimes L'^{-1} \otimes (\bigoplus_{i=o}^{st-3} L'^{-i}) \cong (L^{-s} \otimes (\bigoplus_{i=o}^{st-3} L'^{-1-i})) \oplus$$
$$\oplus (L^{-s} \otimes (\bigoplus_{i=o}^{st-3} L'^{-2-i})).$$

Thus, (6) follows if we prove that $\deg(L^{-s} \otimes L'^{-j}) < o$ for every $1 \leqslant j \leqslant st-1$, or else $-s \cdot \deg(L) + je < o$ for every $1 \leqslant j \leqslant st-1$. Recalling that $N_{Y,X} \cong$ $\cong O_Y(t) \otimes p^*(L)$ is ample then by [21], page 382, Proposition 2.2o, $\deg(L) > te$. Therefore $-s \cdot \deg(L) + je \leqslant -s \deg(L) + (st-1)e = -s(\deg(L)-te) - e < o$ since $\deg(L)-te < o$, $s \geqslant 1$ and $e \leqslant o$.

The proof of Theorem 6 is complete. Q.E.D.

Remark. Theorem 6 would remain also valid in case where E is an indecomposable vector bundle of rank 2 over B if we could prove (6) in this case.

§3. Grassmann varieties as ample divisors

In this section we shall prove the following result by refining and simplifying the method used by Fujita in [15].

Theorem 7. Let Y be the Grassmann variety $G_{n,r}$ of r-dimensional subspaces of the n-dimensional vector space V. Assume that $n \geqslant 5$ and $1<r<n-1$ (i.e. Y is neither a projective space, nor $G_{4,2}$) and that Y is contained in the normal projective variety X as an ample Cartier divisor. Then X is isomorphic to the projective cone $C(Y,N_{Y,X})$ and Y is contained in X as the infinite section.

The proof of this theorem uses several lemmae.

Lemma 4. Let X be a normal projective variety of dimension $\geqslant 4$ and Y an ample Cartier divisor on X. Assume that Y is smooth, and let E be a vector bundle of rank $r \geqslant 1$ on Y such that $H^2(Y,\mathrm{End}(E)\otimes N_{Y,X}^{-t}) = 0$ for every $t>0$. Then there exists a Zariski open neighbourhood U of Y in X and a coherent sheaf E' on X such that $E'/Y \cong E$, E'/U is locally free (of rank r) and for every $x \in X-U$, $\mathrm{depth}_x(E'_x) \geqslant 2$.

Proof. If $r = 1$, lemma 4 is well-known and follows from [20]. Assume therefore $r \geqslant 2$. Using the hypothesis that $H^2(Y,\mathrm{End}(E)\otimes N_{Y,X}^{-t}) = 0$ for every $t>0$, Fujita proved that there is a vector bundle E_1 on the formal completion $\hat{X}$ of X along Y, such that $E_1/Y \cong E$ (see [15], proposition 2.1). On the other hand, by [20], exposé X, example 2.2, the pair (X,Y) satisfies the effective Lefschetz condition, Leff(X,Y). In particular, this implies that there is an open neighbourhood U of Y in X and a vector bundle E_2 of rank r on U, such that $\hat{E}_2 \cong E_1$, and hence $E_2/Y \cong E$, where $\hat{E}_2$ stands for the formal completion of E_2 along Y. Since X-U is finite and X is normal, the sheaf $E' = j_*(E_2)$ satisfies the desired conclusion, where $j:U \hookrightarrow X$ is the canonical inclusion. Q.E.D.

Lemma 5 ([15], corollary 1.4). In the notations of lemma 4, let F be a coherent O_X-module such that there is a Zariski open neighbourhood U of Y in X with the property that F/U is locally free and $\mathrm{depth}_x(F_x) \geqslant 2$ for every $x \in X-U$. Assume moreover that the following maps of restriction $H^o(X,F) \longrightarrow H^o(Y,F/Y)$ and $H^o(X,O_X(Y)) \longrightarrow H^o(Y,N_{Y,X})$ are both surjective and that the natural maps $H^o(Y,F\otimes N_{Y,X}^t)\otimes H^o(Y,N_{Y,X}) \longrightarrow H^o(Y,F\otimes N_{Y,X}^{t+1})$ are all surjective for for every $t \geqslant 0$. Then the natural maps $H^o(X,F\otimes O_X(tY))\otimes H^o(X,O_X(Y)) \longrightarrow$

$H^o(X,F\otimes O_X((t+1)Y))$ *are also surjective for every* $t\geqslant o$. *In particular, F is generated by its global sections.*

Assume now that Y is the Grassmann variety $G_{n,r}$ of r-linear subspaces in the n-dimensional vector space V. If $m=\binom{n}{r}-1$, let $u:Y\hookrightarrow P^m$ be the Plücker embedding and $O_Y(1)$ the sheaf of hyperplane sections of Y with respect to u. Then $O_Y(1)$ generates Pic(Y). Moreover, a result proved independently by Hochster, Kempf, Laksov and Musili (see e.g. [27]) asserts that Y is arithmetically Cohen-Macaulay in P^m. Denote by V_Y the trivial vector bundle of rank n over Y and by E the universal quotient bundle of V_Y of rank n-r.

Lemma 6. In the above notations one has:

i) $H^o(Y,E)=V$ *and* $H^o(Y,E\otimes O_Y(-t))=o$ *for every* $t>o$.

ii) $H^1(Y,E\otimes O_Y(t))=o$ *for every* $t\in\mathbb{Z}$.

iii) *If* $r\geqslant 3$ *then* $H^2(Y,\underline{End}(E)\otimes O_Y(-t))=o$ *for every* $t>o$.

iv) *The natural maps* $H^o(Y,E\otimes O_Y(t))\otimes H^o(Y,O_Y(1))\longrightarrow H^o(Y,E\otimes O_Y(t+1))$ *are surjective for every* $t\geqslant o$.

Lemma 6 is proved in [15] (see 4.11, 4.17, 4.18 and 4.2o) and relies essentially on some results of Kempf [27].

Corollary (of lemmas 4,5,6 and 1). Assume that $Y=G_{n,r}$ *is an ample Cartier divisor on the normal projective variety X. If* $n\geqslant 5$ *and* $3\leq r<n-1$ *then the universal bundle E can be extended to a coherent sheaf E' on X such that E/U is locally free (of rank n-r) and* $depth_x(E'_x)\geqslant 2$ *for every* $x\in X-U$, *with U a suitable open neighbourhood of Y in X. Moreover, the restriction map* $H^o(X,E')\longrightarrow H^o(Y,E)$ *is an isomorphism and E' is generated by its global sections.*

Proof. Since $O_Y(1)$ generates Pic(Y), there is a (unique) $s\in\mathbb{Z}$ such that $N_{Y,X}=O_Y(s)$, and since Y is ample in X, $s>o$. By lemma 6, iii) and lemma 4, E can be extended to an E' with the properties stated in the first part of the corollary. To see that the restriction map $H^o(X,E')\longrightarrow H^o(Y,E)$ is an isomorphism, apply lemma 1 and lemma 6, i) and ii) to the sheaf $F=E'\otimes O_X(-Y)$ in order to deduce that $H^p(X,E'\otimes O_X(-Y))=o$ for p = o,1. The fact that E' is generated by its global sections follows from lemma 5 and lemma 6, iv). Q.E.D.

Proof of Theorem 7. First of all observe that there is no loss of generality in assuming that $r\geqslant 3$. Indeed, there is a canonical isomorphism between

$G_{n,r}$ and $G_{n,n-r}$ such that the universal quotient bundle of rank r of $G_{n,n-r}$ is identified to $\mathrm{Ker}(V_Y \longrightarrow E)$, and so, if r = 2, then $n-r \geqslant 3$ (in the hypotheses of our Theorem). Therefore we can apply the above corollary and deduce that there is an extension E' of E with all the properties stated there. Since E' is generated by its global sections and the map $H^o(X,E') \longrightarrow H^o(Y,E) = V$ is an isomorphism, E' is a quotient of the trivial bundle V_X of rank n over X. Recalling that E'/U is locally free of rank n-r, the universal property of $G_{n,r}$ yields a unique morphism $\pi : U \longrightarrow Y = G_{n,r}$ such that $\pi^*(E) \cong E'/U$. Since $E'/Y \cong E$ we have $\pi/Y = \mathrm{id}$. Now we can apply lemma 3 to derive the conclusion. Indeed, we just verified condition ii) of that lemma, while condition i) follows (in arbitrary characteristic) since Y is arithmetically Cohen-Macaulay in P^m $(m = \binom{n}{r} - 1)$. Q.E.D.

Remarks. 1. Fujita proved in [15] that the Grassmann variety $G_{n,r}$ ($n \geqslant 5$ and $1 < r < n-1$) cannot occur as an ample divisor on any smooth projective variety (see also [14] for another proof). Theorem 7 above should be considered as a strengthening of this result.

2. In the hypotheses of Theorem 7, write $N_{Y,X} \cong O_Y(s)$ for some $s > o$. Di Fiore and Freni also proved a result which is equivalent to Theorem 7 in the case s = 1, by extending a classical method of G. Scorza [13]. On the other hand, if $s \geqslant 3$ Theorem 7 is a direct corollary of Theorem 4 in [4] because $G_{n,r}$ is given in P^m (scheme theoretically) by quadratic equations.

3. Theorem 7 is also valid when $n \geqslant 3$ and $(n,r) = (n,1)$ or $(n,r) = (n,n-1)$, i.e. if Y is a projective space of dimension $\geqslant 2$. This result was proved in [3], Theorem 1, but it turned out to be classical and due to C. Segre and G. Scorza (see [32]).

4. The last exception in Theorem 7 is the one when $Y = G_{4,2}$. In this case Y is isomorphic to a (smooth) hyperquadric in P^5 and the conclusion of Theorem 7 is no longer true in this case. However, one can also enumerate all normal projective varieties X containing $G_{4,2}$ as an ample Cartier divisor. This comes from the following more general result:

Proposition 1. Let Y be a smooth hyperquadric in P^{n+1} $(n \geqslant 3)$ and X a nor- projective variety containing Y as an ample Cartier divisor. Then X is (isomorphic to) one of the following:

i) P^{n+1} and Y is contained in P^{n+1} as a hyperquadric.

ii) A smooth hyperquadric in P^{n+2} and Y is a hyperplane section of it.

iii) The projective cone $C(Y,N_{Y,X})$ and Y is the infinite section.

Proof. If X is smooth, it is known that we have either i) or ii) (see [28], or [37], corollary of §IV). So we can assume X non-smooth, in which case X is isomorphic to a hyperquadric in the weighted projective space $P(\underbrace{1,1,\ldots,1}_{n+2 \text{ times}},s)$ for some $s>0$. Using this and remark 1 (after the proof of Theorem 3) one gets easily iii). Q.E.D.

We shall close this section by two further remarks. The first one shows that the hypothesis of normality is not indispensable in some geometric situations. More precisely:

Proposition 2. Suppose that the smooth subvariety Y of P^n of dimension $\geqslant 2$ has the following property:

(*) Every normal projective variety Z containing Y as an ample Cartier divisor such that $N_{Y,X} \cong O_Y(1)$ (the sheaf of hyperplane sections of Y with respect to P^n), is isomorphic to $C(Y,O_Y(1))$.

Suppose furthermore that X is an arbitrary subvariety of P^{n+1} such that there is a hyperplane H in P^{n+1} with the property that $X\cap H = Y$ (scheme-theoretically). Then there is a point $x\in X$ such that X is the union of all lines of P^{n+1} joining x and an arbitrary point of Y.

Proof. Let $u:\tilde{X}\longrightarrow X$ be the morphism of normalization. Since in our hypotheses Y is contained in the smooth locus of X, Y is also contained in $\tilde{X}$ as an ample Cartier divisor and $N_{Y,\tilde{X}}\cong N_{Y,X}\cong O_Y(1)$. Since by (*) $\tilde{X}$ is isomorphic to $C(Y,O_Y(1))$, let $\tilde{x}$ be the vertex of $\tilde{X}$ and set $x = u(\tilde{x})$. Let y be an arbitrary point of Y and $\tilde{E}$ the generating line of the cone $\tilde{X}$ passing through ($\tilde{x}$ and) y, and set $E = u(\tilde{E})$. Then the curve E passes through x and y, and since the degree of $\tilde{E}$ (with respect to the line bundle $O_{\tilde{X}}(Y)$) is one, and $u^*O_X(Y)\cong O_{\tilde{X}}(Y)$, we infer that the degree of E in P^{n+1} is one, i.e. E is a line in P^{n+1} because E is integral. Q.E.D.

In particular, let us explain a little bit how Theorem 1 of [3] was classically formulated (see [34]). Let $v_s:P^n\hookrightarrow P^m$ (with $m = \binom{n+s}{n}-1$ and $n\geqslant 2$) be the Veronese embedding, and denote by $Y\subset P^m$ the image of v_s. Suppose that we are given a subvariety of P^{m+1} such that there is a hyperplane H in P^{m+1} whose intersection with it is Y. Then this subvariety satisfies the conclusion of Proposition 2. This fact was first observed by C. Segre in the case of the Veronese surface in P^5 and subsequently extended to the general case by G. Scorza (loc. cit.). Using Proposition 2 and some standard facts, it is not difficult to see that this classical result is in fact equivalent to Theorem 1 in [3].

The last remark (which is inspired from [34]) concerns the following situation. Let (Y,L) be a polarized variety of dimension $\geqslant 2$ such that:

a) L is very ample and yields an arithmetically normal embedding of Y in P^n, with $n = \dim |L|$.

b) For every normal projective variety X containing Y as an ample Cartier divisor and such that $N_{Y,X} \cong L$, then X is isomorphic to the projective cone C(Y,L) and Y is the infinite section of C(Y,L).

Put $Y_1 = C(Y,L)$. Then Y is embedded in Y_1 as an ample Cartier divisor (via the infinite section) and $N_{Y,Y_1} \cong L$. Denoting by S the graded k-algebra $\bigoplus_{t=o}^{\infty} H^o(Y_1, O_{Y_1}(tY))$ and by $\sigma \in S_1$ a global equation of Y in Y_1, and using lemma 3, one gets that $S/\sigma S$ is isomorphic (as graded k-algebra) to $\bigoplus_{t=o}^{\infty} H^o(Y,L^t)$. Since we assumed that a) holds, the latter algebra is generated by its homogeneous part of degree one, and thus S has the same property. Moreover, since $\mathrm{depth}(S/\sigma S) \geqslant 2$, we have $\mathrm{depth}(S) \geqslant 3$. Therefore a) implies:

a') Y is a very ample divisor on Y_1 and yields an arithmetically normal embedding of Y in P^{n+1} ($n+1 = \dim |Y|$); moreover, $H^1(Y_1, O_{Y_1}(tY)) = o$ for every integer t.

Now we want to show that a) and b) together imply:

b') For every normal projective variety Y_2 containing the cone Y_1 as an ample Cartier divisor and such that $N_{Y_1,Y_2} \cong O_{Y_1}(Y)$, then Y_2 is isomorphic to the cone $C(Y_1, O_{Y_1}(Y))$.

Proof of b'). Using a') and lemma 1, one easily gets that $H^1(Y_2, O_{Y_2}(tY_1)) = o$ for every integer t. Again from this we deduce that there is an isomorphism of graded k-algebras $S'/\tau S' \cong S$, where S' is $\bigoplus_{t=o}^{\infty} H^o(Y_2, O_{Y_2}(tY_1))$ and $\tau \in S'_1$ is a global equation of Y_1 on Y_2. From this we infer that S' is generated by S'_1, $\mathrm{depth}(S') \geqslant 4$, and that the divisor Y_1 is very ample on Y_2 and yields an embedding of Y_2 in P^{n+2} such that there is a hyperplane H (in P^{n+2}) with the property that $Y_2 \cap H = Y_1 = C(Y,L)$. Let then H' be another hyperplane in P^{n+2}

such that $Y_2 \cap H \cap H' = Y_1 \cap H = Y$. Since $Y_2 = \mathrm{Proj}(S')$ and $\mathrm{depth}(S') \geqslant 4$, Y_2 has the property S_4 of Serre (recall that a local ring A has property S_k if $\mathrm{depth}(A) \geqslant \inf\{k, \dim(A)\}$, and a scheme Z has property S_k if for every point $z \in Z$ the local ring $O_{Z,z}$ has property S_k), and therefore $X' = Y_2 \cap H'$ has property S_3. Moreover, X' supports Y as an ample Cartier divisor and is regular in codimension 1. Using Serre's criterion of normality we then deduce that X' is normal. Applying b) we have $X' = C(Y,L)$, and let x' be the vertex of the cone X'. Now varying H' in the pencil of hyperplanes containing the linear subspace $L = H' \cap H$, the geometric locus of x' is a curve C, which is easily seen to be a line in P^{n+2}. Then it is clear that Y_2 is just the join of the line C with Y_2, which is exactly b'). Q.E.D.

Since in this paper as well as in [3] and [4] we provided many examples of varieties Y satisfying the property b) with respect to every ample line bundle over Y, we can apply (b') to several situations, e.g. when Y is a projective space, or a Grassmann variety, etc. In particular, if we take $(Y,L) = (P^n, O(s))$, with $n \geqslant 2$ and $s \geqslant 1$, we get that the cone $Y_1 = C(P^n, O(s))$ satisfies b'). In order to state more precisely what we can get using this example, it is convenient to use the language of weighted projective spaces (see [12]).

Start with the interpretation $P(\underbrace{1,1,\ldots,1}_{n+1 \text{ times}},s) = C(P^n, O(s))$. Since the sheaf $O_{P(1,\ldots,1,s)}(t)$ is invertible iff s divides t, we get that a normal projective cone over $P(\underbrace{1,1,\ldots,1}_{n+1 \text{ times}},s)$ is a weighted projective space of type $P(\underbrace{1,1,\ldots,1}_{n+1 \text{ times}},s,t)$, with t a multiple of s, and so on. Sumarizing the above discussion for $(Y,L) = (P^n, O(s))$ $(n \geqslant 2)$ and using induction, one gets the following variant of Theorem 1 in [3]:

<u>Proposition 3.</u> <u>Let $Y_o \subset Y_1 \subset Y_2 \subset \ldots$ be a sequence of normal projective varieties such that $Y_o = P^n$, with $n \geqslant 3$ (or $n = 2$ and $\mathrm{char}(k) = o$), and for every $i \geqslant 1$ Y_{i-1} is an ample Cartier divisor on Y_i. Then there exists a sequence of positive integers $q_1, q_2, \ldots$ such that for every $i \geqslant 2$ q_{i-1} divides q_i, and Y_i is isomorphic to the weighted projective space $P(\underbrace{1,1,\ldots,1}_{n+1 \text{ times}},q_1,\ldots,q_i)$ for $i \geqslant 1$. Furthermore, the inclusion $Y_{i-1} \subset Y_i$ corresponds to the natural surjection of polynomial k-algebras $k[T_o,\ldots,T_{n+i}] \longrightarrow k[T_o,\ldots,T_{n+i-1}]$, which maps T_{n+i} into zero and leaves the other variables fixed.</u>

Remark. A more general case of the problem of of weighted projective spaces as ample divisors was considered in [10], as a natural extension of Theorem 1 in [3]. In particular, one proves a more general result than proposition 3 above.

§4. Applications to deformations of projective cones

It is well known that the classification of certain subvarieties of a projective space can give interesting informations concerning the deformation theory. For example the classification of all non-degenerate subvarieties of degree 3 in projective spaces (see [38]) yields in particular an elementary proof of the non-smoothability of the cone in P^6 over $P^1 \times P^2$ via the Segre embedding. Schlessinger constructed many examples of non-smoothable or rigid affine cones over certain projective varieties of dimension ≥ 2 (see [33]). Using and refining Schlessinger's idea, Mumford provided examples of affine cones over certain smooth curves of genus ≥ 2 which are not smoothable (see [30]). In his thesis Pinkham obtained more precise results (see [32]). Hartshorne discussed some conditions for smoothing a subvariety of P^n (see [23]). More recently, Sommese [36] and Fujita [14] gave further examples of non-smoothable projective cones, using results about the impossibility of certain projective manifolds of being ample divisors in another manifolds.

In the spirit of [14] and [36] in this section we are going to apply some results of ours about ample divisors to deformations of certain projective cones. Although our setting is slightly different from Schlessinger's, we cannot claim getting essentially new results. The only reason of presenting them lies in the fact that the proofs are different from the usual ones.

To fix our setting we need some definitions. Let X be a closed subscheme of P^p. An (embedded) deformation of X in P^n is a closed subscheme $U \subset P^n \times T$ which is flat over the parameter space T and such that there is a k-rational point $o \in T$ with the property that the fibre X_o of U over the point o is isomorphic to X. Such a deformation will be simply denoted by (U,T,o), or by $\{X_t\}_{t\in T}$, where $X_t = U \cap (P^n \times \{t\})$ is the fibre of U over t, if no danger of confusion is possible. X is said to be smoothable in P^n if there exists a deformation (U,T, o) of X in P^n such that $\dim(T) > o$, T connected and X_t is smooth for every $t \neq o$. X is rigid in P^n if for every deformation (U,T,o) of X in P^n there exists a Zariski open neighbourhood T' of o in T such that for every k-rational point

$t \in T'$, X_t is isomorphic to X.

Let X be a closed subscheme of P^n having a certain property (P), e.g. to be a complete intersection in P^n of type $(d_1,\dots,d_r)$, etc. We say that every small deformation of X in P^n has also the property (P) if for every deformation (U,T,o) of X in P^n there is a Zariski open neighbourhood T' of o in T such that X_t is also a subscheme of P^n having the property (P) for every k-rational point $t \in T'$.

Let X be an arbitrary proper scheme over k. We say that X is (algebraically) rigid if for every proper flat morphism $f:U \longrightarrow T$ of algebraic schemes over k, with T reduced and connected, such that there is a k-rational point $o \in T$ with the property that $f^{-1}(o)$ is isomorphic to X, there is a Zariski open neighbourhood T' of o in T such that $f^{-1}(t)$ is also isomorphic to X for every k-rational point $t \in T'$. In the case where k is the complex field $\mathbb{C}$, we also say that X is (analytically) rigid if for every proper flat morphism $f:U \longrightarrow T$ of complex-analytic spaces, with T connected and reduced, such that $f^{-1}(o)$ is isomorphic to X for a point $o \in T$, then $f^{-1}(t)$ is also isomorphic to X for every point t belonging to a complex open neighbourhood T' of o in T. Whenever the term "rigid" is used, it is understood both in the algebraic and analytic sense (provided that $k = \mathbb{C}$).

<u>Proposition 4. Let $f:U \longrightarrow T$ be a proper flat morphism of algebraic schemes over k (resp. of complex-analytic spaces), with T connected and reduced. Assume that there is a k-rational point (resp. a point) $o \in T$ such that the fibre $X_o = f^{-1}(o)$ is isomorphic to a closed subscheme of P^n having a certain property (P). Assume moreover that every small deformation of X_o in P^n has also the property (P), $H^i(X_o, O_{X_o}) = o$ for $i = 1,2$ and $H^1(X_o, O_{X_o}(1)) = o$, where $O_{X_o}(1)$ is the sheaf of hyperplane sections of X_o with respect to the inclusion of X_o in P^n. Then there exists a Zariski open neighbourhood (resp. a complex open neighbourhood) T' of o in T such that the fibre $X_t = f^{-1}(t)$ over every k-rational point (resp. over every point) $t \in T'$ is isomorphic to a closed subscheme of P^n having the property (P).</u>

<u>Proof.</u> Claim: There is an étale (resp. an open in the complex topology) neighbourhood $T_1 \longrightarrow T$ (resp. $T_1 \subseteq T$) of the point $o \in T$ and an invertible sheaf L on $U_1 = U \times_T T_1$ inducing on X_o the sheaf $O_{X_o}(1)$.

In order to prove the claim we shall distinguish between the algebraic and

and the analytic one.

First let us fix some notations. Set $A = O_{T,o}$ (the local ring of T at o), m - the maximal ideal of A, $\hat{A}$ the m-adic completion of A, and $\tilde{A}$ the henselization of A with respect to m. For every $p \geqslant o$ let $U_p = (X_o, O_U/m^{p+1}O_U)$ denote the p^{th} infinitesimal neighbourhood of X_o in U ($U_o = X_o$) and set $N(p) = \dim_k(m^{p+1}/m^{p+2})$. In both cases, for every $p \geqslant o$ consider the standard exponential sequence:

$$o \longrightarrow m^{p+1}O_U/m^{p+2}O_U \cong O_{X_o}^{N(p)} \longrightarrow O^*_{U_{p+1}} \longrightarrow O^*_{U_p} \longrightarrow 1,$$

and taking the cohomology we get the exact sequence

$$H^1(O_{X_o}^{N(p)}) \longrightarrow \mathrm{Pic}(U_{p+1}) \xrightarrow{\ \varepsilon_p\ } \mathrm{Pic}(U_p) \longrightarrow H^2(O_{X_o}^{N(p)}) .$$

Since we assumed that $H^i(O_{X_o}) = o$ for $i = 1,2$, we get that the maps ε_p are isomorphisms for every $p \geqslant o$ (in the analytic case we implicitly used a result of GAGA-type).

Now consider the analytic case. We need the following result (see [6]):

<u>Theorem E</u> (Banica, Bingener, Kuhlmann). <u>Let $f: U \longrightarrow T$ be a proper morphism of complex-analytic spaces and $o \in T$ a point. In the above notations, consider the following natural map (defined in an obvious way):</u>

$$\lambda : \operatorname*{dir\,lim}_{U'} \mathrm{Pic}(U') \longrightarrow \operatorname*{inv\,lim}_{p \geqslant o} \mathrm{Pic}(U_p) ,$$

<u>where U' runs over the set of all complex neighbourhoods of X_o in U. Then λ injective and its image is dense in the topology of the inverse limit.</u>

In our case we just showed that the maps ε_p are isomorphisms for every $p \geqslant o$, and hence $\operatorname*{inv\,lim}_{p \geqslant o} \mathrm{Pic}(U_p)$ reduces to $\mathrm{Pic}(X_o)$. From this we infer that the map λ is an isomorphism, proving the claim in the analytic case.

The algebraic case is more subtle. Consider the following cartesian diagram:

$$\begin{array}{ccccccc}
X_o & \subset & \hat{U} & \longrightarrow & \tilde{U} & \longrightarrow & U \\
\downarrow & & \downarrow \hat{f} & & \downarrow \tilde{f} & & \downarrow f \\
o & \subset & \hat{T} = \mathrm{Spec}(\hat{A}) & \longrightarrow & \tilde{T} = \mathrm{Spec}(\tilde{A}) & \longrightarrow & T
\end{array}$$

First we observe that the restriction map $\mathrm{Pic}(\hat{U}) \longrightarrow \mathrm{Pic}(X_o)$ is an isomorphism. This is a consequence of Theorem 5.1.4 from [19], chapter III and the fact that ε_p is an isomorphism for every $p \geqslant o$. Now by Theorem 3.5 in [1] the map $\mathrm{Pic}(\tilde{U}) \longrightarrow \mathrm{Pic}(\hat{U})$ is also an isomorphism. Applying Corollary 2.2 in [1] we get the claim in the algebraic case. Note that the main point in the proof of the claim in the algebraic case was Artin's approximation theory.

The claim being proved, we can use the base change theorems (see e.g. [21], chap. III, §12 in the algebraic case, and [7] in the analytic case) and the assumption that $H^1(X_o, O_{X_o}(1)) = o$ in order to deduce that there is a Zariski open (resp. an open in the complex topology) neighbourhood T_2 of o in T_1 such that:

i) If $g = f \times_{T_1} T_1$ then $g_*(L)/T_2$ is a free O_{T_2}-module of rank n+1, and

ii) $L/g^{-1}(T_2)$ is very ample with respect to $g/g^{-1}(T_2)$ and yields an embedding of $g^{-1}(T_2)$ in $P^n \times T_2$ (over T_2).

Therefore, in the algebraic case we got an (embedded) deformation of X_o in P^n parametrized by T_2, and accordingly (by our assumptions) there is a Zariski open neighbourhood T_3 of o in T_2 such that $X_t \cong g^{-1}(t)$ is a subscheme of P^n having the property (P) for every k-rational point $t \in T_3$. Since u is étale, $u(T_3)$ is a Zariski open neighbourhood of o in T, and hence we conclude (in the algebraic case) by taking $T' = u(T_3)$.

Finally, in the analytic case one proceeds similarly, but using the GAGA-type fact that the Hilbert scheme H parametrizing all closed subschemes of P^n with the same Hilbert polynomial as X_o, represents both the Hilbert functor defined on the algebraic category and the Hilbert functor defined on the analytic category. Q.E.D.

Corollary. Let $f: U \longrightarrow T$ be a proper flat morphism of algebraic schemes over k (resp. of complex-analytic spaces), with T reduced and connected, such that the fibre $X_o = f^{-1}(o)$ over a k-rational point (resp. over a point) $o \in T$ is isomorphic to a complete intersection in P^n of type $(d_1, \ldots, d_r)$ and of dimension ≥ 3. Then there is a Zariski open (resp. an open in the complex topology) neighbourhoof T' of o in T such that for every k-rational point (resp. point) $t \in T'$ the fibre $X_t = f^{-1}(t)$ is also isomorphic to a complete intersection in P^n of type $(d_1, \ldots, d_r)$.

Proof. Since X_o is a complete intersection in P^n of dimension ≥ 3, we have $H^i(X_o, O_{X_o}(t)) = o$ for every i = 1,2 and for every integer t. On the other hand, it is well known that every small (embedded) deformation in P^n of a complete intersection of type $(d_1, \ldots, d_r)$ is also a complete intersection in P^n of the same type (see e.g. [25]). Thus the hypotheses of Propos. 4 are fulfilled, and the corollary follows applying this proposition. Q.E.D.

Remarks. 1. We are indebted to C. Bănică for showing us his paper [6], which was useful in the proof of the analytic part of Proposition 4.

2. The above corollary is of folklore type. It was included as an illustration of the usefulness of Proposition 4 on one hand, and because it will be used in the proof of Theorem 8 below on the other hand. The analytic part of it was proved in [35] in the framework of Kodaira-Spencer's deformation theory (in the case where everything is smooth, but - except the K-3 surfaces - also including the two-dimensional case).

3. The proof of the corollary works also in the case when X_o is one of the following: a quadric or a cubic in P^3, or a complete intersection of type (2,2) in P^4. Indeed, we have $H^1(X_o, O_{X_o}(t)) = o$ for every integer t (this holds for every surface which is a complete intersection in P^n), and in the above cases also $H^2(X_o, O_{X_o}) = o$.

In order to state the next result let Y be a complete intersection in P^n, and denote by $O_Y(1)$ its sheaf of hyperplane sections. If $\dim(Y) \geq 3$ Lefschetz's theorem says that Pic(Y) is generated by $O_Y(1)$.

Theorem 8. Let $f: U \longrightarrow T$ be a proper flat morphism of algebraic k-schemes (resp. of complex-analytic spaces), with T reduced, and assume that the fibre X_o of f over a k-rational point (resp. over a point) $o \in T$ is isomorphic to the cone $C(Y, O_Y(s))$, where Y is a smooth complete intersection in P^n of type $(d_1, \ldots, d_r)$ such that $n-r \geq 3$ and $s > \max(d_1, \ldots, d_r)$. Then there is a Zariski open (resp. an open in the complex topology) neighbourhood T' of o in T such that for every k-rational point (resp. for every point) $t \in T'$ the fibre X_t over t is isomorphic to the cone $C(Y_t, O_{Y_t}(s))$, where Y_t is a smooth complete intersection in P^n of type $(d_1, \ldots, d_r)$ (but may be not isomorphic to $Y_o = Y$).

Proof. Let $Y \hookrightarrow P^{n(s)}$ be the closed embedding given by the complete linear system $|O_Y(s)|$ $(n(1) = n)$. Then the cone $X = C(Y, O_Y(s))$ lies in $P^{n(s)+1}$.

Step 1. Every small (embedded) deformation of X in $P^{n(s)+1}$ is again a cone of type $C(Y_t, O_{Y_t}(s))$, where Y_t is a complete intersection in P^n of type $(d_1, \ldots, d_r)$.

Proof of step 1. Let $\{X_t\}_{t \in B}$ be an (algebraic, embedded) family of closed subschemes of $P^{n(s)+1}$ parametrized by B, such that $X_o = X$ for a k-rational point $o \in B$. Let $H = P^{n(s)}$ be the hyperplane at infinity of $P^{n(s)+1}$. Putting

$Y_t = X_t \cap H$ we get another algebraic family of subschemes of $P^{n(s)}$ parametrized eventually by a Zariski open neighbourhood of o in B. Since the problem is local around o, one can assume that it is parametrized by B itself. Since $Y_o = X_o \cap H = C(Y,O_Y(s)) \cap H = Y$, we can apply the corollary of Proposition 4 to deduce that Y_t is again (isomorphic to) a complete intersection in P^n of type $(d_1,\dots,d_r)$ for every k-rational point $t \in B$ (always shrinking B to a Zariski open neighbourhood if necessary). Since $\dim(Y_t) = \dim(Y) \geqslant 3$, the Lefschetz's theorem allows us to write $N_{Y_t,X_t} \cong O_{Y_t}(s_t)$ for some $s_t > o$. Since $H^1(Y,O_Y(s)) = o$ we can use the base change theorems to infer that

$$\dim H^o(Y,O_Y(s)) = \dim H^o(Y_t,O_{Y_t}(s_t))$$

for every k-rational point $t \in B$ (shrinking again B if necessary). Recalling that Y and Y_t are both isomorphic to complete intersections of the same type in P^n, we then get $s_t = s$ for every k-rational point $t \in B$.

On the other hand, since X_o is normal, X_t can be assumed to be also normal ([19], chapter IV, 12.1.6). And now comes the main point of the whole proof of Theorem 8! Since X_t is normal, $X_t \cap H = Y_t$ and $s_t = s > \max(d_1,\dots,d_r)$, X_t is the cone $C(Y_t,O_{Y_t}(s))$ by Theorem 4 in [4] (or also by the corollary of Theorem 6 in [4]).

Step 1 is proved.

<u>Step 2 (Conclusion)</u>. We have just to apply Proposition 4. In our situation the property (P) of a normal subvariety W of $P^{n(s)+1}$ is the following: "there exists a smooth complete intersection Y' in P^n of type $(d_1,\dots,d_r)$ such that $W = C(Y',O_{Y'}(s))$ ". By step 1, every small deformation of X in $P^{n(s)+1}$ also has the property (P) (note that by the very definition X has the property (P)). To apply Proposition 4 we have to know that $H^i(X,O_X) = o$ for $i = 1,2$ and $H^1(X,O_X(Y)) = o$. But this follows from the discussion preceding Proposition 3.

The proof of Theorem 8 is complete. Q.E.D.

<u>Corollary.</u> i) <u>The cone $C(P^n,O(s))$ is rigid for every $n \geqslant 3$ and $s \geqslant 1$. The same conclusion holds for $n = 2$ if char(k) = o.</u>

ii) <u>Let Y be a smooth hyperquadric in P^n, with $n \geqslant 4$. Then the cone $C(Y,O_Y(s))$ is rigid for every $s \geqslant 3$.</u>

<u>Proof.</u> Part i) follows in case $n \geqslant 3$ from Theorem 8. The case $n = 2$ can be treated in a similar way. Part ii) also follows from Theorem 8 remarking that any two smooth hyperquadrics in P^n are projectively isomorphic. Q.E.D.

Using Proposition 3 and the same method as in the proof of Theorem 8, part i) of the above corollary can be generalized in the following way.

Theorem 9. Let $n, q_1,\dots,q_i$ be natural numbers such that $n \geqslant 2$ and q_{j-1} divides q_j for every $j = 2,\dots,i$. If $n = 2$ assume moreover that $\mathrm{char}(k) = o$. Then the weighted projective space $P(\underbrace{1,1,\dots,1}_{n+1 \text{ times}},q_1,\dots,q_i)$ is rigid.

Theorem 1o ($\mathrm{Char}(k) = o$). i) Let Y be $P^1 \times P^1$. Then the cone $C(Y,O(a,b))$ is rigid for every $a \geqslant 2$ and $b \geqslant 2$, unless $a = b = 2$.

ii) Let Y be F_1 and $p: Y \longrightarrow P^1$ the canonical projection of Y. Then the cone $C(Y,O_Y(b) \otimes p^* O(a))$ is rigid for every $a > b \geqslant 2$.

For the proof use Theorems 3 and 4, Proposition 4, the rigidity of $P^1 \times P^1$ and F_1 and the same method as in the proof of Theorem 8.

Theorem 11. Let Y be the Grassmann variety $G_{n,r}$, with $n \geqslant 5$ and $1 < r < n-1$. For every $s \geqslant 1$ the cone $C(Y,O_Y(s))$ is rigid.

Use Theorem 7, Proposition 4, the rigidity of the Grassmann variety and the same method as in the proof of Theorem 8.

Theorem 12 ($\mathrm{Char}(k) = o$). Let Y be an elliptic curve, L a line bundle on Y of degree $\geqslant 1o$ and X the projective cone $C(Y,L)$. Let $f: U \longrightarrow T$ be a proper flat morphism of algebraic schemes over k (resp. of complex-analytic spaces) such that the fibre of f over a k-rational point (resp. over a point) $o \in T$ is isomorphic to X. Then there is a Zariski open (resp. a complex open) neighbourhood T' of o in T such that the fibre of f over t is isomorphic to the cone $C(Y_t,L_t)$ over a polarized elliptic curve (Y_t,L_t), with $\deg(L_t) = \deg(L)$. The same kind of conclusion holds if Y is a smooth projective curve of genus $g \geqslant 2$ and $\deg(L) \geqslant 4g+5$.

For the proof observe that the two types of surfaces from situations a) and b) of Theorem 1 (resp. Theorem 2) have different Euler-Poincaré characteristics, and therefore cannot fit in the same family. Then use Theorems 1 and 2, Remark 2 (after the proof of Theorem 1), Proposition 4 and the same method as in the proof of Theorem 8.

Remark. Compare with a result of Pinkham from [32], which states that (in the situation of Theorem 12) the affine cone of (Y,L) has no smooth deformations.

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