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A FINITE ELEMENT DISCRETIZATION OF BOUNDARY CONTROL OF A TWO-PHASES

(preliminary version)

by

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A FINITE ELEMENT DISCRETIZATION OF BOUNDARY CONTROL OF A TWO-PHASES STEFAN PROBLEM

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1. INTRODUCTION

This paper is concerned with the approximation of the boundary control problem:

$$(P) \text{ Minimize } \int_0^T \left\{ \frac{1}{2} | y-d |^2_{L^2(\Omega)} + \frac{1}{2} |u|^2_{L^2(\partial\Omega)} \right\} dt$$

subject to:

(1.1) $v_t(t,x) - \Delta y(t,x) = f(t,x)$ a.e. Q,

$v(t,x) \in \beta(y(t,x))$ a.e. Q,

(1.2) $\frac{\partial y}{\partial n} = u$ a.e. Σ,

(1.3) $v(0,x) = v_0(x)$ a.e. Ω.

Ω is a bounded domain with smooth boundary $\partial\Omega$ and $Q = \Omega \times]0,T[$ is a cylinder with lateral face Σ.

We assume $v_0 \in L^2(\Omega)$, $d, f \in L^2(Q)$ and β is a strongly maximal monotone graph in $R \times R$, bounded on bounded sets. When β is given by:

$$(1.4) \quad \beta(r) = \begin{cases} r - r_0 & r > r_0 \\ [-\delta, 0] & r = r_0 \\ K(r-r_0) - \delta, & r < r_0 \end{cases}$$

where $K, \delta > 0$, we obtain a two-phases Stefan free boundary problem (J.L.Lions [11], p.196).

Consider the regularized problem:

$$(P_\varepsilon) \text{ Minimize } \int_0^T \left\{ \frac{1}{2} |y-d|^2_{L^2(\Omega)} + \frac{1}{2} |u|^2_{L^2(\partial\Omega)} \right\} dt$$

subject to:

$$(1.5) \quad \beta^\varepsilon(y(t,x))_t - \Delta y(t,x) = f(t,x) \quad \text{a.e. } Q,$$

and (1.2), (1.3), where

$$(1.6) \quad \beta^\varepsilon(y) = y + \int_{-\infty}^{\infty} \gamma_\varepsilon(y - \varepsilon^2\theta)\rho(\theta)d\theta, \quad \varepsilon > 0,$$

with γ_ε the Yosida approximation of the maximal monotone graph $\gamma(y) = \beta(y) - y$ (it is assumed for convenience that β is strongly monotone of constant greater than one) and ρ is a Friedrichs mollifier, i.e. $\rho \in C^\infty(R)$, $\text{supp}\rho \subset [-1,1]$, $\rho(-\theta) = \rho(\theta)$, $\rho \geq 0$ and $\int_{-\infty}^{\infty} \rho(\theta)d\theta = 1$.

This smoothing technique is mainly due to V.Barbu [2], [3] in the study of necessary optimality conditions for control problems governed by variational inequalities.

In a previous paper [15] the existence of at least one optimal control u^* in $L^2(\Sigma)$ is established for problem (P). Denote by u_ε an optimal control for problem (P_ε) and by $\Pi(u^*)$, $\Pi_\varepsilon(u_\varepsilon)$ the optimal values of problem (P), respectively (P_ε). The following result is known from [15]:

<u>Theorem 1.1. We have the convergences:</u>

$$(1.7) \quad \Pi(u_\varepsilon) \to \Pi(u^*)$$

$$(1.8) \quad \Pi_\varepsilon(u_\varepsilon) \to \Pi(u^*)$$

when $\varepsilon \to 0$.

Therefore, one has to discretize the "better" problem (P_ε) and to compute u_ε.

In the thesis of C.Saguez [13] a semi-discretization method with respect to the time variable is studied for the control of Stefan free boundary problems. V.Arnăutu [1], performs the discretization with respect to the space variables in certain nonconvex parabolic control problems. Our aim is to "reduce" (P_ε) to finite dimensional problems and we discretize both to respect to t and x. The converge result is given in Theorem 4.1.

For related problems or methods, both from numerical and theoretical point of vue, we quote J.F.Bonnans [4], K.H.Hoffman and J.Sprekels [10], I.Pawlow and M.Niezgodka [12], D.Tiba and Z.Meike [14], R.S.Falk [7].

Our algorithm can be compared with the work of C.M. Elliott and J.R.Ockendon [6] in the case of two-phases Stefan problems but the questions and the methods are certainly different.

As a last remark this is the first part in a forthcoming paper of D.Tiba and P.Neitaanmaki [17] concerning numerical results for problem (P).

2. THE DISCRETIZED PROBLEM

For the sake of simplicity assume that Ω is a convex polygonal domain in R^2. Ω is covered by a union of triangles $\mathcal{T}^h$, where h is the length of the largest triangle edge and $\mathcal{T}^h$ verifies the Zlamal conditions (P.G.Ciarlet [5], C.Fix and G.Strang [7]).

Let us denote $H=L^2(\Omega)$, $V=H^1(\Omega)$ with norms $|\cdot|_0$, $|\cdot|_1$.
V_h is the space of continuous functions which are linear on each triangle, with the norm $|\cdot|_h$ induced by the modified $L^2(\Omega)$ inner product:

$$(2.1)\quad (w,v)_h=\frac{1}{3}\sum_{i\leq I} W_i u_i w_i \quad u,w\in V^h.$$

Here I is the number of vertices associated with $\mathcal{T}^h$, W_i is the sum of the areas of the triangles with a vertex in i and u_i, w_i are the values of u,w at node i.

It is wellknown that:

$$(2.2)\quad |v|_0\leq|v|_h\leq C|v|_0 \quad \forall v\in V^h,$$

$$(2.3)\quad |(w,v)_h-(w,v)|\leq Ch^2|w|_1|v|_1\leq Ch|v|_0|w|_1 \quad \forall v,w\in V^h$$

where C is used for different constants and (,) is the H inner product.

V^h is a finite dimensional space of dimension I and we denote $\{b_i\}_{i\leq I}$ a basis in V^h, defined by

$$(2.4)\quad b_j(i)=\delta_{ji}, \quad i,j\leq I$$

and δ_{ij} the Kronecker symbol. Let L^h be the space of traces of functions from V^h endowed with the $L^2(\partial\Omega)$ norm.

Assume that the interval $[0,T]$ is divided in N equal subintervals of length $k>0$.

The discretized control problem is:

$$(2.5)\quad \text{Minimize } \frac{k}{2}\sum_{n=1}^{N}\left\{|y^n-d^n|_h^2+|u^n|^2_{L^2(\partial\Omega)}\right\}$$

subject to $u\in(L^h)^N$, $y\in(V^h)^N$, (2.6) $\left(\frac{v^{n+1}-v^{n}}{k},v\right)_h+\int_{\Omega}\nabla y^{n+1}.\nabla v-\int_{\partial\Omega}v.u^{n+1}=(f^{n+1},v)_h \quad \forall v\in V^h$

(2.7) $v_i^n=\beta^{\varepsilon}(y_i^n),\ i\leq I,\ n\leq N,$

and we denote it by $(P_{h,k})$. Here f^n, u^n, d^n are given vectors of appropriate dimensions. For every $h,k>0$, we prove the existence of an optimal control for $(P_{h,k})$. First we deal with the continuity properties in the nonlinear algebraic system (2.6), (2.7). By a variational argument (C.M.Elliott, J.R.Ockendon[6]) it is known that the system (2.6), (2.7) has a unique solution.

Let J denote the number of nodes associated with $\mathcal{T}^h$ which belong to $\partial\Omega$. We assume that all the vertices of the "polygon" Ω are vertices in $\mathcal{T}^h$, which is a natural condition.

Proposition 2.1. Let $u_p\to u$ in $R^{J\times N}$. The the corresponding solutions y_p, y of (2.6), (2.7) satisfy:

$$y_p\to y \quad \text{in } R^{I\times N},$$

Proof

Let $n_0\leq N$ be fixed and assume we have proved

$$y_p^n\to y^n,\ v_p^n\to v^n,\ \forall\, n\leq n_0-1$$

in R^I, when $p\to\infty$. This is true for $n_0=1$ since, as we shall see in the sequel, v^0 is fixed $\forall p$ by the discretization of the initial condition $v_0(x)$ in (1.3).

We write (2.6) for n_0:

$$(v_p^{n_0},v)_h+k\int_{\Omega}\nabla y_p^{n_0+1}.\nabla v=k\int_{\partial\Omega}u_p^{n_0+1}.v+(v_p^{n_0-1},v)_h+(f^{n_0+1},v)_h.$$

From (2.7) one can see at once that $\{y_p^{n_0}\},\{v_p^{n_0}\}$ are bounded in V^h with respect to p. By taking subsequences we have

$$y_p^n \longrightarrow y^n$$

$$v_p^n \longrightarrow v^n$$

when $p \to \infty$. Since the solution y corresponding to u is unique, the convergence is on the whole sequence.

Proposition 2.2. Problem $(P_{h,k})$ has at least one optimal pair $[y_{h,k}, u_{h,k}]$.

Proof.

Let $\{u_p\}$ be a minimizing sequence for $(P_{h,k})$. Since k is fixed, by (2.5) we get:

$$\left\{ \sum_{n=0}^{N} |u_p^n|^2_{L^2(\partial\Omega)} \right\}$$

bounded with respect to p. Then $|u_p^n|_{L^1(\partial\Omega)}$ is bounded with respect to n,p.

As u_p^n is a piecewise linear function on $\partial\Omega$ (in dimension 1) and $\mathcal{T}^h$ is fixed, it yields $\{|u^n_{p,j}|\}$ bounded for $j \leq J$, $n \leq N$ and every p. Therefore, on a subsequence p_k:

$$u^n_{p_k,j} \longrightarrow \tilde{u}^n_j$$

and by Proposition 2.1, $y^n_{p_k,j} \to \tilde{y}^n_j$, the solution of (2.6), (2.7) corresponding to $\tilde{u}$. It is easy to infer that $[\tilde{y}, \tilde{u}]$ is an optimal pair. It may not be unique because we take subsequences.

3. THE ADJOINT SYSTEM

Let $\theta : (L^h)^N \longrightarrow (V^h)^N$ be the mapping $u \longrightarrow y$ defined by

(2.6), (2.7). It is a nonlinear mapping and we show that it is Gâteaux differentiable.

Proposition 3.1. θ is Gâteaux differentiable and for every $u,w\in(L^h)^N$, $r=\nabla\theta(u)w$ satisfies:

$$(3.1)\quad \frac{1}{k}(\nabla\beta^{\varepsilon}(y^{n+1})r^{n+1}-\nabla\beta^{\varepsilon}(y^n)r^n,v)_h+\int_\Omega\nabla r^{n+1}\nabla v\; - \; -\int_{\partial\Omega}w^{n+1}.v=0,\quad\forall v\in V^h,\ n\leq N-1$$

$$(3.2)\quad r^0=0.$$

Proof.

We have:

$$\nabla\theta(u)w=\lim_{\lambda\to 0}\frac{\theta(u+\lambda w)-\theta(u)}{\lambda}\quad\text{in } (V^h)^N.$$

Denote $y=\theta(u)$, $\tilde{y}=\theta(u+\lambda w)$ and subtract the corresponding equations:

$$\frac{1}{k}\left(\frac{\tilde{v}^{n+1}-v^{n+1}}{\lambda},v\right)_h+\int_\Omega\nabla\frac{y^{n+1}-y^{n+1}}{\lambda}.\nabla v=\int_{\partial\Omega}w^{n+1}.v\;+$$
$$+\frac{1}{k}\left(\frac{\tilde{v}^n-v^n}{\lambda},v\right)_h,\quad v\in V^h.$$

Take $v=\frac{1}{\lambda}(\tilde{y}^{n+1}-y^{n+1})$. An easy computation involving (2.2) and the Lipschitz property of β^{ε} implies:

$$\frac{1}{k}\left|\frac{\tilde{y}^{n+1}-y^{n+1}}{\lambda}\right|_0^2+\int_\Omega\left|\nabla\frac{\tilde{y}^{n+1}-y^{n+1}}{\lambda}\right|^2\leq C\left|\frac{\tilde{y}^{n+1}-y^{n+1}}{\lambda}\right|_1+\frac{1}{k}\cdot\frac{1}{\varepsilon}\left|\frac{\tilde{y}^n-y^n}{\lambda}\right|_h\cdot$$
$$\cdot\left|\frac{\tilde{y}^{n+1}-y^{n+1}}{\lambda}\right|_h$$

Since the initial condition (1.3) is the same for $\theta(u)$ and $\theta(u+\lambda w)$ and N is fixed, we get:

$$\left|\frac{\tilde{y}^n - y^n}{\lambda}\right|_1 \quad \text{bounded for } \lambda > 0,\ n \leq N,$$

$$\left|\frac{\tilde{v}^n - v^n}{\lambda}\right|_0 \quad \text{bounded for } \lambda > 0,\ n \leq N.$$

Therefore $\tilde{y}^n \to y^n$ strongly in $L^2(\Omega)$ when $\lambda \to 0$.

We denote $r = \lim_{\lambda \to 0} \frac{\tilde{y}^n - y^n}{\lambda}$ in the strong topology of $L^2(\Omega)$. Since $\mathcal{T}^h$ is fixed, it yields as in the proof of <u>P2.2</u> that $r \in V^h$ and

$$\frac{\tilde{y}_i^n - y_i^n}{\lambda} \to r_i\,, \quad i \leq I.$$

We can pass to the limit. The first term gives:

$$\left(\frac{\tilde{v}^n - v^n}{\lambda}, v\right)_h = \frac{1}{3} \sum_{i \leq I} W_i \frac{\beta^\varepsilon(\tilde{y}_i^n) - \beta^\varepsilon(y_i^n)}{y_i^n - y_i^n} \cdot \frac{\tilde{y}_i^n - y_i^n}{\lambda} \cdot v_i$$

$$\frac{1}{3} \sum_{i \leq I} W_i \nabla \beta^\varepsilon(y_i^n) \cdot r_i^n \cdot v_i = (\nabla \beta^\varepsilon(y^n) r^n, v)_h$$

and (3.1) is proved.

We denote $\Pi_{h,k} : (L^h)^N \to]-\infty, +\infty]$ the value of the functional associated with $(P_{h,k})$. Then $u_{h,k}$ is a minimum point for $\Pi_{h,k}$ and the Gâ teaux differential vanishes:

$$(3.3)\ \nabla \Pi_{h,k}(u_{h,k}) w = 0, \quad \forall w \in V^h.$$

But, we have:

$$\lim_{\lambda \to 0} \frac{\Pi_{h,k}(u + \lambda w) - \Pi_{h,k}(u)}{\lambda} = k \sum_{n=1}^{N} \left\{ (y^n - d^n, r^n)_h + (u^n, w^n)_{L^2(\partial\Omega)} \right\} =$$

$$= k \sum_{n=1}^{N} \left\{ \left(\left[\nabla \theta(u)^* (y - d) \right]^n, w^n \right)_h + (u^n, w^n)_{L^2(\partial\Omega)} \right\}.$$

By (3.3) and the above equality we get the adjoint system:

$$(3.4)\ \nabla\theta(u_{h,k})^{*}(y_{h,k}-d)=-u_{h,k}$$

in abstract form.

We define the adjoint state $p_{h,k}\in V^h$ by:

$$(3.5)\ (\nabla\beta^{\varepsilon}(y^n_{h,k})\cdot\frac{p^n-p^{n+1}}{k},v)_h+\int_\Omega \nabla p^n\cdot\nabla v=$$

$$=-(y^n_{h,k}-d^n,v)_h,\ \forall v\in V^h,\ n=\overline{0,N},$$

$$(3.6)\ p^N=0.$$

This is a linear system in implicit form and obviously there is a unique solution $p_{h,k}\in V^h$.

<u>Proposition 3.4.</u> <u>Relation</u> (3.4) <u>is equivalent with:</u>

$$(3.7)\ u^n_{h,k}=p^n_{h,k}\Big|_{\partial\Omega},\ n\leq N.$$

<u>Proof</u>.

We have $\nabla\theta(u_{h,k})^{*}:(V^h)^N\to(L^h)^N$. According to (3.4) we show that $-\nabla\theta(u_{h,k})^{*}(y_{h,k}-d)=p_{h,k}\Big|_{\partial\Omega}$, that is:

$$(3.8)\ \sum_{n=0}^{N-1}(p^{n+1}_{h,k},w^{n+1})_{L^2(\partial\Omega)}=-\sum_{n=0}^{N-1}(y^{n+1}_{h,k}-d^{n+1},[\nabla\theta(u_{h,k})w]^{n+1})_h,$$

$$\forall\ w\in(L^h)^N.$$

By (3.1) this is equivalent with

$$(3.9)\ \sum_{n=0}^{N-1}(p^{n+1}_{h,k},w^{n+1})_{L^2(\partial\Omega)}=-\sum_{n=0}^{N-1}(y^{n+1}_{h,k}-d^{n+1},r^{n+1})_h.$$

In order to prove (3.9)and implicity (3.8)we put in (3.1) $v=p^{n+1}_{h,k}$:

$$(3.10)\quad \sum_{n=0}^{N-1}\left(\frac{\nabla\beta^{\varepsilon}(y^{n+1})r^{n+1}-\nabla\beta^{\varepsilon}(y^{n})r^{n}}{k},p^{n+1}\right)_h +$$

$$+\sum_{n=0}^{N-1}\int_{\Omega}\nabla r^{n+1}.\nabla p^{n+1}=\sum_{n=0}^{N-1}\int_{\partial\Omega}w^{n+1}.p^{n+1}$$

(we omit the subscripts h,k).

Summing by parts in the first term of (3.10), we have:

$$(3.11)\quad \sum_{n=0}^{N-1}\int_{\partial\Omega}w^{n+1}.p^{n+1}=\sum_{n=0}^{N-1}\int_{\Omega}\nabla r^{n+1}.\nabla p^{n+1}+\sum_{n=1}^{N-1}\left(\nabla\beta^{\varepsilon}(y^{n}).\frac{p^{n}-p^{n+1}}{k},r^{n}\right)_h$$

$$=\sum_{n=0}^{N-1}\int_{\Omega}\nabla r^{n}.\nabla p^{n}+\sum_{n=0}^{N-1}\left(\nabla\beta^{\varepsilon}(y^{n}).\frac{p^{n}-p^{n+1}}{k},r^{n}\right)_h$$

from (3.2) and (3.6).

By (3.5) we finish the proof.

4. THE CONVERGENCE

We show that, in a certain sense, the solution of problem $(P_{h,k})$ approximates the solution of problem (P_ε) when $h,k\to 0$. As a general remark we underline that our result is much more general and it can be applied, for instance, at other types of discretization or other finite elements, etc.

For the sake of simplicity we assume that r_o, d,f are defined pointwise and v_i^o, d_i^n, f_i^n denote the value at the moment n.k and at the vertex i of $\mathcal{T}^h$.

We recall the discretized control problem:

$$(P_{h,k})\quad \text{Minimize } \frac{k}{2}\sum_{n=1}^{N}\left\{|y^n-d^n|_h^2+|u^n|^2_{L^2(\partial\Omega)}\right\}$$

subject to:

$$(2.6)\quad \left(\frac{v^{n+1}-v^n}{k},v\right)_h+\int_\Omega \nabla y^{n+1}\nabla v-\int_{\partial\Omega} v.u^{n+1}=(f^{n+1},v)_h$$

$$(2.7)\quad v_i^n=\beta^\varepsilon(y_i^n),\ i\leq I,\ n\leq N.$$

<u>Theorem 4.1</u>. <u>Under the monotonicity and boundedness assumptions for</u> β <u>we have</u>:

$$(4.1)\quad \lim_{k\to 0}\lim_{h\to 0}\Pi_{h,k}(u_{h,k})=\Pi_\varepsilon(u_\varepsilon).$$

<u>Proof</u>.

For every $h,k>0$, it yields

$$(4.2)\quad \frac{k}{2}\sum_{n=1}^N\{\|y^n_{h,k}-d^n|^2_h+|u^n_{h,k}|^2_{L^2(\partial\Omega)}\}\leq\frac{k}{2}\sum_{n=1}^N\{\|y^n-d^n|^2_h+|u^n|^2_{L^2(\partial\Omega)}\}$$

any $u\in(L^h)^N$ and $y=\theta(u)\in(V^h)^N$.

Let $u_o\in W^{1,2}(0,T;L^2(\partial\Omega))$ be fixed and $u^{h,k}\in(L^h)^{N+1}$ be the discretization of u_o, $y^{h,k}=\theta(u^{h,k})$. Then, it is known from D.Tiba, M.Tiba [16] that:

$$(4.3)\quad \lim_{h,k\to 0}u^{h,k}=u_o \text{ in } L^2(\Sigma),$$

$$(4.4)\quad \lim_{h,k\to 0}y^{h,k}=y_o \text{ in } L^2(\Omega),$$

the solution of (1.5), (1.2), (1.3) corresponding to u_o.

By (2.3) we see that $\lim_{h,k\to 0}\Pi_{h,k}(u^{h,k})=\Pi_\varepsilon(u_o)$ and from (4.2) we conclude that

$$\frac{k}{2}\sum_{n=1}^N\{\|y^n_{h,k}-d^n|^2_h+|u^n_{h,k}|^2_{L^2(\partial\Omega)}\}$$

is bounded as $h,k\to 0$. Therefore

(4.5) $\frac{k}{2}\sum_{n=1}^{N}|u^n_{h,k}|^2_{L^2(\partial\Omega)}$,

(4.6) $\frac{k}{2}\sum_{n=1}^{N} y^n_{h,k}$.

are bounded with respect to $h,k>0$.

For other estimates we put $v=y^{n+1}_{h,k}$ in (2.6). In the sequel we omit the subscripts h,k:

(4.7) $$\frac{1}{k}(y^{n+1}-y^n,\ y^{n+1})_h+\frac{1}{k}(w^{n+1}-w^n,y^{n+1})_h+|\nabla y^{n+1}|^2_0=$$

$$=\int_{\partial\Omega} y^{n+1}\cdot u^{n+1}+(f^{n+1},y^{n+1})_h\ .$$

Here $w^n_{h,k}=\beta^\varepsilon(y^n_{h,k})-y^n_{h,k}=\gamma^\varepsilon(y^n_{h,k})$ according to (1.6).

By a device due to O.Grange and F.Mignot [9], it is known that:

(4.8) $$\sum_{n=0}^{p-1}(w^{n+1}-w^n,\ y^{n+1})\geq C.$$

From (4.7), (4.8) we get:

$$\frac{1}{k}\sum_{n=0}^{p-1}(y^{n+1}-y^n,y^{n+1})_h+\frac{C}{k}+\sum_{n=0}^{p-1}|\nabla y^{n+1}|^2_0\leq\sum_{n=0}^{p-1}\int_{\partial\Omega} y^{n+1}\cdot u^{n+1}+$$

$$+\sum_{n=0}^{p-1}(f^{n+1},y^{n+1})_h.$$

Then:

$$\frac{1}{2k}|y^p|^2_h+\frac{1}{2}\sum_{n=0}^{p-1}|\nabla y^{n+1}|^2_0\leq\frac{C}{k}+C\sum_{n=0}^{p-1}|u^n|^2_{L^2(\partial\Omega)}+$$

$$+\frac{1}{2}\sum_{n=0}^{p-1}|y^{n+1}|^2_0$$

and (4.5), (4.6) yield:

(4.9) $|y^n|_0^2 \leq C, \forall n,h,k,$

(4.10) $\sum_{n=0}^{N-1} k|y^{n+1}|^2 \leq C, \forall h,k.$

Let $k>0$ be fixed. By taking subsequences we infer:

(4.11) $u^n_{h,k} \xrightarrow{h\to 0} u^n_k$ weakly in $L^2(\partial\Omega)$,

(4.12) $y^n_{h,k} \xrightarrow{h\to 0} y^n_k$ weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$.

Since β^ε is Lipschitz continuous, we have

(4.13) $v^n_{h,k} \xrightarrow{h\to 0} v^n_k = \beta^\varepsilon(y^n_k)$

strongly in $L^2(\Omega)$.

We pass to the limit with respect to $h\to 0$ in (2.6), (2.7

(4.14) $\left(\frac{v_k^{n+1}-v_k^n}{k}, v\right) + \int_\Omega \nabla y_k^{n+1} \cdot \nabla v - \int_{\partial\Omega} v.u_k^{n+1} = \int_\Omega f_k^{n+1}.v, \forall v\in H^1(\Omega).$

We consider the control problem

(P_k) Minimize $\frac{k}{2}\sum_{n=1}^{N}\left\{|y^n-d^n|_0^2+|u^n|^2_{L^2(\partial\Omega)}\right\}$

subject to (4.13), (4.14). This can be compared with the semi-discretization method of C.Saguez [13].

Here d^n, y^n, u^n, f^n, v^n denote the function on Ω obtained for $t=nk\leq T$.

By the same argument as in Section 2 one can see that (P_k) has at least one solution $\in L^2(\Omega)^N \times L^2(\partial\Omega)^N$.

Using the weak lower semicontinuity of the norm and

(4.2)-(4.4), (4.11), (4.12) we get:

$$(4.15)\quad \frac{k}{2}\sum_{n=1}^{N}\{|y_k^n-d^n|_0^2+|u_k^n|^2_{L^2(\partial\Omega)}\}\leq\frac{k}{2}\sum_{n=1}^{N}\{|y_o^n-d^n|_0^2+|u_o^n|^2_{L^2(\partial\Omega)}\}$$

for every $u_o\in W^{1,2}(0,T;L^2(\partial\Omega))$.

By an easy calculation it yields:

$$(4.16)\quad \sum_{n=1}^{N} k|u_k^n|^2_{L^2(\partial\Omega)}\leq C\ ,\forall k>0,$$

$$(4.17)\quad |y_k^n|_0^2\leq C,\ \forall k,n$$

$$(4.18)\quad \sum_{n=1}^{N} k|y_k^n|_1^2\leq C,\ \forall k>0.$$

Next, by the same reasoning as in D.Tiba, M.Tiba [16], from (4.14) we get

$$\sum_{n=0}^{N}k\left|\frac{\beta^{\varepsilon}(y_k^{n+1})-\beta^{\varepsilon}(y_k^n)}{k}\right|^2_{H^1(\Omega)^*}\leq C$$

and we can pass to the limit when $k\to 0$:

Denote by $\tilde{u}$, $\tilde{y}$, $\tilde{v}$ the weak limits of the mesh functions defined on $[0,T]$ in the usual manner from the vectors u_k, y_k, v_k.

We have $\tilde{y}$ solution to (1.5), (1.2), (1.3) corresponding to $\tilde{u}$ and $\tilde{v}=\beta^{\varepsilon}(y)$. Moreover, by (4.15) one gets:

$$(4.19)\quad \int_0^T\left\{\frac{1}{2}|\tilde{y}-d|_0^2+\frac{1}{2}|\tilde{u}|^2_{L^2(\partial\Omega)}\right\}dt\leq$$

$$\leq\int_0^T\left\{\frac{1}{2}|y_o-d|_0^2+\frac{1}{2}|u_o|^2_{L^2(\partial\Omega)}\right\}dt$$

for every $u_o\in W^{1,2}(0,T;L^2(\partial\Omega))$.

The following lemma is needed:

<u>Lemma 4.2.</u> <u>Assume that</u> $u_1,u_2\in L^2(\Sigma)$

and satisfy $|u_1-u_2|_{L^2(\Sigma)} < \delta$, $\delta > 0$ being given. Then y_1, y_2 the solutions of (1.5), (1.2), (1.3) corresponding to u_1, u_2 satisfy

(4.20) $|y_1-y_2|_{L^2(Q)} < C.\delta$

where C is independent of δ.

Proof of Lemma 4.2.

Denote $w_i = \int_0^t y_i$, $i=1,2$. Then

$$\beta^\varepsilon((w_i)_t) - \Delta w_i = w_0 + \int_0^t f \qquad \text{a.e.} Q,$$

$$w_i(0,x) = 0 \qquad \text{a.e.} \Omega,$$

$$\frac{\partial w_i}{\partial n} = \int_0^t u_i \qquad \text{a.e.} \Sigma,$$

for $i=1,2$.

Subtract the corresponding equations and multiply by $(w_1-w_2)_t$:

$$\int_0^t \int_\Omega |(w_1)_t - (w_2)_t|^2 - \frac{1}{2}\int_\Omega \Delta(w_1(t)-w_2(t))(w_1(t)-w_2(t))dx \leq 0, \quad \forall t \in [0,T].$$

Henceforth:

$$\int_0^t \int_\Omega |(w_1)_t - (w_2)_t|^2 + \frac{1}{2}\int_\Omega |\nabla(w_1-w_2)|^2 dx \leq$$

$$\frac{1}{2}\int_{\partial\Omega} (w_1(t,\sigma)-w_2(t,\sigma)) . \int_0^t (u_1-u_2) ds d\sigma \leq$$

$$\leq C\delta |w_1(t)-w_2(t)|_1$$

The conclusion follows at once.

Proof of Theorem 4.1 (continued)

By (4.19) and (4.20) we get:

$$(4.21)\quad \int_0^T \left\{ \frac{1}{2}|\tilde{y}-d|_0^2 + \frac{1}{2}|\tilde{u}|^2_{L^2(\partial\Omega)} \right\} dt \leq$$

$$\leq \int_0^T \left\{ \frac{1}{2}|y_0-d|_0^2 + \frac{1}{2}|u_0|^2_{L^2(\partial\Omega)} \right\} dt$$

for every $u_0 \in L^2(\Sigma)$, that is the pair $[\tilde{y},\tilde{u}]$ is optimal for problem (P_ε).

We have shown that, on a subsequence, we have:

$$(4.22)\quad \lim_{k\to 0} \lim_{h\to 0} y_{h,k} = \tilde{y}$$

$$(4.23)\quad \lim_{k\to 0} \lim_{h\to 0} u_{h,k} = \tilde{u}$$

in the weak, respectively strong topology, where $[\tilde{y},\tilde{u}]$ is an optimal pair for (P_ε).

By means of the adjoint system we prove that the above convergences are strong:

$$(3.5)\quad (\nabla\beta^\varepsilon(y^n_{h,k}) \cdot \frac{p^n_{h,k}-p^{n+1}_{h,k}}{k}, v)_h + \int_\Omega \nabla p^n_{h,k} \cdot \nabla v =$$

$$= -(y^n_{h,k}-d^n, v)_h \quad \forall v \in V^h,\ n=0,N-1,$$

$$(3.6)\quad p^N_{h,k}=0,$$

$$(3.7)\quad u^n_{h,k}=p^n_{h,k} \quad ,\ n\leq N.$$

Put $v=p^n_{h,k}-p^{n+1}_{h,k}$:

$$k\left|\frac{p^n-p^{n+1}}{k}\right|_h^2 + \int_\Omega \nabla p^n(\nabla p^n-\nabla p^{n+1}) = -(y^n-d^n, p^n-p^{n+1})_h$$

(we omit the subscripts h,k).

Summing with respect to n, after an easy computation, we obtain:

$$\frac{1}{2}\sum_{n=s}^{N-1} k\left|\frac{p^n-p^{n+1}}{k}\right|_h^2+\frac{1}{2}\int_\Omega|\nabla p^s|^2\leq C\sum_{n=s}^{N-1}k|y^n-d^n|_h^2 .$$

Therefore:

$$(4.24)\quad |p^n_{h,k}|_1\leq C\quad \forall n,h,k$$

$$(4.25)\quad \sum_{n=0}^{N-1}k\left|\frac{p^n_{h,k}-p^{n+1}_{h,k}}{k}\right|_h^2\leq C,\ \forall h,k.$$

By (3.7) and (4.24), (4.25) we see that the convergence i (4.23) is (on a subsequence of the iterated limit) in the stron topology of $L^2(\Sigma)$.

We recall that when k is fixed $y_{h,k}$ is strongly convergen with respect to h (on a subsequence) and $y_{h,k}\rightarrow y_k$:

$$(4.14)\quad \left(\frac{v_k^{n+1}-v_k^n}{k},v\right)+\int_\Omega\nabla y_k^{n+1}\cdot\nabla v-\int_{\partial\Omega}v\cdot u_k^{n+1}=\int_\Omega f_k^{n+1}\cdot v,\ v\in H^1(\Omega),$$

$$(4.13)\quad v_k^n=\beta^\varepsilon(y_k^n).$$

Summing in (4.14) for $n=\overline{0,s-1}$, it yields:

$$(\beta^\varepsilon(y_k^s),v)+\int_\Omega k\sum_0^{s-1}\nabla y_k^{n+1}\cdot\nabla v=\int_{\partial\Omega}v\cdot k\sum_0^{s-1}u_k^{n+1}+$$

$$+\int_\Omega v\cdot k\sum_0^{s-1}f_k^{n+1}+(v^0,v).$$

Taking, the corresponding sum for y_m for $n=\overline{0,q-1}$ and subtracting, we infer:

$$(4.26)\quad (\beta^\varepsilon(y_k^s)-\beta^\varepsilon(y_m^q),v)+\int_\Omega\left(\sum_0^{s-1}k\nabla y^{n+1}-\sum_0^{q-1}m\nabla y^{n+1}\right)\cdot\nabla v=$$

REFERENCES

1. V. Arnăutu - "Approximation of Optimal Distributed Control Problems Governed by Variational Inequalities", Numer. Mat.38, 393-416 (1982).

2. V.Barbu - "Necessary conditions for distributed control problems governed by parabolic variational inequalities", SIAM J.Control Optimiz., vol.19, no.1(1981).

3. V.Barbu - "Boundary Control Problems with Nonlinear State Equation", SIAM J.Control Optimiz., 20(1), 125-144, (1982).

4. J.F.Bonnans - Thèse de docteur ingineur, Univ.de Tech. de Compiègne, 1982.

5. P.G.Ciarlet - "The finite element method for elliptic problems", North Holland, Amsterdam (1978).

6. C.M.Elliott
J.R.Ockendon - "Weak and variational methods for moving boundary problems", Pitman (1982).

7. R.S.Falk - "Approximations of an optimal control problem", "Applied nonlinear functional analysis", R.Gorenflo and K.-H.Hoffman eds. in Methoden und Verfahren der mathematischen Physik, band 25, Peter Lang (1983).

8. G.Fix
G.Strang - "An analysis of the finite element method", Prentice-Hall (1973).

9. O.Grange
F.Mignot - "Sur la resolution d'une equation et d'une inequation paraboliques non linéaires", J.Funct.Analysis, 11, 77-93 (1972).

10. K.-H.Hoffman
J.Sprekels - "Automatic delay-control in a two-phases Stefan Problem", in "Differential-Difference Equations", L.Collatz, G.Meinardus, W.Wetterling eds., Birkhäuser (1983).

11. J.L.Lions - "Quelques methodes de resolution des problèmes aux limites nonlinéaires", Dunod (1969).

12. I.Pawlow
M.Niezgodka - "Optimal control for parabolic systems with free boundaries" in Lect.Notes Control an Information Sciences 22, p.412-421, Springer (1980).

13. C.Saguez - Thèse, Univ.de Technologie de Campiegne (1980).

14. D.Tiba
Z.Meike - "Optimal control for a Stefan problem" in "Analysis and Optimization of Systems", A.Bensoussan and J.L.Lions eds, Lect.Notes in Control and Information Sciences 44 (1982).

15. D.Tiba - "Boundary Control for a Stefan problem", Proceedings of the Oberwolfach Conference, K.-H.Hoffman and W.Krabs eds., Birkhäuser (to appear).

16. D.Tiba
M.Tiba - "Regularity of the boundary data and the convergence of the finite element discretization in two-phases Stefan problems" (to appear).

17. D.Tiba
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