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Abstract. The present paper is concerned with some variational problems defined on a sequence of non-empty, weakly closed subsets of a Hilbert space, that satisfy certain properties. The solutions of these problems form a sequence that is weakly convergent to the solution of a similar problem, but with a supplementary term. These results are transposed in the particular cases of equations and variational inequalities. Some corrector results are proved.

INTRODUCTION

The first paragraph of the paper deals with the convergence of the variational problems:

$$F(u_\varepsilon)=\inf_{v\in K_\varepsilon} F(v)$$
$$F(v)=\frac{1}{2}a(v,v)-\langle f,v\rangle$$

where $K_\varepsilon$ are subsets of a Hilbert space, weakly closed that satisfy some hypotheses. It is proved that if $u_\varepsilon$ is weakly convergent to $u\in K$, then $u_\varepsilon$ satisfies:

$$G(u)=\inf_{v\in K} G(v)$$
$$G(v)=F(v)\ +\frac{1}{2}\phi(v)$$

where $\phi$ is a functional that depends on $K_\varepsilon$ and K.

In the particular case when $K_\varepsilon$ and K are subspaces these results were proved by D.Cioränescu [1]. The remarks 1.3 and 1.5 intend to stress the link between the hypotheses used in this paper and the hypotheses used in the case of the subspaces.

The hypotheses (1.)-(4.) and (1'.)-(4'.) are in fact direct consequences of the properties of $K_\varepsilon$ and K. The uniqueness of $\beta_\varepsilon$ and $\phi$ that satisfy these hypotheses, for given $K_\varepsilon$ and K, are the results of the theorems 1.2 and 1.4.

The corollaries 1.1 and 1.2 give some corrector results that improve the convergence of $u_\varepsilon$.

<u>The second paragraph</u> is concerned with the study of the convergence of the generalized (Sobolev) solutions of the problems:

$$A_\varepsilon u=f_\varepsilon$$

where $A_\varepsilon:\mathcal{D}(A_\varepsilon)\subset H\rightarrow H$, $A_\varepsilon=A_{/\mathcal{D}(A_\varepsilon)}$ where A is a linear, symmetric and positive definite operator. It is proved that the solutions of the above problems are weakly convergent to the generalized solution of:

$$Au+\ G^{-1}Bu=f$$

i.e. a similar equation, but with a supplementary term.

In this paragraph we make use of the variational caracterization of the generalized solution by means of the energetic spaces and of the Friedrichs' extension of a linear and positive definite operator.

The examples make use of the functions $w_\varepsilon$ and $\mu$ that are, for various domains, constructed in D.Cioränescu, F.Murat [3],[4]. The first example is concerned with the Dirichlet's problem for $\Delta$ and the second one with the Stokes' problem. The first example was studied also in D.Cioränescu [2] and, by means of the energy method, in D.Cioränescu, F.Murat [3],[4]. In the second example the Brinkman's law is obtained for the flow in a porous medium with a critical size of particles; we obtain by this method the same condition for the diameters of the particles as the one in E.Sanchez-Palencia [7] and Th.Levy [6].

In the first part of the third paragraph we study the general result for variational inequalities of the type:

$$\begin{cases} u_\varepsilon \in K_\varepsilon \\ a(u_\varepsilon, v-u_\varepsilon) \geqslant \langle f, v-u_\varepsilon\rangle, \quad \text{for all } v \in K_\varepsilon \end{cases}$$

where $K_\varepsilon$ are non-empty closed convex sets and a bilinear, continuous, coercive, symmetric functional; then $u_\varepsilon$ is weakly convergent to $u$ the solution of:

$$\begin{cases} u \in K \\ a(u,v-u) + \langle \phi'(u), v-u\rangle \geqslant \langle f, v-u\rangle, \quad \text{for all } v \in K \end{cases}$$

The first example studied is the one for the variational inequalities with strongly oscillating constraints. We obtain the same results as those obtained by means of the energy method in D.Cioränescu, F.Murat[3],[4].

The second example studies the variational inequalities with bilateral constraints.

## 1. FUNCTIONALS

Throughout this paragraph we will denote by: V a Hilbert space, $K, K_\varepsilon \subset V$ non-empty subsets, closed in the weak topology of V, $a: V\times V \to R$ a bilinear, symmetric, coercive, bounded functional and $F: V \to R$ the functional defined by:

$$F(v)=\frac{1}{2}a(v,v)-\langle f,v\rangle \tag{1.1}$$

where $f \in V'$.

Remark 1.1. It is easily verified that $F$ is weakly lower semicontinuous and that:

$$\lim_{\|v\|\to\infty} F(v)=\infty \tag{1.2}$$

Then $F$ is bounded below and achieves its infimum on every $K_\varepsilon$. We will denote by $u_\varepsilon$ one of the solutions of the problem:

$$\begin{cases} u_\varepsilon \in K_\varepsilon \\ F(u_\varepsilon)=\inf\limits_{v\in K_\varepsilon} F(v) \end{cases} \tag{1.3}$$

Theorem 1.1. If for every $\varepsilon$ there exists $\beta_\varepsilon : K \to R$ and $\phi: K \to R_+$ such that the following conditions are satisfied:

(1.) $(1-\beta_\varepsilon)K \subset K_\varepsilon$

(2.) for each $v \in K$, $\beta_\varepsilon v \to 0$ in V weakly

(3.) for each $v \in K$ and for each $\{v_\varepsilon\}$, $v_\varepsilon \in K_\varepsilon$, $v_\varepsilon \to v$ in V weakly, we have:

$$\overline{\lim_{\varepsilon \to 0}}\ a(\beta_\varepsilon v, v_\varepsilon) \leq -\phi(v)$$

(4.) for each $v \in K$ $\overline{\lim_{\varepsilon \to 0}}$ $a(\beta_\varepsilon v, \beta_\varepsilon v) \leq \phi(v)$

and if there exists $u \in K$ such that:

$$u_\varepsilon \longrightarrow u \quad \text{in } V \text{ weakly} \tag{1.4}$$

then u is a solution of the following problem:

$$\begin{cases} u \in K \\ G(u) = \inf_{v \in K} G(v) \end{cases} \tag{1.5}$$

where:

$$G(v) = \frac{1}{2}a(v,v) + \frac{1}{2}\phi(v) - \langle f, v \rangle \tag{1.6}$$

Proof

Let $\alpha_\varepsilon$ be an element of V given by:

$$u_\varepsilon = (1-\beta_\varepsilon)u + \alpha_\varepsilon \tag{1.7}$$

Hence, by (1.5) and (1.):

$$\alpha_\varepsilon \rightharpoonup 0 \text{ in } V \text{ weakly} \tag{1.8}$$

Using (1.7) we get:

$$\begin{aligned} a(u_\varepsilon, u_\varepsilon) = a(u,u) + a(\alpha_\varepsilon, \alpha_\varepsilon) - 2a(\beta_\varepsilon u, u_\varepsilon) - \\ - a(\beta_\varepsilon u, \beta_\varepsilon u) + 2a(\alpha_\varepsilon, u) \end{aligned} \tag{1.9}$$

The functional a being positive and by using (3.), (4.) and (1.8) we obtain:

$$\underline{\lim}_{\varepsilon\to 0} \ a(u_\varepsilon, u_\varepsilon) \geq a(u,u)+\phi(u) \tag{1.10}$$

For all $w\in K$ we define the elements:

$$w_\varepsilon = (1-\beta_\varepsilon)w \tag{1.11}$$

Thus we deduce, out of (1.) and (2.), that:

$$w_\varepsilon \in K_\varepsilon, \quad w_\varepsilon \rightharpoonup w \ \text{ in V weakly} \tag{1.12}$$

If we pass to the limit in:

$$a(w_\varepsilon, w_\varepsilon) = a(w,w)+a(\beta_\varepsilon w, \beta_\varepsilon w)-2a(w,\beta_\varepsilon w) \tag{1.13}$$

we obtain:

$$\overline{\lim}_{\varepsilon\to 0} \ a(w_\varepsilon, w_\varepsilon) \leq a(w,w)+\phi(w) \tag{1.14}$$

Because $w_\varepsilon \in K_\varepsilon$ from (1.4) we get:

$$F(u_\varepsilon) \leq F(w_\varepsilon)$$

and by using (1.10) and (1.14) we obtain:

$$\begin{aligned} \frac{1}{2}a(u,u)+\frac{1}{2}\phi(u)-\langle f,u\rangle &\leq \underline{\lim}_{\varepsilon\to 0} \ F(u_\varepsilon) \leq \overline{\lim}_{\varepsilon\to 0} \ F(u_\varepsilon) \leq \\ &\leq \frac{1}{2}\ a(w,w)+\frac{1}{2}\phi(w)-\langle f,w\rangle \end{aligned} \tag{1.15}$$

for all $w\in K$. Hence we have proved that:

$$G(u)=\inf_{w\in K} G(w)$$

REMARK 1.2. If the hypotheses (1.)-(4.) are satisfied then for all $v \in K$:

$$\underline{\lim}_{\varepsilon \to 0} a(\beta_\varepsilon v, \beta_\varepsilon v) = \overline{\lim}_{\varepsilon \to 0} a(\beta_\varepsilon v, \beta_\varepsilon v) = \phi(v) \tag{1.16}$$

Proof

For all $v \in K$:

$$v - \beta_\varepsilon v \in K_\varepsilon , \quad v - \beta_\varepsilon v \longrightarrow v \quad \text{in } V \text{ weakly}$$

Then by (3.):

$$\overline{\lim}_{\varepsilon \to 0} \ a(\beta_\varepsilon v, \ v - \beta_\varepsilon v) \leqslant -\phi(v)$$

Hence:

$$\underline{\lim}_{\varepsilon \to 0} \ a(\beta_\varepsilon v, \beta_\varepsilon v) \geqslant \phi(v)$$

and using hypothesis (4.) we obtain (1.16).

Next we shall improve the convergence of $u_\varepsilon$

Corollary 1.1. In the hypotheses of Theorem 1.1:

$$u_\varepsilon - (1 - \beta_\varepsilon) u \to 0 \text{ in } V \text{ strongly}$$

Proof

If in (1.15) we make $w = u$ we prove that there exists $\lim_{\varepsilon \to 0} F(u_\varepsilon)$ and:

$$\lim_{\varepsilon \to 0} F(u_\varepsilon) = \frac{1}{2} a(u,u) + \frac{1}{2} \phi(u) - \langle f, u \rangle \tag{1.18}$$

Hence there exists also:

$$\lim_{\varepsilon\to 0} a(u_\varepsilon, u_\varepsilon) = a(u,u) + \phi(u) \qquad (1.19)$$

From (1.9) and (1.16) we get:

$$-2\underline{\lim}_{\varepsilon\to 0}\ a(\beta_\varepsilon u, u_\varepsilon) + \overline{\lim}_{\varepsilon\to 0}\ a(\alpha_\varepsilon, \alpha_\varepsilon) = 2\phi(u) \qquad (1.20)$$

From (3.) and the positivity of a we have:

$$0 \leq \underline{\lim}_{\varepsilon\to 0}\ a(\alpha_\varepsilon, \alpha_\varepsilon) \leq \overline{\lim}_{\varepsilon\to 0}\ a(\alpha_\varepsilon, \alpha_\varepsilon) \leq 0 \qquad (1.21)$$

and hence $\alpha_\varepsilon$ is strongly convergent to zero in V.

In the following theorem we study the uniqueness of the functionals $\phi$ and the quasi-uniqueness of the operators $\beta_\varepsilon$ that satisfy the hypotheses of Theorem 1.1:

Theorem 1.2. If there exist also $\gamma_\varepsilon : K \longrightarrow V$ and $\psi : K \longrightarrow R_+$ that satisfy (1.)-(4.) then:

a) $\phi = \psi$

b) $\lim_{\varepsilon\to 0} \|(\beta_\varepsilon - \gamma_\varepsilon)v\| = 0$, for all $v \in K$

Proof

Let $v \in K$. We have:

$$(1-\gamma_\varepsilon)v \in K_\varepsilon,\quad (1-\gamma_\varepsilon)v \rightharpoonup v \text{ in } V \text{ weakly.}$$

Thus:

$$\overline{\lim}_{\varepsilon\to 0}\ a(\beta_\varepsilon v, v - \gamma_\varepsilon v) \leq -\phi(v) \qquad (1.22)$$

and by using (2.):

$$-\underline{\lim}_{\varepsilon\to 0}\, a(\beta_\varepsilon v, \gamma_\varepsilon v) \leq -\phi(v) \qquad (1.23)$$

In the same way we prove that:

$$-\underline{\lim}_{\varepsilon\to 0}\, a(\gamma_\varepsilon v, \beta_\varepsilon v) \leq -\psi(v) \qquad (1.24)$$

From the last two inequalities we deduce that:

$$\overline{\lim}_{\varepsilon\to 0}\, a(\beta_\varepsilon v - \gamma_\varepsilon v, \beta_\varepsilon v - \gamma_\varepsilon v) \leq 0$$

and thus we obtain both statements of the theorem.

The following remark intends to stress the link between the hypothesis (3.) of Theorem 1.1 and the one of D.Cioranescu [1].

Remark 1.3. If K and $K_\varepsilon$ are subspaces of V, then the hypothesis (3.) of Theorem 1.1 is equivalent to the following:
- for each $v \in K$ and for each $\{w_\varepsilon\}$, $w_\varepsilon \in K_\varepsilon$, $w_\varepsilon \rightharpoonup 0$ in V weakly, we have:

$$\lim_{\varepsilon\to 0}\, a(\beta_\varepsilon v, w_\varepsilon) \leq 0 \qquad (1.25)$$

Proof

For proving the necessity we define $v_\varepsilon$, by:

$$v_\varepsilon = (1-\beta_\varepsilon)v + w_\varepsilon \qquad (1.26)$$

Thus $v_\varepsilon \rightharpoonup v$ in V weakly. By writing that:

$$\overline{\lim}_{\varepsilon\to 0}\, a(\beta_\varepsilon v, v_\varepsilon) \leq -\phi(v) \qquad (1.27)$$

and using (1.26) we obtain (1.25).

The sufficiency can be proved in a similar way.

The next theorem is an alternative to Theorem 1.1.

Theorem 1.4. Let $\mathcal{K} \subset K$ be a dense subset. If for every $\varepsilon$ there exists a continuous operator $\beta_\varepsilon : K \to V$ and a continuous functional $\phi : K \to R_+$ such that the following conditions are satisfied:

(1!) $(1-\beta_\varepsilon)\mathcal{K} \subset K_\varepsilon$

(2!) for each $v \in \mathcal{K}$, $\beta_\varepsilon v \to 0$ in $V$ weakly

(3!) for each $v \in K$, and for each $\{\underline{v}_\eta\} \subset \mathcal{K}$, $\underline{v}_\eta \to v$ in $V$ strongly, for each $\{v_\varepsilon\}$, $v_\varepsilon \in K_\varepsilon$, $v_\varepsilon \to v$ in $V$ weakly, we have:

$$\lim_{\eta \to 0} \overline{\lim_{\varepsilon \to 0}} \; a(\beta_\varepsilon \underline{v}_\eta, v_\varepsilon) \leq -\phi(v)$$

(4!) for each $v \in \mathcal{K}$, $\overline{\lim\limits_{\varepsilon \to 0}} \; a(\beta_\varepsilon v, \beta_\varepsilon v) \leq \phi(v)$

and if there exists $u \in K$ such that:

$$u_\varepsilon \to u \quad \text{in } V \text{ weakly} \tag{1.28}$$

then $u$ is a solution of the problem (1.5).

Proof

There exists $\{\underline{u}_\eta\} \subset \mathcal{K}$ so that:

$$\underline{u}_\eta \to u \quad \text{in } V \text{ strongly} \tag{1.29}$$

Let $\alpha_{\varepsilon\eta}$ be an element of $V$ given by:

$$u_\varepsilon = (1-\beta_\varepsilon)\underline{u}_\eta + \alpha_{\varepsilon\eta} \tag{1.30}$$

Hence:

$$\begin{aligned} a(u_\varepsilon, u_\varepsilon) &= a(\underline{u}_\eta, \underline{u}_\eta) + a(\alpha_{\varepsilon\eta}, \alpha_{\varepsilon\eta}) - 2a(\beta_\varepsilon \underline{u}_\eta, u_\varepsilon) - \\ &- a(\beta_\varepsilon \underline{u}_\eta, \beta_\varepsilon \underline{u}_\eta) + 2a(\alpha_{\varepsilon\eta}, \underline{u}_\eta) \end{aligned} \tag{1.31}$$

We deduce that:

$$\underline{\lim}_{\varepsilon\to 0}\, a(u_\varepsilon, u_\varepsilon) \geq a(\underline{u}_\eta, \underline{u}_\eta) - 2\, \underline{\lim}_{\varepsilon\to 0}\, a(\beta_\varepsilon \underline{u}_\eta, u_\varepsilon) -$$
$$-\phi(\underline{u}_\eta) + 2a(u-\underline{u}_\eta, \underline{u}_\eta) \tag{1.32}$$

and hence, when $\eta$ tends to zero, we get:

$$\underline{\lim}_{\varepsilon\to 0}\, a(u_\varepsilon, u_\varepsilon) \geq a(u,u) + \phi(u) \tag{1.33}$$

If $w \in K$, there exists $\{\underline{w}_\eta\} \subset \mathcal{K}$ so that:

$$\underline{w}_\eta \longrightarrow w \text{ in } V \text{ strongly} \tag{1.34}$$

We define:

$$w_{\varepsilon\eta} = (1-\beta_\varepsilon)\underline{w}_\eta \tag{1.35}$$

and we observe, by (1') and (2') that:

$$w_{\varepsilon\eta} \longrightarrow w_\eta \text{ in } V \text{ weakly}, \quad w_{\varepsilon\eta} \in K_\varepsilon \tag{1.36}$$

Thus:

$$\overline{\lim}_{\varepsilon\to 0}\, a(w_{\varepsilon\eta}, w_{\varepsilon\eta}) \leq a(\underline{w}_\eta, \underline{w}_\eta) + \phi(\underline{w}_\eta) \tag{1.37}$$

Because of (1.36) and (1.3) we deduce that:

$$F(u_\varepsilon) \leq F(w_{\varepsilon\eta}) \tag{1.38}$$

and by using (1.33) and (1.37) we obtain:

$$\frac{1}{2}a(u,u) + \frac{1}{2}\phi(u) - \langle f, u\rangle \leq \underline{\lim}_{\varepsilon\to 0}\, F(u_\varepsilon) \leq \overline{\lim}_{\varepsilon\to 0}\, F(u_\varepsilon) \leq$$
$$\leq \frac{1}{2}a(\underline{w}_\eta, \underline{w}_\eta) + \frac{1}{2}\phi(\underline{w}_\eta) - \langle f, \underline{w}_\eta\rangle \tag{1.39}$$

By passing with $\eta$ to the limit we prove that:

$$G(u) = \inf_{w \in K} G(w)$$

The proof of the following remark is similar with the one of Remark 1.2, if in (3') we take $\underline{v}_\eta = v$ for all $\eta$.

Remark 1.4. If the hypotheses (1')-(4') are satisfied, then for all $v \in \mathcal{K}$:

$$\underline{\lim}_{\varepsilon \to 0} a(\beta_\varepsilon v, \beta_\varepsilon v) = \overline{\lim}_{\varepsilon \to 0} a(\beta_\varepsilon v, \beta_\varepsilon v) = \phi(v) \qquad (1.40)$$

Corollary 1.2. In the hypotheses of Theorem 1.3:

$$\lim_{\eta \to 0} \overline{\lim}_{\varepsilon \to 0} \| u_\varepsilon - (1-\beta_\varepsilon)\underline{u}_\eta \| = \lim_{\eta \to 0} \underline{\lim}_{\varepsilon \to 0} \| u_\varepsilon - (1-\beta_\varepsilon)\underline{u}_\eta \| = 0 \qquad (1.41)$$

where $\{\underline{u}_\eta\} \subset \mathcal{K}$, $\underline{u}_\eta \longrightarrow u$ in $V$ strongly.

If $u \in \mathcal{K}$ then:

$$u_\varepsilon - (1-\beta_\varepsilon) u \longrightarrow 0 \text{ in } V \text{ strongly.}$$

Proof

If in (1.39) we make $\underline{w}_\eta = \underline{u}_\eta$, by passing with $\eta$ to the limit we get:

$$\overline{\lim}_{\varepsilon \to 0} F(u_\varepsilon) = \underline{\lim}_{\varepsilon \to 0} F(u_\varepsilon) = \frac{1}{2} a(u,u) + \frac{1}{2}\phi(u) - \langle f, u \rangle \qquad (1.42)$$

and hence:

$$\lim_{\varepsilon \to 0} a(u_\varepsilon, u_\varepsilon) = a(u,u) + \phi(u) \qquad (1.43)$$

Next we deduce from (1.31) and from the hypothesis (3') that:

$$\lim_{\eta\to 0}\ \overline{\lim_{\varepsilon\to 0}}\ a(\alpha_{\varepsilon\eta},\alpha_{\varepsilon\eta})\leq 0 \qquad (1.44)$$

Thus, using the positivity of a, we obtain (1.41).

We shall state next the equivalents of Theorem 1.2 and of Remark 1.3 in the hypotheses (1!)-(4!). The proofs are similar.

Theorem 1.4. If there exist also $\gamma_\varepsilon : K \to V$ and $\psi : K \to R_+$ that satisfy (1!)-(4!) then:

a) $\phi = \psi$ (1.45)

b) $\lim_{\varepsilon\to 0} \|(\beta_\varepsilon - \gamma_\varepsilon)v\| = 0$, for all $v \in \mathcal{K}$ (1.46)

Remark 1.5. If K and $K_\varepsilon$ are subspaces of $V$, then the hypothesis (3!) of Theorem 1.3 is equivalent to the following:

- for each $v \in K$ and for each $\{v_\eta\} \subset \mathcal{K}$, $v_\eta \to v$ in $V$ strongly, for each $\{w_\varepsilon\}$, $w_\varepsilon \in K_\varepsilon$, $w_\varepsilon \to 0$ in $V$ weakly, we have:

$$\lim_{\eta\to 0}\ \overline{\lim_{\varepsilon\to 0}}\ a(\beta_\varepsilon v_\eta, w_\varepsilon)\leq 0 \qquad (1.47)$$

Remark 1.6. If K is a closed subspace of $V$, $\beta_\varepsilon \in \mathcal{L}(K,V)$ (resp. $\{\|\beta_\varepsilon\|\}$ bounded) and the hypothesis (4.) (resp. (4!)) is satisfied then:

a) there exists a bilinear, continuous, positive and simmetric functional $\tilde{\phi} : K\mathrm{x}K \to R$ so that:

$$\lim_{\varepsilon\to 0} a(\beta_\varepsilon v, \beta_\varepsilon w) = \tilde{\phi}(v,w) \qquad (1.48)$$

b) $\phi$ is continuous

c) there exists $B \in \mathcal{L}(K,K)$ so that:

$$\tilde{\phi}(v,w) = (Bv,w), \quad \text{for all } v, w \in K \qquad (1.49)$$

Proof

Because:

$$a(\beta_\varepsilon v, \beta_\varepsilon w) = \frac{1}{2}\left(a\left(\beta_\varepsilon(v-w), \beta_\varepsilon(v-w)\right) + a\left(\beta_\varepsilon(v+w), \beta_\varepsilon(v+w)\right)\right)$$

and because there exists (in both cases) the limit:

$$\lim_{\varepsilon\to 0} a(\beta_\varepsilon v, \beta_\varepsilon v), \text{ for all } v \in K$$

we observe that there exists the limit:

$$\lim_{\varepsilon\to 0} a(\beta_\varepsilon v, \beta_\varepsilon w), \text{ for all } v \in K, w \in K$$

Therefore we define:

$$\lim a(\beta_\varepsilon v, \beta_\varepsilon w) = \tilde{\phi}(v,w) \qquad (1.50)$$

It is easy to prove that $\tilde{\phi}$ is bilinear and symmetric. For proving the continuity we observe that if C and M are the positive constants given by:

$$|a(v,w)| \leq M \|v\|_V \|w\|_V, \quad \|\beta_\varepsilon v\|_V \leq C\|v\|_V \qquad (1.51)$$

then:

$$|\tilde{\phi}(v,w)| \leq MC^2 \|v\|_V \|w\|_V \qquad (1.52)$$

Observing that in order to deduce (1.10) and (1.33) we have used only the hypotheses that $u_\varepsilon \in K_\varepsilon$ and that $u_\varepsilon$ is weakly convergent we can state:

Remark 1.7. If the hypotheses (1.)-(4.), (resp. (1')-(4')) are satisfied then, for every $\{v_\varepsilon\}$ that satisfies:

$$v_\varepsilon \in K_\varepsilon\,,\quad v_\varepsilon \longrightarrow v \quad \text{in } V \text{ weakly} \tag{1.53}$$

we have:

$$\underline{\lim}_{\varepsilon \to 0}\ a(v_\varepsilon, v_\varepsilon) \geq a(u,u) + \phi(u) \tag{1.54}$$

## 2. EQUATIONS

### 2.1. General framework

Throughout this paragraph we will denote by: $H$ a Hilbert space; $\mathcal{D}(A) \subset H$ a dense subspace; $H_\varepsilon \subset H$ closed subspaces; $\mathcal{D}(A_\varepsilon) \subset \mathcal{D}(A)$, $\mathcal{D}(A_\varepsilon) \subset H_\varepsilon$ dense subspaces of $H_\varepsilon$; $A : \mathcal{D}(A) \to H$ a linear, symmetric and positive definite operator; $A_\varepsilon : \mathcal{D}(A_\varepsilon) \to H_\varepsilon$, $A_\varepsilon = A/_{\mathcal{D}(A_\varepsilon)}$; $f$ an element of $H$; $f_\varepsilon$ the projection of $f$ on $H_\varepsilon$ (i.e. $(f_\varepsilon, u)_H = (f,u)_H$ for all $u \in H$ ).

We will consider the energetic spaces:

$H_A$ the completion of $\mathcal{D}(A)$ with respect to $\|\cdot\|_{H_A}$

$H_{A_\varepsilon}$ the completion of $\mathcal{D}(A_\varepsilon)$ with respect to $\|\cdot\|_{H_{A_\varepsilon}}$

where we have denoted by:

$$(u,v)_{H_A} = (Au, v)_H \quad \text{for all } u, v \in \mathcal{D}(A) \tag{2.1}$$

$$(u,v)_{H_{A_\varepsilon}} = (A_\varepsilon u, v)_H \quad \text{for all } u, v \in \mathcal{D}(A_\varepsilon) \tag{2.2}$$

It can be easely seen that:

$$(u,v)_{H_{A_\varepsilon}} = (u,v)_{H_A} \qquad \text{for all } u, v \in \mathcal{D}(A_\varepsilon) \tag{2.3}$$

$$\|u\|_{H_{A_\varepsilon}} = \|u\|_{H_A} \geq \delta \|u\|_H \tag{2.4}$$

(where $\delta$ is the constant from: $(Au,u)_H \geq \delta^2 \|u\|^2$ )

We define $\varphi : H_A \to H$ in the following way: if $u \in H_A$ there exists a sequence $\{u_n\} \subset \mathcal{D}(A)$ convergent to $u$; hence $\{u_n\}$ is a Cauchy sequence in $\|\cdot\|_{H_A}$; from (2.4) we deduce that $\{u_n\}$ is a Cauchy sequence in $\|\cdot\|_H$ and hence there exists $u_0 \in H$, the limit in $H$ of $\{u_n\}$; we define $\varphi(u) = u_0$.

It can be proved (Dincă[5]) that $\varphi$ is a linear, continous, injective and dense imbedding of $H_A$ into H.

In an analogous manner we define $\varphi_\varepsilon : H_{A_\varepsilon} \to H_\varepsilon$ the linear, continuous, injective and dense imbeddings of the energetic spaces $H_{A_\varepsilon}$ into $H_\varepsilon$.

Using (2.4) we prove that:

$$H_{A_\varepsilon} \subset H_A \text{ and } \varphi_\varepsilon = \varphi_{/H_{A_\varepsilon}} \tag{2.5}$$

We denote by $G:H \to H_A$ a linear, bounded and injective operator defined by:

$$Gf=u_0 \quad , \text{ for all } f\in H \tag{2.6}$$

where $u_0$ is given by the Riesz Theorem:

$$(\varphi(u),f)_H=(u,u_0)_{H_A} \quad , \text{ for all } u\in H_A \tag{2.7}$$

and in a similar way $G_\varepsilon : H_\varepsilon \to H_{A_\varepsilon}$.

We consider the problems:

$$A_\varepsilon u=f_\varepsilon \tag{2.8}$$

Each of these problems has a unique generalized (Sobolev) solution:

$$u_\varepsilon \in H$$

It can be proved (Dincă[5]) that $u_\varepsilon$ is the classical solution of the equation:

$$\tilde{A}_\varepsilon u=f_\varepsilon \tag{2.9}$$

where $\tilde{A}_\varepsilon$ is the Friedrichs' extension of $A_\varepsilon$ and is given by:

$$\tilde{A}_\varepsilon=(\varphi\circ G\ )^{-1}:\ (G_\varepsilon(H_\varepsilon))\subset H_\varepsilon \xrightarrow{\text{onto}} H \qquad (2.10)$$

Theorem 2.1. If for every $\varepsilon$ there exists $\beta_\varepsilon\in\mathcal{L}(H_A,H$ and $\phi: H_A\to\mathbb{R}_+$ that satisfy the hypotheses (1.)-(4.) (resp. (1!)-(4!)) (with $V=K=H_A$, $K_\varepsilon=H_{A_\varepsilon}$) and if

$$R(B)\subset R(G) \qquad (2.11)$$

(B given by (1.49)) then the generalized (Sobolev) solutions of the problems (2.8) satisfy:

$$u_\varepsilon \longrightarrow u \qquad ,\ \text{in } H \text{ weakly} \qquad (2.12)$$

where $u$ is the generalized (Sobolev) solution of:

$$Au+G^{-1}Bu=f \qquad (2.13)$$

Proof

We consider the functional $F_\varepsilon: H_{A_\varepsilon}\to\mathbb{R}$

$$F_\varepsilon(v)=\frac{1}{2}\ \|v\|^2_{H_{A_\varepsilon}}-(\varphi_\varepsilon(v),f_\varepsilon)_H=$$
$$=\frac{1}{2}\ \|v\|^2_{H_{A_\varepsilon}}-(\varphi(v),f)_H \qquad (2.14)$$

By the fundamental variational theorem and the interpretation of the generalized solution, if $u_{\varepsilon 0}$ is the unique solution of the problem:

$$\begin{cases} u_{\varepsilon 0}\in H_{A_\varepsilon} \\ F_\varepsilon(u_{\varepsilon 0})=\min\limits_{v\in H_{A_\varepsilon}} F_\varepsilon(v) \end{cases} \qquad (2.15)$$

then $u_\varepsilon=\varphi(u_{\varepsilon 0})$. From (2.15) we deduce that:

$$\| u_{\varepsilon o} \|_{H_A} \leq \frac{1}{\gamma} \| f \|_{H} \tag{2.16}$$

Consequently there exists a subsequence of $\{u_{\varepsilon o}\}$ (denoted by $\{u_{\varepsilon' o}\}$ weakly convergent to an element of $H_A$:

$$u_{\varepsilon' o} \longrightarrow u_o \quad , \text{ in } H_A \text{ weakly} \tag{2.17}$$

By Theorem 1.1 (resp.1.3) $u_o$ is the unique solution of the problem:

$$\begin{cases} u_o \in H_A \\ G(u_o) = \min\limits_{v \in H_A} G(v) \end{cases} \tag{2.18}$$

where:

$$G(v) = \frac{1}{2} a(v,v) + \frac{1}{2} \phi(v) - (\varphi(v), f)_H \tag{2.19}$$

The weak convergence of $\{u_{\varepsilon o}\}$ results by the uniqueness of $u_o$.

If we denote by $u = \varphi(u_o)$ then by the linearity of $\varphi$ and by (2.7) we get:

$$(\varphi(u_{\varepsilon o}) - \varphi(u_o), w)_H = (u_{\varepsilon o} - u_o, Gw)_{H_A} \tag{2.20}$$

for all $w \in H$, and hence:

$$u_\varepsilon \longrightarrow u \quad , \text{ in } H \text{ weakly} \tag{2.21}$$

We shall prove next that $u$ is the generalized solution of the equation (2.13).

From (2.18) we deduce that:

$$(u_o + Bu_o - Gf, v)_{H_A} = 0, \text{ for all } v \in H_A \tag{2.22}$$

Therefore $\upsilon$ is the classical solution of:

$$G^{-1}(I+B)\varphi^{-1}\upsilon=f \tag{2.23}$$

We shall prove that $G^{-1}(I+B)\varphi^{-1}$ is the Friedrichs' extension of $A+G^{-1}B$. We define on $\mathcal{D}(A)$ the following scalar product:

$$(v,w)_{H_{A+G^{-1}B}}=(Av,w)_H+(G^{-1}Bv,w)_H \tag{2.24}$$

Hence:

$$(v,w)_{H_{A+G^{-1}B}}=(v,w)_{H_A}+(Bv,w)_{H_A} \tag{2.25}$$

Then if we denote by $H_{A+G^{-1}B}$ the completion of $\mathcal{D}(A)$ in the norm given by (2.24) because:

$$\| v \|^2_{H_A} \leq \| v \|^2_{H_{A+G^{-1}B}} \leq (MC^2+1)\| v \|^2_{H_A} \tag{2.26}$$

we obtain:

$$H_A \equiv H_{A+G^{-1}B} \tag{2.27}$$

If we apply Riesz Theorem, there exists $\bar{G}:H \to H_A$ so that:

$$(\varphi(v),f)_H=(v,\bar{G}f)_{H_{A+G^{-1}B}}=$$

$$=(v,\bar{G}f)_{H_A}+(Bv,\bar{G}f)_{H_A}=(v,(I+B)\bar{G}f)_{H_A} \tag{2.28}$$

Using (2.7) we get:

$$Gf=(I+B)\bar{G}f \text{ , for all } f\in H \tag{2.29}$$

and hence the Friedrichs' extension of $A+G^{-1}B$ is:

$$\widetilde{A+G^{-1}B}=(\varphi\circ\bar{G})^{-1}=G^{-1}(I+B)\varphi^{-1} \qquad (2.30)$$

Therefore $u$ is the generalized solution of (2.13).

The following corollaries are direct consequence of the linearity and continuity of $\varphi$ and of Corollary 1.1 and 1.2.

Corollary 2.1. If the hypotheses (1.)-(4.) are satisfied (with $V=K=H_A$, $K_\varepsilon=H_{A_\varepsilon}$) then:

$$\begin{aligned} &u_{\varepsilon 0}-(1-\beta_\varepsilon)u_0 \to 0, \text{ in } H_A \text{ strongly}\\ &u_\varepsilon-u+(\varphi\circ\beta_\varepsilon\circ\varphi^{-1})u \to 0, \text{ in } H \text{ strongly} \end{aligned} \qquad (2.31)$$

Corollary 2.2. If the hypotheses (1')-(4') are satisfied (with $V=K=H_A$, $K_\varepsilon=H_{A_\varepsilon}$) then:

$$\begin{aligned} &\lim_{\eta\to 0}\ \overline{\lim_{\varepsilon\to 0}}\ \| u_{\varepsilon 0}-(1-\beta_\varepsilon)\underline{u}_{\eta 0}\|_{H_A}=\lim_{\eta\to 0}\ \underline{\lim_{\varepsilon\to 0}}\ \| u_{\varepsilon 0}-(1-\beta_\varepsilon)\underline{u}_{\eta 0}\|_{H_A}=0\\ &\lim_{\eta\to 0}\ \overline{\lim_{\varepsilon\to 0}}\ \| u_\varepsilon-\underline{u}_\eta+(\varphi\circ\beta_\varepsilon\circ\varphi^{-1})\underline{u}_\eta\|_H=\\ &\qquad =\lim_{\eta\to 0}\ \underline{\lim_{\varepsilon\to 0}}\ \| u_\varepsilon-\underline{u}_\eta+(\varphi\circ\beta_\varepsilon\circ\varphi^{-1})\underline{u}_\eta\|_H=0 \end{aligned} \qquad (2.32)$$

Remark 2.1. Hypothesis (2.11) is in fact equivalent with: there exists a linear operator $C:H_A\to H$, so that:

$$\tilde{\phi}(u,v)=(Cu,\varphi(v)), \quad \text{for all } u,v\in H_A \qquad (2.33)$$

where $C=G^{-1}B$.

## 2.2. Examples

Let $\Omega$ be a bounded subset of $\mathbb{R}^n$ and, for every $\varepsilon$, $T_i^\varepsilon$ closed subsets of $\Omega$, $1\le i\le N(\varepsilon)$. We define:

$$\Omega^\varepsilon=\Omega-\bigcup_{i=1}^{N(\varepsilon)}T_i^\varepsilon \qquad (2.34)$$

We suppose that $\Omega^\varepsilon$ is chosen in such a way that there exists $w_\varepsilon$ and $\mu$ that satisfy:

(H0) $0 \le w_\varepsilon \le 1$ a.e. in $\Omega$

(H1) $w_\varepsilon \in H^1(\Omega)$

(H2) $w_\varepsilon = 0$ on $T_i^\varepsilon$, $1 \le i \le N(\varepsilon)$

(H3) $w_\varepsilon \rightharpoonup 1$, in $H^1(\Omega)$ weakly

(H4) $\mu \in W^{-1,\infty}(\Omega)$

(H5) $$\begin{cases} \text{for each } v_\varepsilon \text{ and } v \text{ that satisfy:} \\ v_\varepsilon \longrightarrow v, \text{ in } H^1(\Omega) \text{ weakly} \\ v_\varepsilon = 0 \text{ on } T_i^\varepsilon, \ 1 \le i \le N(\varepsilon) \\ \text{and for every } \varphi \in \mathcal{D}(\Omega) \\ \langle -\Delta w, \varphi v_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)} \longrightarrow \langle \mu, \varphi v \rangle_{H^{-1}, H_0^1(\Omega)} \end{cases}$$

(the existence of such $w_\varepsilon$ and $\mu$ is proved for various $\Omega^\varepsilon$ in D.Cioränescu, F.Murat[3],[4]).

Example 2.1.

The first application of the previous theorem will be for the Dirichlet problem:

$$\begin{cases} u_\varepsilon \in H_0^1(\Omega^\varepsilon) \\ -\Delta u_\varepsilon = f \quad \text{in } \Omega^\varepsilon \end{cases} \tag{2.35}$$

where $f \in L^2(\Omega)$. This problem was also studied by D.Cioränescu [2] and, by means of the energy method, in D.Cioränescu and F.Murat [3],[4]. In the latter it is proved a corrector result which will be found again as a consequence of Corollary 2.2.

Theorem 2.2. If $u_\varepsilon$ are the unique solutions of (2.35) then:

$$u_\varepsilon \rightharpoonup u \quad , \text{ in } H_0^1(\Omega) \text{ weakly} \tag{2.36}$$

where $u$ is the unique solution of:

$$\begin{cases} u \in H_0^1(\Omega) \\ -\Delta u+\mu u=f \quad \text{in} \quad \Omega \end{cases} \tag{2.37}$$

Moreover:

$$u_\varepsilon - w_\varepsilon u \longrightarrow 0 \text{ , in } W_0^{1,1}(\Omega) \text{ strongly} \tag{2.38}$$

Proof

We shall prove that the conditions of Theorem 2.1. are satisfied with:

$$\begin{aligned} &H=L^2(\Omega),\ \mathcal{D}(A)=\{u\in C^2(\bar\Omega)/u_{/\partial\Omega}=0\},\ A=-\Delta \\ &H_\varepsilon=L^2(\Omega^\varepsilon), \mathcal{D}(A_\varepsilon)=\{u\in C^2(\overline{\Omega^\varepsilon})/u_{/\partial\Omega^\varepsilon}=0\} \\ &\beta_\varepsilon\varphi=(1-w_\varepsilon)\varphi \quad \text{for all } \varphi\in\mathcal{K}=\mathcal{D}(\Omega)\subset H_0^1(\Omega) \end{aligned} \tag{2.39}$$

It is obvious that:

$$H_A=H_0^1(\Omega), \qquad H_{A_\varepsilon}=H_0^1(\Omega^\varepsilon) \text{ , } \varphi=\text{id.} \tag{2.40}$$

The first two hypotheses (1') and (2') are obviously satisfied and:

$$\lim_{\eta\to 0}\ \lim_{\varepsilon\to 0}\ (\beta_\varepsilon \underline{v}_\eta , v_\varepsilon)_{H_0^1(\Omega)} = -\langle \mu v, v\rangle_{H^{-1},H_0^1(\Omega)} \tag{2.41}$$

for every $\{\underline{v}_\eta\}\subset\mathcal{D}(\Omega)$, $v_\eta \to v$ in $H_0^1(\Omega)$ strongly and $v_\varepsilon \in H_0^1(\Omega^\varepsilon)$, $v_\varepsilon \longrightarrow v$ in $H_0^1(\Omega)$ weakly, and also:

$$\lim_{\varepsilon\to 0}\ \|\beta_\varepsilon v\|_{H_0^1(\Omega)} = \langle \mu v, v\rangle_{H^{-1},H_0^1(\Omega)} \tag{2.42}$$

We observe that:

$$\tilde{\phi}(u,v)=(\mu u,v)_{L^2(\Omega)} \tag{2.43}$$

hence $G^{-1}B=\mu$ and this yields (2.36).

If $\{\underline{u}_\eta\} \subset \mathcal{D}(\Omega)$ so that $\underline{u}_\eta \longrightarrow u$ in $H_0^1(\Omega)$ strongly, by (2.32) and by passing to the limit in:

$$\| u_\varepsilon - w_\varepsilon u \|_{W_0^{1,1}(\Omega)} \leq \| u_\varepsilon - w_\varepsilon \underline{u}_\eta \|_{W_0^{1,1}(\Omega)} + \| w_\varepsilon (\underline{u}_\eta - u) \|_{W_0^{1,1}(\Omega)}$$

we get (2.37).

By Remark 1.7 we can prove the following lemma:

<u>Lemma 2.1.</u> If $\{v_\varepsilon\} \subset H_0^1(\Omega^\varepsilon)$ so that $v_\varepsilon \longrightarrow v$ in $H_0^1(\Omega)$ weakly, then:

$$\varliminf_{\varepsilon \to 0} \| v_\varepsilon \|^2_{H_0^1(\Omega)} \geq \| v \|^2_{H_0^1(\Omega)} + \langle \mu, v^2 \rangle_{W^{-1,\infty}, W_0^{1,1}(\Omega)} \qquad (2.44)$$

<u>Example 2.2.</u>

We shall study next the Stokes problem:

$$\begin{cases} v_\varepsilon \in (H_0^1(\Omega^\varepsilon))^n, \ p_\varepsilon \in L^2(\Omega^\varepsilon) \\ -\Delta v_\varepsilon = f - \operatorname{grad} p_\varepsilon \quad \text{in } \Omega^\varepsilon \\ \operatorname{div} v_\varepsilon = 0 \quad \text{in } \Omega^\varepsilon \end{cases} \qquad (2.45)$$

where $f \in (L^2(\Omega^\varepsilon))^n$. By passing to the limit we shall obtain a Brinkman's law.

<u>Theorem 2.3.</u> If $v_\varepsilon, p_\varepsilon$ are the unique solutions of (2.45) then:

$$v_\varepsilon \longrightarrow v \quad , \text{in } (H_0^1(\Omega))^n \text{ weakly} \qquad (2.46)$$

where v,p are the solutions of:

$$\begin{cases} v\in(H_0^1(\Omega)) \ , \ p\in L^2(\Omega) & \\ -\Delta v+\mu v=f-\text{grad } p & \text{in } \Omega \\ \text{div} v=0 & \text{in } \Omega \end{cases} \qquad (2.47)$$

Proof

We denote by $E=\{w\in(H_0^1(\Omega))^n / \text{div } w=0\}$.

The equivalent variational formulation of (2.45) is:

$$(v_\varepsilon,w)_{H_0^1(\Omega)}=(f,w) \ , \ \text{for all } w\in E \qquad (2.48)$$

Consequently if $P:(H_0^1(\Omega))^n \to E$ is the projection operator of $(H^1(\Omega))^n$ onto E, then $Pu_\varepsilon=v_\varepsilon$, where $u_\varepsilon$ are the solutions of (2.35).

We denote $v=Pu$. Therefore by Theorem 2.2:

$$v_\varepsilon \rightharpoonup v \quad , \text{ in } (H_0^1(\Omega))^n \text{ weakly} \qquad (2.49)$$

Because $u$ is the unique solution of (2.37) then:

$$(u,w)_{(H_0^1(\Omega))^n}+\langle\mu u,w\rangle=(f,w)_{(L^2(\Omega))^n} \ , \ \text{for all } w\in(H_0^1(\Omega))^n$$

and hence $v$ is the unique solution of:

$$(v,w)_{(H_0^1(\Omega))^n}+\langle\mu u,w\rangle=(f,w)_{(L^2(\Omega))^n} \ , \ \text{for all } w\in E$$

Therefore there exists $p\in(L^2(\Omega))^n$ such that :

$$-\Delta v+\mu v=f - \text{grad } p \qquad (2.50)$$

Remark 2.2. We obtain, by this method, the discussion concerning the critical size of the particles for which Brinkman's law occurs (see also E.Sanchez-Palencia [7] and T.Léwy [6]). For instance if n=3 and if we denote by $a_\varepsilon$ the diameter of the particles, in order to obtain Brinkman's law, we must impose:

$$a_\varepsilon = c\,\varepsilon^3 \tag{2.51}$$

## 3. VARIATIONAL INEQUALITIES

### 3.1. General framework

Let K, $K_\varepsilon \subset V$ be non-empty, closed, convex subsets of the Hilbert space V.

We denote by $u_\varepsilon$ the unique solutions of the following variational inequalities:

$$\begin{cases} u_\varepsilon \in K_\varepsilon \\ a(u_\varepsilon, v-u_\varepsilon) \geq \langle f, v-u_\varepsilon\rangle, \quad \text{for all } v \in K_\varepsilon \end{cases} \tag{3.1}$$

We want to study the convergence of $\{u_\varepsilon\}$ when $\varepsilon$ tends to zero, if $K_\varepsilon$ satisfy the hypotheses of the first paragraph.

Theorem 3.1. If for every $\varepsilon$ there exists a continuous operator $\beta_\varepsilon : K \to V$ and if there exists $\phi : \mathrm{Sp}K \to R_+$ a continuous, Gâteaux differentiable functional on K, that satisfy the hypotheses (1.)-(4.) (resp. (1')-(4')) and if:
(5.) every weakly convergent subsequence of $\{u_\varepsilon\}$ has its limit in K, then:

$$u_\varepsilon \rightharpoonup u \quad \text{in V weakly} \tag{3.2}$$

where $u$ is the unique solution of the variational inequality:

$$\begin{cases} u \in K \\ a(u, v-u) + \langle \phi'(u), v-u\rangle \geq \langle f, v-u\rangle, \quad \text{for all } v \in K \end{cases} \tag{3.3}$$

Proof

We shall prove first that $\{u_\varepsilon\}$ is bounded. Let $v \in K$ (resp. $v \in \mathcal{K}$) and:

$$v_\varepsilon = (1-\beta_\varepsilon)v \qquad (3.4)$$

Then, by the hypotheses of the theorem:

$$v_\varepsilon \in K_\varepsilon, \quad v_\varepsilon \longrightarrow v \quad \text{in V weakly} \qquad (3.5)$$

By writting the variational inequality (2.1) for $v=v_\varepsilon$ we get:

$$a(u_\varepsilon, u_\varepsilon) \leq a(u_\varepsilon, v_\varepsilon) - \langle f, v_\varepsilon - u_\varepsilon \rangle \qquad (3.6)$$

and hence, from the coercivity and boundedness of a:

$$\gamma^2 \| u_\varepsilon \|_V^2 \leq M \| u_\varepsilon \|_V \cdot \| v_\varepsilon \|_V + \| f \|_{V'} (\| u_\varepsilon \|_V + \| v_\varepsilon \|_V) \qquad (3.7)$$

Using that $\{v_\varepsilon\}$ is bounded we conclude that $\{u_\varepsilon\}$ is bounded too.

Then there exists a subsequence of $\{u_\varepsilon\}$ weakly convergent to an element $u \in V$. By the hypothesis (5.) we get $u \in K$.

Because $u_\varepsilon$ is the unique solution of (1.3), by Theorem 1.1. (resp.1.3.), we obtain that u is a solution of:

$$\begin{cases} u \in K \\ G(u) = \inf\limits_{v \in K} G(v) \end{cases} \qquad (3.8)$$

where G is given by (1.6). By Remark 1.6. we deduce that u is the unique solution of (3.8). Consequently we get (3.2) and also (3.3) by equivalence with the problem (3.8).

The following corollaries are obvious consequences of Corollary 1.1. and Corollary 1.2.:

Corollary 3.1. If the hypotheses (1.)-(4.) are satisfied then:

$$u_\varepsilon - (1-\beta_\varepsilon)u \to 0, \text{ in } V \text{ strongly} \qquad (3.9)$$

Corollary 3.2. If the hypotheses (1'.)-(4'.) are satisfied then:

$$\lim_{\eta\to 0} \overline{\lim_{\varepsilon\to 0}} \| u_\varepsilon - (1-\beta_\varepsilon)\underline{u}_\eta \|_V = \lim_{\eta\to 0} \lim_{\varepsilon\to 0} \| u_\varepsilon - (1-\beta_\varepsilon)\underline{u}_\eta \|_V = 0 \qquad (3.10)$$

(where $\{\underline{u}_\eta\} \subset \mathcal{K}$, $\underline{u}_\eta \longrightarrow u$ in $V$ strongly).

## 3.2. Examples

Example 3.1.

The example that we shall study first will be the example of the variational inequalities with strongly oscillating unilateral constraints. This example was also studied by D.Cioränescu F.Murat[3],[4], we will obtain the same results by using Theorem 3.1.

We define the following closed, convex sets:

$$K = \{ v \in H_0^1(\Omega) \,/\, v \geq \psi \text{ a.e. in } \Omega \} \qquad (3.11)$$

$$K = \{ v \in H_0^1(\Omega) \,/\, v \geq \psi_\varepsilon \text{ a.e. in } \Omega \} \qquad (3.12)$$

where $\Omega$ is a bounded subset of $\mathbb{R}^n$ and $\psi$ is a measurable function defined on $\Omega$ and:

$$\psi_\varepsilon = \begin{cases} \psi & \text{in } \Omega^\varepsilon \\ 0 & \text{on } T_i^\varepsilon \end{cases} \qquad (3.13)$$

( $\Omega^{\varepsilon}$ and $T_i^{\varepsilon}$ are defined in the paragraph 2.2). We suppose that K is non-empty and that

$$\mathcal{K} = K \cap \mathcal{D}(\Omega) \tag{3.14}$$

is a dense subset of K. We take $V=H_0^1(\Omega)$.

<u>Theorem 3.2.</u> If $f \in H^{-1}(\Omega)$ and $u_\varepsilon$ are the unique solutions of the variational inequalities:

$$\begin{cases} u_\varepsilon \in K_\varepsilon \\ \displaystyle\int_\Omega \operatorname{grad} u_\varepsilon . \operatorname{grad}(v-u_\varepsilon) \geq \langle f, v-u_\varepsilon \rangle_{H^{-1}, H_0^1(\Omega)}, \text{ for all } v \in K_\varepsilon \end{cases} \tag{3.15}$$

then:

$$u_\varepsilon \longrightarrow u \quad , \text{ in } H_0^1(\Omega) \text{ weakly} \tag{3.16}$$

where $u$ is the unique solution of:

$$\begin{cases} u \in K \\ \displaystyle\int_\Omega \operatorname{grad} u . \operatorname{grad}(v-u) - \langle \mu u^-, v-u \rangle \geq \langle f, v-u \rangle \\ \qquad\qquad \text{for all } v \in K \end{cases} \tag{3.17}$$

Moreover:

$$u_\varepsilon^+ \longrightarrow u^+ \quad , \text{ in } H_0^1(\Omega) \text{ strongly} \tag{3.18}$$

$$u_\varepsilon^- - w_\varepsilon u^- \longrightarrow 0 \, , \text{ in } W_0^{1,1}(\Omega) \text{ strongly} \tag{3.19}$$

<u>Proof</u>

We shall prove that if we define the operators $\beta_\varepsilon$ in such a way that:

$$\beta_\varepsilon \varphi = (-1+w_\varepsilon)\varphi^- \quad , \text{ for every } \varphi \in \mathcal{K} \qquad (3.20)$$

and extend them over K by continuity then the hypotheses (1!)-(4!) are satisfied.

The first two hypotheses are easily verified.

In order to prove the third one we observe that for every $\{\underline{v}_\eta\} \subset \mathcal{K}$ , $\underline{v}_\eta \to v$ strongly and for every $\{v_\varepsilon\} \subset K_\varepsilon$ , $v_\varepsilon \rightharpoonup v$ weakly, we get:

$$\int_\Omega \operatorname{grad} \beta_\varepsilon \underline{v}_\eta \cdot \operatorname{grad} v_\varepsilon = -\int_\Omega \operatorname{grad} \underline{v}_\eta^- \cdot \operatorname{grad} v_\varepsilon + \langle \Delta w_\varepsilon , \underline{v}_\eta^- v_\varepsilon \rangle -$$

$$- \int_\Omega v_\varepsilon \operatorname{grad} w_\varepsilon \cdot \operatorname{grad} \underline{v}_\eta^- + \int w_\varepsilon \operatorname{grad} \underline{v}_\eta^- \cdot \operatorname{grad} v_\varepsilon \qquad (3.21)$$

and hence:

$$\lim_{\eta \to 0} \lim_{\varepsilon \to 0} \int_\Omega \operatorname{grad} \beta_\varepsilon \underline{v}_\eta \cdot \operatorname{grad} v_\varepsilon = \langle \mu , v^- v \rangle - 2\int_\Omega |\operatorname{grad} v^-|^2 \leq$$
$$\leq \langle \mu v^- , v \rangle_{H^{-1}, H_0^1(\Omega)} \qquad (3.22)$$

In a similar way we prove that for every $\varphi \in \mathcal{K}$

$$\lim_{\varepsilon \to 0} \int_\Omega |\operatorname{grad} \beta_\varepsilon \varphi|^2 = -\langle \mu \varphi^- , \varphi \rangle_{H^{-1}, H_0^1(\Omega)} \qquad (3.23)$$

Hence the functional $\phi$ is defined by:

$$\phi(v) = -\langle \mu v^- , v \rangle \quad \text{for all } v \in K \qquad (3.24)$$

$\phi$ is Gâteaux differentiable and (D.Ciorănescu, F.Murat [3], [4] lemma 4.2):

$$\langle \phi'(u) , v \rangle = -2\langle \mu u^- , v \rangle \text{ for all } u,v \in K \qquad (3.25)$$

We prove the hypothesis (5.) by observing that:

$$u_\varepsilon \geq w_\varepsilon \psi^+ - \psi^- \qquad \text{a.e. in } \Omega \qquad (3.26)$$

and by passing to the limit.

Then, from (3.2) $u_\varepsilon$ is weakly convergent to $u$, where $u$ is the unique solution of (3.17).

By passing to the limit in the inequality:

$$\| u_\varepsilon - u^+ + w_\varepsilon u^- \|_{W_0^{1,1}(\Omega)} \leq \| u_\varepsilon - \underline{u}_\eta + w_\varepsilon \underline{u}_\eta^- \|_{W_0^{1,1}(\Omega)} +$$

$$+ \| \underline{u}_\eta^+ - u^+ \|_{W_0^{1,1}(\Omega)} + \| w_\varepsilon (u^- - \underline{u}_\eta^-) \|_{W_0^{1,1}(\Omega)} \qquad (3.27)$$

and by virtue of Corollary 3.2, we get:

$$u_\varepsilon - u + (-1 + w_\varepsilon) u^- \longrightarrow 0 \text{ , in } W_0^{1,1}(\Omega) \text{ strongly} \qquad (3.28)$$

We observe that because $u_\varepsilon \in K_\varepsilon$ we get $u_\varepsilon^+ \in K_\varepsilon$, $u_\varepsilon^- \in H_0^1(\Omega^\varepsilon)$. Thus, by Remark 1.7 and by Lemma 2.1, we obtain:

$$\underline{\lim}_{\varepsilon \to 0} \| u_\varepsilon^+ \|^2_{H_0^1(\Omega)} \geq \| u^+ \|^2_{H_0^1(\Omega)} \qquad (3.29)$$

$$\underline{\lim}_{\varepsilon \to 0} \| u_\varepsilon^- \|^2_{H_0^1(\Omega)} \geq \| u^- \|^2_{H_0^1(\Omega)} + \langle \mu, u^{-2} \rangle \qquad (3.30)$$

Taking into account that if $u_\varepsilon$ are the solutions of the variational inequalities (3.15) we get (1.43), then it results that in (3.29) and (3.30) the limits exist and the inequality transforms into equality. Hence:

$$u_\varepsilon^+ \longrightarrow u^+ \quad \text{, in } H_0^1(\Omega) \text{ strongly} \qquad (3.31)$$

and from (3.28) and (3.31) we get (3.19).

<u>Example 3.2.</u>

We shall study next the convergence of the solutions of some variational inequalities with bilateral constraints

We define the following closed, convex sets:

$$K = \{ v \in H^1_0(\Omega) / \, 0 \leq v \leq \psi \text{ a.e. in } \Omega \} \tag{3.32}$$

$$K = \{ v \in H^1_0(\Omega^\varepsilon) / \, 0 \leq v \leq \psi \text{ a.e. in } \Omega^\varepsilon \} \tag{3.33}$$

where $\Omega$ is a measurable function defined on the bounded set $\Omega \subset R^n$.
We suppose that K is non-empty and that:

$$\mathcal{K} = K \cap \mathcal{D}(\Omega) \tag{3.34}$$

is a dense subset of K. We take $V = H^1_0(\Omega)$

Theorem 3.3. If $f \in H^{-1}(\Omega)$ and $u_\varepsilon$ are the unique solutions of the variational inequalities:

$$\begin{cases} u_\varepsilon \in K_\varepsilon \\ \displaystyle\int_\Omega \operatorname{grad} u_\varepsilon \cdot \operatorname{grad}(v - u_\varepsilon) \geq \langle f, v - u_\varepsilon \rangle \quad , \text{ for all } v \in K_\varepsilon \end{cases} \tag{3.35}$$

then:

$$u_\varepsilon \rightharpoonup u \quad , \text{ in } H^1_0(\Omega) \text{ weakly} \tag{3.36}$$

where $u$ is the unique solution of:

$$\begin{cases} u \in K \\ \displaystyle\int_\Omega \operatorname{grad} u \cdot \operatorname{grad}(v - u) - \langle \mu u, v - u \rangle \geq \langle f, v - u \rangle \\ \qquad \text{for all } v \in K. \end{cases} \tag{3.37}$$

Moreover:

$$u_\varepsilon - w_\varepsilon u \to 0 \quad , \text{ in } W^{1,1}_0(\Omega) \text{ strongly} \tag{3.38}$$

Proof

We define $\beta_\varepsilon$ in the following way:

$$\beta_\varepsilon \varphi = (1 - w_\varepsilon u) \quad , \text{ for all } \varphi \in \mathcal{K} \qquad (3.39)$$

The first two hypotheses are obviously satisfied.

In order to prove the hypotheses (3') and (4') we obtain the same results as in Example 2.1; hypothesis (5) is also satisfied. We define:

$$\phi(u) = \langle \mu u, u \rangle_{H^{-1}, H_0^1(\Omega)} \text{ , for all } u \in SpK \quad (3.40)$$

and hence:

$$\langle \phi'(u), v \rangle = \langle \mu u, v \rangle_{H^{-1}, H_0^1(\Omega)}$$

The conclusions of the theorem are thus immediate.

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REFERENCES

[1] D.Cioranescu, Calcul de variation sur de sous-espaces variable, C.R.Acad.Sc.Paris, série A, t.291, 1980, p.19-22.

[2] D.Cioranescu, Calcul de variation sur de sous-espaces variable. Applications, C.R.Acad.Sc.Paris,série A, t.291, 1980, p.87-90.

[3] D.Cioranescu, F.Murat, Un terme étrange venu d'ailleurs I, II, Nonlinear partial differential equations and their applications, Collège de France Seminar, ed.H.Brezis, J.L.Lions Research Notes in Mathematics, Pitman, London, 1982, vo-

[4] D.Cioränescu, F.Murat, Un terme étrange venu d'ailleurs, Univ.Pierre et Marie Curie, Laboratoire d'analyse numérique, no.82.005.

[5] G.Dincă, Variational Methods and Applications, Tech. Publ. Comp., Bucharest, 1980, (in romanian).

[6] Th.Levy, Fluid Flow Through an Array of Fixed Particles, Int. J.Engng.Sci., vol.21, no.1, 1983, p.11-23.

[7] E.Sanchez-Palencia, On the Asymptotics of the Fluid Flow Past an Array of Fixed Obstacels, Int.J.Engng. Sci., vol.20, no.12, 1982, p.1291-1301.