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by

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THE WEAK FATOU PROPERTY AND THE EGOROFF PROPERTY

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The aim of this paper is to prove that for every if for every extended Riesz norm ρ on an order complete Riesz space E the seminorm ρ_L has the weak Fatou property then E has the Egoroff property; a result of this kind was already announced in [4] but the proof was not correct. Results of similar nature, but considering the Fatou property instead of the weak Fatou property were given in [1] and [2].

A Riesz space E has the Egoroff property if for any $x \in E_+$ and any double sequence $(x_{nk})_{n,k \geq 1} \subset E_+$ such that $x_{nk} \uparrow_k x$ for every $n \geq 1$ there is $\varphi : \mathbb{N} \to \mathbb{N}$ and a sequence $(x_n)_{n \geq 1} \subset E$ such that $x_n \leq x_{n,\varphi(n)}$ and $x_n \uparrow x$.

An extended Riesz seminorm on the Riesz space E is a function $\rho : E \to \mathbb{R}_+ \cup \{\infty\}$ such that:

i) $\rho(x+y) \leq \rho(x) + \rho(y)$, $\rho(ax) = a\rho(x)$ for $x, y \in E$ and $a \in \mathbb{R}_+$.

ii) $\rho(x) \leq \rho(y)$ for $x, y \in E$ and $|x| \leq |y|$.

An extended Riesz seminorm ρ is a Riesz norm if $x \neq 0$ implies $\rho(x) \neq 0$.

The extended Riesz seminorm ρ has the weak Fatou property if there is $c \geq 1$ such that from $0 \leq x_n \uparrow x$ it follows that $\rho(x) \leq c \sup_{n \geq 1} \rho(x_n)$.

For every extended Riesz seminorm ρ the Lorentz seminorm ρ_L is defined by

$$\rho_L(x) = \inf \left\{ \sup_{n \geq 1} \rho(x_n) \,\middle|\, 0 \leq x_n \uparrow |x| \right\}.$$

Theorem. Assume that the continuum hypothesis holds. Then for every order complete Riesz space E the following are equivalent:

i) For every extended Riesz norm ρ on E the seminorm ρ_L has the weak Fatou property.

ii) E has the Egoroff property.

The proof of the theorem needs some lemmas.

A weakly σ-distributive Riesz space is a σ-order complete Riesz space E such that for every order bounded double sequence $(x_{nk})_{n,k\geq 1} \subset E_+$ increasing in k for every n we have:

$$\inf_{n\geq 1} \sup_{k\geq 1} x_{nk} = \sup_{\varphi: \mathbb{N}\to\mathbb{N}} \inf_{n\geq 1} x_{n,\varphi(n)} .$$

A Riesz space is called order separable if for every $x \in E$ and every $A \subset E$ such that $x = \sup A$ there is a countable subset $B \subset A$ such that $x = \sup B$.

Lemma 1. Let E be a weakly σ-distributive order separable Riesz space. Then E has the Egoroff property.

<u>Proof.</u> Let $0 \leq x_{nk} \uparrow_k x$ for every $n \geq 1$. We want to show that there are $(y_n)_{n\geq 1} \subset E$ and $\psi: \mathbb{N} \to \mathbb{N}$ such that $y_n \leq x_{n,\psi(n)}$ and $y_n \uparrow x$.

As E is weakly σ-distributive we have

$$x = \inf_{n\geq 1} \sup_{k\geq 1} x_{nk} = \sup_{\varphi: \mathbb{N}\to\mathbb{N}} \inf_{n\geq 1} x_{n,\varphi(n)}$$

As E is order separable there is a sequence $(\varphi_m)_{m\geq 1}$ of maps $\varphi_m: \mathbb{N} \to \mathbb{N}$ such that

$$x = \sup_{m\geq 1} \inf_{n\geq 1} x_{n,\varphi_m(n)} .$$

Let $\psi_h: \mathbb{N} \to \mathbb{N}$, $\psi: \mathbb{N} \to \mathbb{N}$ be given by

$$\psi_n = \sup_{1\le k\le n} \varphi_k$$

$$\psi(n) = \psi_n(n).$$

Put

$$y_n = \inf_{m\, 1} x_{m,\psi_n(m)}$$

The sequence $(y_n)_{n\geq 1}$ is increasing. We have

$$x = \sup_{m\geq 1} \inf_{n\geq 1} x_{n,\varphi_m(n)} \leq \sup_{m\geq 1} \inf_{n\geq 1} x_{n,\psi_m(n)} \leq \sup_{m\geq 1} y_m \leq x.$$

Hence $y_n \uparrow x$. On the other side

$$y_n = \inf_{m\geq 1} x_{m,\psi_n(m)} \leq x_{n,\psi_n(n)} = x_{n,\psi(n)}.$$

The proof is complete.

A result of Pinsker and Amemiya ([3], theorem 75.5) states that under the continuum hypothesis, every order complete Riesz space with the Egoroff property is order separable. It is also easy to prove that a σ-order complete Riesz space with the Egoroff property is weakly σ-distributive. Hence, the preceeding lemma gives a converse to these statements.

We say that a Riesz space has property i) if it satisfies i) from the statement of the theorem.

If E is a σ-order complete space we denote by P_x the projection on the band generated by x.

Lemma 2. Let E be an Archimeadian Riesz space. Suppose there is a σ- order complete Riesz space F, a positive σ-order continuous linear map $T: E \rightarrow F$, an element $x \in E_+$, a sequence $(x_n)_{n\geq 1} \subset E_+$ and a double sequence $(x_{nk})_{n,k\geq 1} E_+$ such that:

i) $x_n \uparrow x$.

ii) $x_{nk} \uparrow_k x$ for every $n \geq 1$

iii) $P_{Tx_m} Tx \neq Tx$ for every $n \geq 1$

iv) $\inf_{n \geq 1} f x_{n,\varphi(n)} = 0$ for every $\varphi : \mathbb{N} \to \mathbb{N}$.

Then E has not the property i).

<u>Proof</u>. We may assume that x_{nk} is decreasing in n for every $k \geq 1$ (otherwise replace x_{nk} by $\inf_{1 \leq m \leq n} x_{mk}$).

Let K be the solid convex hull of $\{x_{nk} \wedge x_n \mid n,k \geq 1\}$. Define the extended Riesz seminorm ρ by

$$\rho(y) = \inf\{a \mid a \geq 0,\ y \in aK\}$$

(it is understood that $\inf \emptyset = \infty$).

If $v \in K$ then $|v| \leq x$. Hence if $\rho(y) = 0$ then $|y| \leq ax$ for every $a > 0$. As E is Archimeadian, $y = 0$. Thus ρ is an extended Riesz norm.

Suppose that E has the property i). Then ρ_L has the weak Fatou property. As $x_{nk} \wedge x_n \in K$ it follows that

$$\rho_L(x_{nk} \wedge x_n) \leq \rho(x_{nk} \wedge x_n) \leq 1$$

for every $n,k \geq 1$. Hence $\rho_L(x_n) \leq c$ for every $n \geq 1$. Applying once again the weak Fatou property we get $\rho_L(x) \leq c^2$. Put $d = c^2 + 1$. There is $(y_m)_{m \geq 1} \subset E_+$ such that $0 \leq y_m \uparrow x$ and $\rho(y_m) < d$ for $m \geq 1$. If $z_m = \frac{1}{d} y_m$ then $z_m \uparrow \frac{1}{d} x$ and $z_m \in K$. Hence, there is a triple sequence $(\lambda^m_{nk})_{m,n,k \geq 1}$ such that:

i) For every $m \geq 1$ the set $\{(n,k) \mid \lambda^m_{nk} \neq 0\}$ is finite,

ii) $z_m \leq \sum_{n,k \geq 1} \lambda^m_{nk} (x_{nk} \wedge x_n)$.

iii) $\sum_{n,k \geq 1} \lambda^m_{nk} = 1$ for every $m \geq 1$.

Define $\psi : \mathbb{N} \to \mathbb{N}$ by

$$\psi(m) = \sup\{k \mid k \geq 1, \ (\exists n)\lambda^m_{nk} \neq 0\}.$$

Put

$$\lambda^m_n = \sum_{k \geq 1} \lambda^m_{nk}$$

As $\lambda^m_{nk} \neq 0$ implies $k \leq \psi(m)$ we have

$$\sum_{n,k \geq 1} \lambda^m_{nk}(x_{nk} \wedge x_n) \leq \sum_{n,k \geq 1} \lambda^m_{nk}(x_{n,\psi(m)} \wedge x_n) = \sum_{n \geq 1} \lambda^m_n (x_{n,\psi(m)} \wedge x_n).$$

Hence

$$z_m \leq \sum_{n \geq 1} \lambda^m_n (x_{n,\psi(m)} \wedge x_n),$$

$$\sum_{n \geq 1} \lambda^m_n = 1.$$

Now we prove the following:

(A) For every $\varepsilon > 0$ there is $m \geq 1$ such that for every $m_1 > m$ there is $m_2 > m_1$ such that $\sum_{m \leq n < m_1} \lambda^{m_2}_n \leq \varepsilon$.

Otherwise there would be an $\varepsilon > 0$ such that for every $m \geq 1$ there is $m_1 > m$ such that $m_2 > m_1$ implies $\sum_{m \leq n < m_1} \lambda^{m_2}_n > \varepsilon$. Let $k \geq 1$ be such that $k\varepsilon > 1$. Choose $1 = m_0 < m_1 \ldots < m_k$ inductively such that

$$\sum_{m_i \leq n < m_{i+1}} \lambda^m_n > \varepsilon \text{ for } 0 \leq i \leq k-1 \text{ and } m > m_{i+1}$$. Let $m = m_k + 1$. Then

$$1 = \sum_{n \geq 1} \lambda^m_n \geq \sum_{0 \leq i \leq k-1} \ \sum_{m_i \leq n < m_{i+1}} \lambda^m_n > k\varepsilon$$

which is a contradiction.

Now take $\varepsilon = \frac{1}{2d}$ and get m with the property in the statement of (A). Choose inductively an increasing sequence $(m_p)_{p\geq 1}$ such that $m_1 = m+1$ and $\sum_{m\leq n<m_{p-1}} \lambda_n^{m_p} \leq \varepsilon$ for $p\geq 2$. Fix $p\geq 2$. We have for $q\geq p$

$$Tz_{m_p} \leq Tz_{m_q} \leq \sum_{n\geq 1} \lambda_n^{m_q} T(x_{n,\psi(m_q)} \wedge x_n).$$

But

$$\sum_{1\leq n<m} \lambda_n^{m_q} T(x_{n,\psi(m_q)} \wedge x_n) \leq \Big(\sum_{1\leq n<m} \lambda_n^{m_q}\Big) Tx_m$$

$$\sum_{m\leq n<m_{q-1}} \lambda_n^{m_q} T(x_{n,\psi(m_q)} \wedge x_n) \leq \Big(\sum_{m\leq n<m_{q-1}} \lambda_n^{m_q}\Big) Tx \leq \frac{1}{2d} Tx$$

$$\sum_{m_{q-1}\leq n} \lambda_n^{m_q} T(x_{n,\psi(m_q)} \wedge x_n) \leq Tx_{m_{q-1},\psi(m_q)}.$$

Hence

$$Tz_{m_p} \leq \sum_{1\leq n\leq m} \lambda_n^{m_q} Tx_m + \frac{1}{2d} Tx + Tx_{m_{q-1},\psi(m_q)}$$

Applying $I-P_{Tx_m}$ we get

$$(I-P_{Tx_m})\Big(Tz_{m_p} - \frac{1}{2d} Tx\Big) \leq Tx_{m_{q-1},\psi(m_q)}$$

Define $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ by

$$\varphi(n) = \psi(m_p) \quad \text{for } 1\leq n\leq m_{p-1}$$

$$\varphi(n) = \psi(m_{q+1}) \text{ for } m_{q-1} < n \leq m_q,\ q\geq p.$$

Then

$$0 \le \inf_{q \ge p} Tx_{m_{q-1}, \psi(m_q)} \le \inf_{n \ge 1} Tx_{n, \psi(n)} = 0.$$

It follows that

$$(I - P_{Tx_m})(Tz_{m_p} - \frac{1}{2d}Tx) \le 0.$$

As $z_{m_p} \uparrow \frac{1}{d}x$ we have $Tz_{m_p} \uparrow \frac{1}{d}Tx$. Hence

$$(I - P_{Tx_m})(\frac{1}{2d}Tx) = 0$$

which together with $P_{Tx_m}Tx \ne Tx$ gives a contradiction.

Lemma 3. Let E be a σ-order complete Riesz space. Let $0<a<1$, $x \in E_+$ and $(x_n)_{n \ge 1} \subset E_+$ be such that $x_n \uparrow x$ and $(ax - x_n)_+ \ne 0$ for every $n \ge 1$. Then there is a sequence $(y_n)_{n \ge 1}$ such that $0 \le y_n \uparrow x$ and $P_{y_n}x \ne x$ for every $m \ge 1$.

Proof. Let $y_n = P_{(x_n - ax)_+}x$. Obviously $y_n \uparrow x$ and $P_{y_n}x = y_n$. Suppose that $P_{y_n}x = x$. Then $y_n = x$. We have

$$0 \le x + P_{(ax-x_n)_+}x = P_{(x_n-ax)_+}x + P_{(ax-x_n)_+}x \le x,$$

thus

$$P_{(ax-x_n)_+}x = 0.$$

Therefore

$$0 \le P_{(ax-x_n)_+}(ax - x_n) = -P_{(ax-x_n)_+}x_n \le 0$$

which implies

$$P_{(ax-x_n)_+}x_n=0.$$

It follows that

$$(ax-x_n)_+=P_{(ax-x_n)_+}(ax-x_n)_+=(P_{(ax-x_n)_+}(ax-x_n))_+=0$$

which is a contradiction.

A Riesz subspace F of the Riesz space E is called relatively σ-order closed if from $x\in E_+$, $x_n\in F$ and $x_n\downarrow x$ it follows that $x\in F$.

Lemma 4. Let E be a σ-order complete Riesz space and let F be a relatively σ-order closed Riesz subspace of E. If E has property i) then F also has property i).

Proof. Let ρ be an extended Riesz norm on F. Define the extended Riesz seminorm $\bar{\rho}$ on E by

$$\bar{\rho}(x)=\inf\{\rho(y)\,|\,y\in F, |x|\leq y\}$$

($\inf\phi=\infty$). As F is relatively σ-order closed, for every $x\in E$ such that $\bar{\rho}(x)<\infty$ there is $y\in F_+$ such that $|x|\leq y$ and $\rho(y)=\bar{\rho}(x)$. In particular it follows that $\bar{\rho}$ is an extended Riesz norm. It will then suffice to prove that $\rho_L(x)=\bar{\rho}_L(x)$ for every $x\in F_+$.

Let $x\in F_+$ and let $x_n\in F_+$ be such that $x_n\uparrow x$ in F. Then $x_n\uparrow x$ in E. We have

$$\bar{\rho}_L(x)\leq\sup_{n\geq1}\bar{\rho}(x_n)\leq\sup_{n\geq1}\rho(x_n)$$

hence

$$\bar{\rho}_L(x) \leq \rho_L(x).$$

To prove the converse inequality, let $x_n \in E_+$ be such that $x_n \uparrow x$. We want to prove that $\sup_{n \geq 1} \bar{\rho}(x_n) \geq \rho_L(x)$. We may assume that $\bar{\rho}(x_n) < \infty$ for $n \geq 1$. There is $y_n \in F_+$ such that $x_n \leq y_n$ and $\rho(y_n) = \bar{\rho}(x_n)$. Put

$$z_n = \inf_{m \geq n} y_m \wedge x.$$

Then $z_n \in F$ and $x_n \leq z_n \leq x$, hence $z_n \uparrow x$. We have

$$\rho(z_n) \leq \rho(y_n) = \bar{\rho}(x_n)$$

hence

$$\rho_L(x) \leq \sup_{n \geq 1} \rho(z_n) \leq \sup_{n \geq 1} \bar{\rho}(x_n).$$

Proof of the theorem.

ii) $\Rightarrow$ i) This is known. In fact, ii) implies that ρ_L has the Fatou property ([2]).

i) $\Rightarrow$ ii) We prove first that E is weakly σ-distributive. If this is not the case, there is by [4] an extended Riesz norm ρ on E such that ρ_L is not a norm. Hence there is $x \in E_+ \setminus \{0\}$ such that $\rho_L(x) = 0$. This implies the existence of $(x_{nk})_{n,k \geq 1}$ E_+ such that $x_{nk} \uparrow_k x$ and $\sup_{k \geq 1} (x_{nk}) \leq \frac{1}{n}$ for every $n \geq 1$. Let $\varphi : \mathbb{N} \to \mathbb{N}$ and let $y = \inf_{n \geq 1} x_{n,\varphi(n)}$. We have

$$\rho(y) \leq \rho(x_{n,\varphi(n)}) \leq \frac{1}{n}$$

hence $\rho(y) = 0$ which implies that $y = 0$.

There is a sequence $(z_n)_{n\geq 1}$ such that $0\leq z_n\uparrow x$ and $(\frac{1}{2}x-z_n)_+\neq 0$ for $n\geq 1$. Otherwise for every $n\geq 1$ there would be a $\varphi(n)\geq 1$ such that $x_{n,\varphi(n)}\geq\frac{1}{2}x$, which is a contradiction. Therefore by lemma 3 there is a sequence $(x_n)_{n\geq 1}$ such that $0\leq x_n\uparrow x$ and $P_{x_n}x\neq x$ for $n\geq 1$.

By lemma 1 applied to $F=E$ and $T=1_E$ we obtain that E has not property i), which is a contradiction. Hence E is weakly σ-distributive

Second we prove that E is order separable. If not, there is an uncountable order bounded set $M\subset E_+$ consisting of disjoint elements. Let B(M) be the Riesz space of all bounded functions $f:M\to\mathbb{R}$. Define $H:B(M)\to E$ by

$$H(f)=\sup_{\substack{F\subset M\\ F\text{ finite}}}\sum_{x\in F}f(x)x$$

for every $f\in B(M)_+$ and then extend H by linearity. Then H is a Riesz isomorphism of B(M) onto H(B(M)) and H(B(M)) is a relatively σ-order closed Riesz subspace of E. By lemma 4, B(M) has property i). As M is uncountable and we assume the continuum hypothesis holds, a result of Banach and Kuratowsky ([3], ch.10) states that there is a double sequence $(M_{nk})_{n,k\geq 1}$ of subsets of M such that $M_{nk}\subset M_{n,k+1}$, $\bigcup_{k\geq 1}M_{nk}=M$ for every $n\geq 1$ and $\bigcap_{n\geq 1}M_{n,\varphi(n)}$ is at most countable for every $\varphi:\mathbb{N}\to\mathbb{N}$. There is also a sequence $(M_n)_{n\geq 1}$ of subsets of M such that $M_n\subset M_{n+1}$, $\bigcup_{n\geq 1}M_n=M$ and $M\setminus M_n$ is not countable. This can be obtained as follows: there is an one-to-one map $g:\mathbb{R}\to M$. Then put $M_n=M\setminus g((n,\infty))$.

Let G be the order ideal of all bounded maps $f:M\to\mathbb{R}$ such that $\{x\mid f(x)\neq 0\}$ is at most countable and let F be the quotient Riesz space $\frac{B(M)}{G}$. Let $T:B(M)\to F$ be the quotient map; T is σ-order continuous Let

$$x_{nk} = \chi_{M_{nk}}$$

$$x_n = \chi_{M_n}$$

$$x = \chi_M$$

(χ_A being the characteristic function of a set $A \subset M$). Then $x_{nk} \uparrow x$ for $n \geq 1$, $x_n \uparrow x$, $\inf Tx_{n,\varphi(n)} = o$ for every $\varphi : \mathbb{N} \to \mathbb{N}$ and

$$P_{Tx_n} Tx = TP_{x_n} x \neq Tx.$$

An application of Lemma 2 shows that B(M) has not property i), which is a contradiction. Hence E is order separable.

By lemma 1, E has the Egoroff property.

R E F E R E N C E S

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