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The classical Hopkins-Levitzki Theorem states that any right Artinian ring with identity element is right Noetherian. Usually this Theorem is proved by the method of factoring through the nilpotent Jacobson radical of the ring. A proof which avoids the concept of the Jacobson radical was first performed by Shock [1]; he obtains also necessary and sufficient conditions for an Artinian module over a ring not necessarily unitary to be Noetherian.

The relativization of the classical Hopkins-Levitzki Theorem with respect to a Gabriel topology was first proved in the commutative case and conjectured in the noncommutative case by Albu and Năstăsescu [1 ; Théorème 4.7, Problème 4.8]. The noncommutative case of the relative Hopkins-Levitzki Theorem was first proved by Miller and Teply [1]. However, their proof is long and complicated; another module-theoretical proof of this Theorem, also hard, is available in Faith [1]. A different way to approach this Theorem is to translate the module-theoretical relative chain conditions occuring in its statement in absolute chain conditions in a suitable Grothendieck category, and to prove thus a general Hopkins-Levitzki Theorem in such a category; this was done by Năstăsescu [1]. Another short proof of this general Hopkins-Levitzki Theorem in a Grothendieck category is also due to Năstăsescu [2], and is somewhat similar to the one given by Shock [1] for modules over Artinian rings not necessarily with identity element.

A discussion on the various forms of the Hopkins-Levitzki Theorem and the connection between them may be found in Albu and Năstăsescu [2].

A short noncategorical proof of the relative Hopkins-Levitzki Theorem does not yet exists. The aim of this paper is to give such a proof by placing the Hopkins-Levitzki Theorem in a latticial setting; moreover, we shall obtain even two different proofs of this lattice-theoretical form of the Hopkins-Levitzki Theorem. Our proofs are inspired by some ideas of Shock [1] and Năstăsescu [1], [2], and are based on the concepts of length and Loewy length of an upper continuous and modular lattice of finite length.

0. PRELIMINARIES

Throughout this paper L will denote an upper continuous and modular lattice. The least (resp. greatest) element of L will be denoted by 0 (resp. 1). The notation and terminology will follow Stenström [1].

Recall that a non-zero element a of L is an atom if whenever $b \in L$ and $b < a$, then $b = 0$. The lattice L is called semi-atomic if 1 is a join of atoms; L is called semi-Artinian if for every $x \in L$, $x \neq 1$ the sublattice $[x,1]$ of L contains an atom. As in the case of modules, it can be shown (see e.g. Năstăsescu [3]) that if L is a semi-atomic lattice, then L is complemented, and for every $x, y \in L$ with $x \leq y$ the interval $[x,y]$ of L is also a semi-atomic lattice.

The ascending Loewy series of L:

$$s_0(L) < s_1(L) < \dots < s_{\lambda(L)}(L) \qquad (*)$$

is defined inductively: $s_0(L) = 0$, $s_1(L) = So(L)$ where $So(L)$ is the socle of L (i.e. the join of all atoms of L), and if the elements $s_\beta(L)$ of L have been defined for all ordinals $\beta < \alpha$, then we set $s_\alpha(L) = \bigvee_{\beta<\alpha} s_\beta(L)$ if α is a limit ordinal, and $s_\alpha(L) = So([s_\gamma(L),1])$ if $\alpha = \gamma + 1$; $\lambda(L)$ is the least ordinal λ such that $s_\lambda(L) = s_{\lambda+1}(L)$. The ordinal $\lambda(L)$ is the Loewy length of L, and it exists always because L is a set. The intervals $[s_\alpha(L), s_{\alpha+1}(L)]$, which are for each $\alpha < \lambda(L)$ semi-atomic lattices, are called the factors of the series (*). As in the case of modules it is easy to show that L is a semi-Artinian lattice if and only if $s_{\lambda(L)}(L) = 1$; moreover, for such a lattice, $So(L)$ is an essential element of L (see e.g. Năstăsescu [3]).

Recall that in the sequel L will always be assumed upper continuous and modular.

<u>0.1. LEMMA</u> Let $x, y \in L$ be such that $x \leq y$. Then

$$s_\alpha([0,x]) = s_\alpha([0,y]) \wedge x$$

for each ordinal α.

<u>Proof.</u> The lemma holds trivially if $\alpha = 0$, so assume it holds for each ordinal $\gamma < \alpha$ and proceed by induction. For the sake of brevity denote $s_\beta([0,x]) = x_\beta$ and $s_\beta([0,y]) = y_\beta$ for each ordinal $\beta \leq \alpha$.

If α is a limit ordinal, then

$$x_\alpha = \bigvee_{\beta<\alpha} x_\beta = \bigvee_{\beta<\alpha} (y_\beta \wedge x) = (\bigvee_{\beta<\alpha} y_\beta) \wedge x = y_\alpha \wedge x.$$

If $\alpha = \beta+1$, then

$[x_\beta , y_\alpha \wedge x] = [y_\beta \wedge x , y_\alpha \wedge x] \simeq [y_\beta , y_\beta \vee (y_\alpha \wedge x)] \subseteq [y_\beta , y_\alpha]$. Since $[y_\beta , y_\alpha]$ is a semi-atomic lattice, so is $[x_\beta , y_\alpha \wedge x]$, and consequently $y_\alpha \wedge x \leq x_\alpha$. On the other hand

$$[y_\beta , y_\beta \vee x_\alpha] \simeq [x_\alpha \wedge y_\beta , x_\alpha] \subseteq [x_\beta , x_\alpha]$$

because $x_\beta \leq y_\beta$ by the induction hypothesis. It follows that $[y_\beta , y_\beta \vee x_\alpha]$ is a semi-atomic lattice, hence $y_\beta \vee x_\alpha \leq y_\alpha$, and so $x_\alpha \leq y_\alpha \wedge x$. ∎

<u>0.2. PROPOSITION</u> Let $(z_i)_{i\in I}$ be a family of elements of L. Then $[0, \bigvee_{i\in I} z_i]$ is a semi-Artinian lattice if and only if $[0,z_i]$ is a semi-Artinian lattice for each $i \in I$, and in this case

$$\lambda([0, \bigvee_{i\in I} z_i]) = \sup_{i\in I} \lambda([0,z_i]).$$

<u>Proof.</u> Suppose that $[0,z_i]$ are all semi-Artinian lattices, and denote $\alpha = \sup_{i\in I} \lambda([0,z_i])$. By Lemma above one has for each $j\in I$

$$z_j = s_\alpha([0,z_j]) \leq s_\alpha([0, \bigvee_{i\in I} z_i]),$$

and so $\bigvee_{j\in I} z_j \leq s_\alpha([0, \bigvee_{i\in I} z_i])$. Hence $\bigvee_{i\in I} z_i = s_\alpha([0, \bigvee_{i\in I} z_i])$, and consequently $[0, \bigvee_{i\in I} z_i]$ is a semi-Artinian lattice and $\lambda([0, \bigvee_{i\in I} z_i]) \leq \alpha$.

Conversely, suppose that $[0, \bigvee_{i\in I} z_i]$ is semi-Artinian, and denote $\lambda([0, \bigvee_{i\in I} z_i]) = \beta$. Then

$$s_\beta([0,z_j]) = z_j \wedge s_\beta([0, \bigvee_{i\in I} z_i]) = (\bigvee_{i\in I} z_i) \wedge z_j = z_j$$

for each $j \in I$, by 0.1. Hence $[0,z_j]$ is semi-Artinian for each $j \in I$ and $\lambda([0,z_j]) \leqslant \beta$. It follows that $\sup_{i\in I} \lambda([0,z_i]) \leqslant \beta$, and the proposition is proved.∎

Recall that a lattice with 0 and 1 is of <u>finite length</u> if there exists a (Jordan-Hölder) composition series between 0 and 1. It is well-known that a modular lattice with 0 and 1 is of finite length if and only if it is both Artinian and Noetherian; the length of such a lattice X will be denoted in the sequel by $\ell(X)$.

The next Lemma is a lattice-theoretical formulation of the Proposition 2 of Shock [1]. For the convenience of the reader we include a proof here.

<u>0.3. LEMMA</u> If A is an Artinian and modular lattice with 1, then there exists an element $a^* \in A$ which is the least element of A such that the sublattice $[a^*, 1]$ of A is of finite length.

<u>Proof</u>. Let $N = \{x \in A \mid [x,1] \text{ is a lattice of finite length}\}$. Since $1 \in N$, $N \neq \emptyset$. If $x_1, x_2 \in N$, then $[x_1 \wedge x_2, x_1] \simeq [x_2, x_1 \vee x_2] \subseteq [x_2, 1]$, hence $[x_1 \wedge x_2, x_1]$ is of finite length, and consequently $[x_1 \wedge x_2, 1]$ is also of finite length because $[x_1, 1]$ is of finite length. It follows that $x_1 \wedge x_2 \in N$. Let a^* be a minimal element of N; if $x \in N$ then $a^* \wedge x \in N$, and so $a^* \wedge x = a^*$ by the minimality of a^*, i.e. $a^* \leqslant x$. Hence a^* is the least element of N.∎

If A is a lattice as in the above Lemma, then $\ell([a^*, 1])$ will be called the <u>reduced length</u> of A, and will be denoted in the sequel by $\ell^*(A)$.

As an easy consequence of 0.3 we obtain the following known result which will be used frequently in this paper:

0.4. COROLLARY Let C be a complemented and modular lattice (e.g. C may be any upper continuous, modular and semi-atomic lattice). Then C is Artinian if and only if C is Noetherian.

Proof. Suppose that C is Artinian, and consider the element $c^* \in C$ defined by 0.3. If c is a complement of c^*, then $[0,c] = [c^* \wedge c, c] \simeq [c^*, c^* \vee c] = [c^*, 1]$, hence the sublattice $[0,c]$ of C is of finite length. Suppose that $c \neq 1$; since C is Artinian there exists $a \in C$ such that a is an atom of the interval $[c,a]$. It follows that $[0,a]$ is of finite length. If b is a complement of a, then $[0,a] \simeq [b,1]$, hence $[b,1]$ is of finite length, and consequently $c^* \leq b$. Then $c^* \wedge a \leq b \wedge a = 0$, and so

$$a = a \wedge 1 = a \wedge (c \vee c^*) = c \vee (a \wedge c^*) = c \vee 0 = c,$$

a contradiction. Hence $c = 1$, and thus C is Noetherian.

If C is Noetherian, then the opposite lattice C^{op} of C is modular, complemented and Artinian, and then, by the proof above, C^{op} must be Noetherian, i.e. C is Artinian. ∎

0.5. PROPOSITION The following properties of an upper continuous and modular lattice L are equivalent:

(1) L is a lattice of finite length.

(2) L is an Artinian lattice with $\lambda(L)$ finite.

(3) L is a Noetherian and semi-Artinian lattice.

Proof. (1)$\Longrightarrow$(2) is clear.

(2)$\Longrightarrow$(3): Let $n = \lambda(L)$; then the ascending Loewy series of L is

$$0 = s_o(L) < s_1(L) < \dots < s_n(L) = 1.$$

For each $i = 0, \dots, n-1$ $[s_i(L), s_{i+1}(L)]$ is Artinian and semi-atomic, hence Noetherian by 0.4. Consequently $L = [0,1]$ is of finite length.

(3)$\Longrightarrow$(1): $s_{\lambda(L)}(L) = 1$ since L is semi-Artinian, and $\lambda(L)$ is a finite ordinal, say n, since L is Noetherian. For each $i = 0, \ldots ,n-1$ $[s_i(L),s_{i+1}(L)]$ is Noetherian and semi-atomic, hence Artinian by 0.4. Hence $L = [0,1]$ is of finite length. ∎

<u>0.6. REMARK</u> From the proof of the above Proposition it follows that if the lattice L is of finite length, then $\lambda(L) \leq \ell(L)$; clearly $\lambda(L) = \ell(L)$ if and only if for each $i = 0, \ldots , \lambda(L)-1$ each $s_{i+1}(L)$ is an atom in the sublattice $[s_i(L),s_{i+1}(L)]$ of L. ∎

1. MAIN RESULTS

<u>1.1. THEOREM</u> Let L be an Artinian, upper continuous and modular lattice, and let

(*) $\quad x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots$

be an ascending chain in L such that the sublattices $[0,x_i]$ of L are Noetherian for all $i \geq 1$. Then, the following two conditions are equivalent:

(1) The chain (*) is stationary.

(2) For each natural number $i \geq 1$ and each $y \in L$ with $y < x_i$ there exists an element $a_{yi} \in L$ such that $a_{yi} \leq x_i$ and $a_{yi} \nleq y$; furthermore, there exists a natural number t such that $\lambda([0,a_{yi}]) < t$ for all $i \geq 1$ and all $y < x_i$.

<u>Proof</u>. (1)$\Longrightarrow$(2): Suppose that $x_n = x_{n+1} = x_{n+2} = \ldots$ and denote $k = \lambda([0,x_n])$; then $\lambda([0,x_i]) < k+1$ for all $i \geq 1$. If $y < x_i$, then clearly (2) holds by choosing $a_{yi} = x_i$.

(2)$\Longrightarrow$(1): Suppose that the chain (*) is strictly ascending. Then the sequence $(\lambda([0,x_i]))_{i\geq 1}$ is unbounded, for otherwise, there exists a natural number m such that $\lambda([0,x_i]) \leqslant m$ for all $i \geqslant 1$; then $\lambda([0, \bigvee_{i=1}^{\infty} x_i]) \leqslant m$ by 0.2, and thus, by 0.5 $[0, \bigvee_{i=1}^{\infty} x_i]$ is a Noetherian lattice, a contradiction. Let $k \geqslant 1$ be such that $\lambda([0,x_k]) > t$. Since $[0,x_k]$ is Noetherian, there exists an element $y < x_k$ maximal with the property $\lambda([0,y]) \leqslant t$. By hypothesis, there exists $a = a_{yk}$ such that $a \leqslant x_k$ and $a \nleqslant y$. According to 0.2, $\lambda([0,a \vee y]) = \sup(\lambda([0,a]), \lambda([0,y])) \leqslant t$. But $y < a \vee y < x_k$ since $a \nleqslant y$ and $\lambda([0,x_k]) > t$; this contradicts the maximality of y, and consequently, the chain $(*)$ must be stationary. ∎

<u>1.2. COROLLARY</u> Let L be an upper continuous and modular lattice satisfying the following condition:

(λ) For each $y < x$ in L there exists an element $a_{yx} \in L$ such that $a_{yx} \leqslant x$, $a_{yx} \nleqslant y$ and $[0,a_{yx}]$ is semi-Artinian; furthermore, there exists a natural number t such that $\lambda([0,a_{yx}]) < t$ for all $y < x$ in L.

If L is Artinian, then L is Noetherian.

<u>Proof</u>. Consider the ascending Loewy series of L:

$$0 = s_0(L) < s_1(L) < \ .....$$

For each $i \geqslant 0$, $[s_i(L), s_{i+1}(L)]$ is semi-atomic and Artinian, hence Noetherian by 0.4; it follows that $[0,s_i(L)]$ is Noetherian for all $i \geqslant 1$. By 1.1, there exists a natural number n such that $s_n(L) = s_{n+1}(L)$. Hence $1 = s_n(L)$ because L is Artinian, and consequently $L = [0,1] = [0,s_n(L)]$ is Noetherian. ∎

<u>1.3. REMARK</u> The condition (λ) about an Artinian, upper continuous and modular lattice L is necessary for L to be Noetherian. Indeed, in this case, for each $y < x$ in L choose $a_{yx} = x$; then $\lambda([0,a_{yx}]) \leqslant \ell([0,a_{yx}]) \leqslant \ell(L)$. ∎

In order to show that the Artinian condition on a lattice L as in 1.2 is actually necessary for L to be Noetherian, we need the following simple

<u>1.4. LEMMA</u> Let L be an upper continuous and modular lattice and $y<x$ elements in L for which there exists $a\in L$ such that $a\leqslant x$, $a\nleqslant y$ and the sublattice $[0,a]$ of L is semi-Artinian. Then, the interval $[y,x]$ of L contains an atom.

<u>Proof</u>. Since L is a modular lattice, it follows that there exists a canonical isomorphism of lattices

$$[a\wedge y,a]\simeq[y,a\vee y].$$

But $a\nleqslant y$, hence $a\wedge y<a$, and so, the interval $[a\wedge y,a]$ contains an atom, because $[0,a]$ is semi-Artinian; if b is an atom of $[a\wedge y,a]$, then the corresponding element z of b by the above isomorphism is clearly an atom of the interval $[y,a\vee y]$, and hence an atom of $[y,x]$. ∎

We are now in a position to give the following

<u>1.5. THEOREM</u> Let L be an upper continuous and modular lattice satisfying the condition (λ) from 1.2. Then L is Artinian if and only if L is Noetherian.

<u>Proof</u>. If L is Artinian, then L is Noetherian by 1.2. Conversely, if L is Noetherian, then L is Artinian by 0.5, since L is semi-Artinian by 1.4. ∎

Recall that if A is an Artinian and modular lattice with 1, we have denoted in the Section 0 by $\ell^*(A)$ the so called reduced length of A; more precisely, $\ell^*(A) = \ell([a^*,1])$, where a^* is the least element of A such that the sublattice $[a^*,1]$ of A is of finite length (see 0.3); if in addition A is upper continuous we can define the <u>reduced Loewy length</u> $\lambda^*(A)$ of A as being $\lambda([a^*,1])$. Clearly $\lambda^*(A) \leq \ell^*(A)$. Note that a^* is also the least element of A such that the sublattice $[a^*,1]$ of A is of finite Loewy length.

We shall consider now to other conditions on a lattice L (upper continuous and modular as always in this paper):

(λ^*) For each $y < x$ in L there exists an element $a_{yx} \in L$ such that $a_{yx} \leq x$, $a_{yx} \nleq y$ and $[0,a_{yx}]$ is Artinian; in addition, there exists a natural number t such that $\lambda^*([0,a_{yx}]) < t$ for all $y < x$ in L.

(ℓ^*) For each $y < x$ in L there exists an element $a_{yx} \in L$ such that $a_{yx} \leq x$, $a_{yx} \nleq y$ and $[0,a_{yx}]$ is Artinian; in addition, there exists a natural number t such that $\ell^*([0,a_{yx}]) < t$ for all $y < x$ in L.

Clearly, if L satisfies the condition (ℓ^*), then L satisfies the condition (λ^*) too. We ignore the other connections between the conditions (λ), (λ^*) and (ℓ^*) on L. However, we have the following result:

<u>1.6. THEOREM</u> If the upper continuous and modular lattice L satisfies the condition (λ^*), then L is Artinian if and only if L is Noetherian.

<u>Proof.</u> The proof may be reduced to the proof 1.5 by using the following obvious fact: $\lambda^*([0,a]) = \lambda([0,a])$ for any $a \in A$ such that $[0,a]$ is of finite length. ∎

We shall investigate now the condition (ℓ^*) on L.

1.7. THEOREM Let L be an upper continuous and modular lattice satisfying the condition (ℓ^*) above. Then L is semi-Artinian and has finite Loewy length.

Proof. First of all, L is semiartinian by 1.4. For each natural number n denote by s_n the term $s_n(L)$ of the ascending Loewy series of L, and suppose that $s_n \neq s_{n+1}$ for all natural numbers n.

Let $x \in L$ be such that $x \leq s_k$ for some natural number k and $[0,x]$ is Artinian. Then $s_k([0,x]) = s_k([0,1]) \wedge x = s_k \wedge x = x$ by 0.1, hence $\lambda([0,x]) \leq k$, and then, by 0.5, $[0,x]$ is of finite length.

If now $n \geq 1$ is a natural number and x is an element of L such that $x \leq s_n$, $x \nleq s_{n-1}$ and $[0,x]$ is Artinian, then we shall prove inductively that $\ell([0,x]) \geq n$. If $n = 1$, then $x \neq 0$, and so $\ell([0,x]) \geq 1$. Let $x \in L$ be such that $x \leq s_{n+1}$, $x \nleq s_n$ and $[0,x]$ is Artinian. Denote $z = x \wedge s_n$ and $y = z \vee s_{n-1}$. Then $s_{n-1} \leq y \leq s_n$ and $y = (x \wedge s_n) \vee s_{n-1} = (x \vee s_{n-1}) \wedge s_n$. But $x \nleq s_{n-1}$, hence $s_{n-1} < s_{n-1} \vee x$, and consequently $(x \vee s_{n-1}) \wedge s_n \neq s_{n-1}$ because the socle s_n of the semi-Artinian lattice $[s_{n-1}, 1]$ is an essential element of the lattice $[s_{n-1}, 1]$. Thus $y \neq s_{n-1}$ and therefore $z \nleq s_{n-1}$; it follows that $z \wedge s_{n-1} < z$. By condition (ℓ^*), there exists $a \in L$ such that $[0,a]$ is Artinian, $a \leq z$ and $a \nleq z \wedge s_{n-1}$. Then $a \leq z \leq s_n$, $a \nleq s_{n-1}$ and $a \leq z < x$, hence $\ell([0,a]) < \ell([0,x])$. On the other hand, by the induction hypothesis $\ell([0,a]) \geq n$, and consequently $\ell([0,x]) \geq n+1$.

Since he have assumed that $s_n \neq s_{n+1}$ for all natural numbers n, it follows that for each $n \geq 1$ there exists $a_n \in L$ such that $a_n \leq s_n$, $a_n \nleq s_{n-1}$, $[0,a_n]$ is Artinian and $\ell^*([0,a_n]) < t$. Then $[0,a_n]$ is of finite length and $\ell([0,a_n]) \geq n$. On the other hand, $n \leq \ell([0,a_n]) = \ell^*([0,a_n]) < t$ for all $n \geq 1$, a contradiction. This completes the proof.■

1.8. COROLLARY If the upper continuous and modular lattice L satisfies the condition (ℓ^*), then L is Artinian if and only if L is Noetherian.

Proof. Apply 1.7 and 0.5.■

1.9. REMARKS (1) An other proof of 1.8 can be obtained from 1.6 since L satisfies clearly the condition (λ^*) too.

(2) The condition (ℓ^*) about an Artinian, upper continuous and modular lattice L is necessary for L to be Noetherian: see 1.3.■

2. APPLICATIONS

Let $\mathcal{C}$ be a Grothendieck category, i.e. an abelian category with exact direct limits and with a generator, and let X be an object of $\mathcal{C}$. $\mathcal{L}(X)$ will denote the lattice of all subobjects of X. It is well-known that $\mathcal{L}(X)$ is a modular and upper continuous lattice (see e.g. Stenström [1]). If U and M are objects of $\mathcal{C}$ then M is said to be U-generated if there exists an epimorphism $U^{(I)} \longrightarrow M$ for some set I, or equivalently, if whenever M' is a subobject of M, $M' \neq M$, there exists $f \in \mathrm{Hom}_{\mathcal{C}}(U,M)$ such that $\mathrm{Im}(f) \nleq M'$. M is said to be strongly U-generated if each subobject of M is U-generated.

2.1. THEOREM (Năstăsescu [1],[2]) Let $\mathcal{C}$ be a Grothendieck category and U an Artinian object of $\mathcal{C}$. If M is an Artinian object of $\mathcal{C}$ which is strongly U-generated, then M is Noetherian.

Proof. By 1.8 it will suffice to check that the lattice $L = \mathcal{L}(M)$ satisfies the condition (ℓ^*). Let $X, Y \in L$ be such that $Y < X$. Since X is U-generated there exists $f \in \mathrm{Hom}_{\mathcal{C}}(U,X)$ such that $A = Im(f) \not\leq Y$. But $A \simeq U/Ker(f)$, hence the lattice $\mathcal{L}(A) = [0,A]$ is isomorphic to the interval $[Ker(f),U]$ of $\mathcal{L}(U)$. Note also that $A \leq X$ and $[0,A]$ is Artinian because U is an Artinian object of $\mathcal{C}$. Thus $\ell^*([0,A]) = \ell^*([Ker(f),U]) \leq \ell^*([0,U]) = \ell^*(\mathcal{L}(U))$, and so $L = \mathcal{L}(M)$ satisfies the condition (ℓ^*). Let us mention that according to 1.7, **any** strongly U-generated object of $\mathcal{C}$ is a Loewy object having finite Loewy length. ∎

Our next aim is to apply 1.8 to get a simple noncategorical proof of the relative Hopkins-Levitzki Theorem. For this, we shall recall briefly some basic definitions, notations and properties concerning the lattice of F-saturated submodules of a module.

Let R be an associative, unitary and nonzero ring, and Mod-R the category of unitary right R-module. If M is a right R-module, then $\mathcal{L}(M)$ will denote the lattice of all submodules of M. Let F be a right Gabriel topology on R, $(\mathcal{T},\mathcal{F})$ the corresponding hereditary torsion theory on Mod-R, and t the torsion radical associated to $(\mathcal{T},\mathcal{F})$. If $M \in$ Mod-R, we shall use the following notation

$$C_F(M) = \{N \in \mathcal{L}(M) \mid M/N \in \mathcal{F}\}.$$

If $P \in \mathcal{L}(M)$, then $\widetilde{P}$ will denote the F-saturation of P in M, i.e. $\widetilde{P}/P = t(M/P)$; note that $P \in C_F(M)$ if and only if $P = \widetilde{P}$, i.e. P is F-saturated. If $(N_i)_{i\in I}$ is a family of elements of $C_F(M)$, then $\bigvee_{i\in I} N_i = \widetilde{\sum_{i\in I} N_i}$ and $\bigwedge_{i\in I} N_i = \bigcap_{i\in I} N_i$ are elements of $C_F(M)$. Moreover, $C_F(M)$ is an upper continuous and modular lattice with respect to the partial ordering given by "$\subseteq$" (inclusion) and with respect to the operations "$\vee$" and "$\wedge$". $C_F(M)$ is called the lattice of all F-saturated submodules of M and is sometimes denoted also by $Sat_F(M)$.

Let us mention the following properties of the lattice $C_F(M)$;

(1) If $N \in \mathcal{L}(M)$ and $N \in \mathcal{T}$, then the lattices $C_F(M)$ and $C_F(M/N)$ are canonical isomorphic; in particular $C_F(M) \simeq C_F(M/t(M))$.

(2) If $N \in \mathcal{L}(M)$ and $M/N \in \mathcal{T}$, then the lattices $C_F(M)$ and $C_F(N)$ are canonical isomorphic; in particular $C_F(N) \simeq C_F(\widetilde{N})$.

(3) If $M \in \mathcal{F}$ and $N \in C_F(M)$, then $C_F(N) = [0,N]$ and $C_F(M/N) \simeq [N,M]$, where the intervals are considered in the lattice $C_F(M)$.

(4) If M and M' are isomorphic R-modules, then the lattices $C_F(M)$ and $C_F(M')$ are isomorphic. ∎

For all these summarized facts on the lattices $C_F(M)$ the reader is referred to Stenström [1] or Albu and Năstăsescu [2].

Recall that $M \in$ Mod-R is said to be <u>F-Noetherian</u> (resp. <u>F-Artinian</u>) if $C_F(M)$ is a Noetherian (resp. Artinian) lattice. R is said to be F-Noetherian (resp. F-Artinian) if the R-module R_R is F-Noetherian (resp. F-Artinian).

<u>2.2. THEOREM</u> (Miller and Teply [1]) Let F be a right Gabriel topology on the ring R such that R is F-Artinian. Then, a right R-module M is F-Artinian if and only if M is F-Noetherian.

<u>Proof</u>. By the property (1) above, $C_F(M) \simeq C_F(M/t(M))$, hence we can suppose that $M \in \mathcal{F}$. According to 1.8 it will suffice to check that the lattice $C_F(M)$ satisfies the condition (ℓ^*). Let $Y < X$ be elements in $C_F(M)$. Then, there exists $x \in X \setminus Y$, and denote $B = xR$, $I = \mathrm{Ann}_R(x)$, $A_{YX} = A = \widetilde{B}$. Clearly $A \in C_F(M)$, $A \leq X$, and $A \nleq Y$. Since $R/I \simeq B \leq M$, it follows that $R/I \in \mathcal{F}$, and so $I \in C_F(R)$. By the properties (2), (3), (4) above one gets:

$$[I,R] \simeq C_F(R/I) \simeq C_F(B) \simeq C_F(A) = [0,A],$$

where the interval $[I,R]$ is considered in $C_F(R)$ and the interval $[0,A]$ in $C_F(M)$. Since $C_F(R)$ is an Artinian lattice, it follows that $[0,A]$ is an Artinian lattice, and then

$$\ell^*([0,A]) = \ell^*([I,R]) \leq \ell^*(C_F(R)),$$

Thus $C_F(R)$ satisfies the condition (ℓ^*). ■

2.3. REMARK When the proofs of 1.6 and 1.8 are carried out on the particular lattice $C_F(M)$, F being a right Gabriel topology on R such that R is F-Artinian, one gets two different short module-theoretical proofs of the relative Hopkins-Levitzki Theórem, quoted in Faith [1] as the Teply-Miller Theorem. ■

The next result has been proved by Năstăsescu and Raianu [1] by using the notion of quotient category. We shall present below a much shorter latticial proof. The terminology involved in all that follows can be found in Năstăsescu and Van Oystaeyen [1].

2.4. THEOREM (Năstăsescu and Raianu [1]) Let G be a group, $R = \bigoplus_{\sigma \in G} R_\sigma$ a graded ring of type G, and F a graded right Gabriel topology on R such that R is gr F-Artinian. Then, a graded right R-module M is gr F-Artinian if and only if M is gr F-Noetherian.

Proof. By definition, M is gr F-Artinian (resp. gr F-Noetherian) if the lattice $C^g_F(M) = \{ N \in \mathcal{L}_g(M) \mid M/N \in \mathcal{F} \}$ is Artinian (resp. Noetherian), where $\mathcal{L}_g(M)$ is the lattice of all graded submodules of M and $(\mathcal{T}, \mathcal{F})$ is the hereditary rigid torsion theory defined by F. Let us preserve the notations from the proof of 2.2; this proof can be adapted to the graded case as follows. The element $x \in X \setminus Y$ can be choosed homogeneous, say of degree τ. Then there exists an isomorphism of graded R-modules $R(\sigma)/I \simeq B$, where $\tau^{-1} = \sigma$ and $R(\sigma)$ is the σ-suspension of R. On the other hand, since the torsion theory $(\mathcal{T}, \mathcal{F})$ is rigid, the correspondence $J \longmapsto J(\sigma)$ yields an isomorphism of lattices $C^g_F(R) \simeq C^g_F(R(\sigma))$. Consequently $\ell^*([0,A]) = \ell^*([I,R(\sigma)]) \leq \ell^*(C^g_F(R(\sigma))) = \ell^*(C^g_F(R))$. ■

2.5. REMARK Applying 2.4 to the particular case $F = \{R_R\}$ one gets another proof, which avoids the concept of the Jacobson graded radical, of the graded version of the Hopkins-Levitzki Theorem (see Năstăsescu and Van Oystaeyen [1]): any right gr-Artinian ring is right gr-Noetherian.∎

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