INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250 3638

A DESINGULARIZATION THEOREM IN NON NORMAL ONE DIMENSIONAL RINGS

by

Vasile NICA

PREPRINT SERIES IN MATHEMATICS

No.43/1983

BUCURESTI

A DESINGULARIZATION THEOREM IN ON NORMAL ONE DIMENSIONAL RINGS

by

Vasile NICA*)

July 1983

*) Ghirlandei 38, Bl.D1, ap.27, 77592, Bucharest, Romania

A DESINGULARIZATION THEOREM IN NON NORMAL ONE DIMENSIONAL RINGS

Vasile Nica

o. Introduction

It is well known the following beautiful result due to Néron:

Theorem(Néron [15]) Let $R \subset R'$ be an unramified, regular inclusion of discrete valuation rings. Then every finite type sub R algebra B of R′ can be embedded in a finite type smooth sub R algebra $B' \subset R'$, or, equivalently, R′ is a filtered inductive limit of its finite type smooth sub R algebras.

This desingularization theorem is very useful in approximation theory because it reduces solvability in R′ of some polynomial systems over R to others for which the implicit function theorem is possible to apply.

The aim of this paper is to prove the following extension of Néron′s theorem to the non normal one dimensional case:

Theorem o.1 Let $u:A \longrightarrow A'$ be a local regular morphism of noetherian domains, A′ being one dimensional and henselian. Suppose that the canonical residue field extension induced by u is separable. Then A′ is a filtered inductive limit of finite type smooth A algebras.

Many generalizations of Néron′s classical desingularization have been already made, but all of these treat the so-called normal case: A is a normal ring [9, 19, 10, 17, 20]

The proof of the theorem o.1 is inspired from [21] and is done in section 4.

The first two sections deal with a special kind of ring morphisms, called desingularization morphisms, which are filtered inductive limits of finitely presented smooth morphisms. Some properties concerning their behaviour are studied and it is shown that, in the noetherian case these morphisms are regular.

Néron's theorem and its known extensions suggest the following general desingularization problem: Is a regular morphism of noetherian rings a desingularization one? [20,21,22]. As we see in section 2, cor. 2.4 an affirmative answer to the above problem solves positively a famous conjecture of the approximation theory, stated by Artin [4]:

"Every excellent, henselian, local ring has the approximation property" i.e. every polynomial system over A has a solution in A iff. it has a solution in the completion $\hat{A}$.

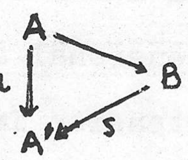
Some technical facts concerning lifting smoothness are included in section 3.

As an application of the theorem 3.11 we give in the last section a slight improvement of a recent desingularization theorem in dimensión two (normal case) due to Artin and Denef [6].

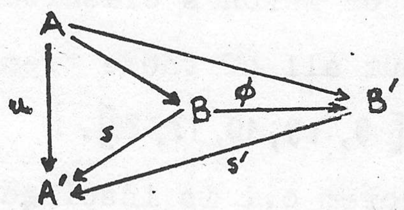
We are indebded to dr. D. Popescu and prof. dr. N. Radu for many helpful conversations on the subject.

1. Desingularization morphisms

(1.1) Definition A ring morphism $u: A \longrightarrow A'$ is called desingularization morphism (shortly d-morphism) if any commutative diagram

(1.1.1) 

in which B is a finitely presented A algebra, can be embedded into a larger one:

(1.1.2) 

in which B' is a finitely presented smooth A algebra.

(1.2) Proposition(Artin [6]) $u: A \longrightarrow A'$ is a desingularization morphism iff u is a filtered inductive limit of finitely presented smooth morphisms.

(1.3) Examples of d-morphisms i) A morphism of the form $A \longrightarrow A[X_i]$

ii) An unramified regular inclusion of discrete valuation rings(cf. Néron's desingularization theorem)

iii) A local formally smooth morphism of local artinian rings (clearly such a morphism is regular).

(1.4) As finitely presented smooth morphisms are flat and flatness is preserved by filtered inductive limits it results that any desingularization morphism is flat. Also d-morphisms are stable under base change and filtered inductive limits.

Now, we shall see that these morphisms are stable under composition. The proof is based on the following:

(1.5) Lemma Consider a commutative diagram of ring morphisms:

$$\begin{array}{ccc} A & \xrightarrow{u} & A' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\phi} & C \end{array}$$

in which B (resp. C) is a finitely presented algebra over A (resp. C). In addition suppose C smooth over A'. Then, there exists a commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{v} & D & \xrightarrow{w} & A' \\ \downarrow & & \downarrow & & \downarrow \\ B & \xrightarrow{\lambda} & E & \xrightarrow{\mu} & C \end{array}$$

such that:

1) $w \circ v = u$, $\mu \circ \lambda = \phi$

2) D is a finitely presented A algebra, E is a finitely presented smooth D algebra and $C = E \otimes_D A'$.

Proof Write:

$$B = A[Y]/f.A[Y] \quad \text{with } Y = (Y_1, \ldots, Y_N),\ f = (f_1, \ldots, f_n$$

$$C = A'[Z]/g.A'[Z] \quad \text{with } Z = (Z_1, \ldots, Z_M) ,\ g = (g_1, \ldots, g_m)$$

Denote $y_i = \text{cls}.Y_i \bmod f.A[Y]$, $z_j = \text{cls}.Z_j \bmod g.A'[Z]$ such that:

$B=A[Y]$, $C=A'[Z]$ and let $P_i \in A'[Z]$ be some polynomials such that $P_i(z)=\phi(y_i)$ $1\leq i\leq N$. Let $\Phi : A[Y] \longrightarrow A[Z]$ be the extension of u defined by $Y_i \rightsquigarrow P_i(Z)$. As Φ induces ϕ , we have :

$$\Phi(f_i)=f_i(P_1(Z),\ldots,P_N(Z)) \in g.A'[Z]$$

i.e.

$$(1.5.1)\quad f_i(P_1(Z),\ldots,P_N(Z))=\sum_{j=1}^{m} l_{ij}(Z)g_{ij}(Z) \qquad 1\leq i\leq n \quad \text{with } l_{ij}\in A[Z] .$$

As A' is a filtered inductive limit of finitely presented A algebras there exists such an A algebra D from which proceed the coefficients of polynomials included in (1.5.1), and such that in $D[Z]$, the identities (1.5.1) hold effectively. We put $E=D[Z]/g.D[Z]$. Then the morphism $\Phi : A[Y] \longrightarrow D[Z]$, $\Phi(Y_i)=P_i(Z)$ which extends the structure morphism $A \longrightarrow D$ induces an A morphism $\lambda : B \longrightarrow E$. Clearly $C=E\otimes_D A'$. It remains to choose D such that E is smooth over D.

For this we recall that C is smooth over A' iff some special polynomial identities over A' are fulfilled. These identities can be lifted in $D[Z]$ choosing D suitably.

We need some preparations.

Let $h=(h_1,\ldots,h_r)$ be a variable system of polynomials from the ideal $g.A'[Z]$ with $r\leq M$ also variable. Let Δ_h be the ideal generated in $A'[Z]$ by all r-minors of the jacobian matrix $\left[\frac{\partial h}{\partial Z}\right]$. Let $(h:g)$ be the ideal of polynomials $v\in A'[Z]$ such that $vg_j \in h.A'[Z]$ $1\leq j\leq m$. Finally consider the ideal

$$H=g.A'[Z]+\sum_h \Delta_h(h:g)$$

where the sum is taken over all systems $h=(h_1,\ldots,h_r)\subset g.A'[Z]$.

By the jacobian criterion of smootheness for finitely presented algebras, C is smooth over A' iff $H=A'[Z]$. Consequently, C is smooth over A' iff there exist polynomials $u_1,\ldots,u_n$ in $A'[Z]$, a finite number of systems $h^{(p)}=(h_1^{(p)},\ldots,h_{r_p}^{(p)})$, $1\leq p\leq s$ in $g.A'[Z]$ with $r_p\leq M$ and for every p a r_p-minor m_p of the jacobian matrix $\left[\frac{\partial h^{(p)}}{\partial Z}\right]$ and a polynomial $v_p\in A'[Z]$ such that $v_p g_j \in h^{(p)}.A'[Z]$, $1\leq j\leq m$, all these satisfying the identity:

$$(1.5.2)\quad \sum_{j=1}^{n} g_j u_j + \sum_{p=1}^{s} m_p v_p = 1$$

As $v_p g_j \in h^{(p)}.A'[Z]$ there exist polynomials $w_{pjk} \in A'[Z]$ such that

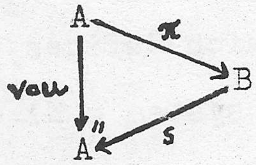
$$(1.5.3)\quad v_p g_j = \sum_{k=1}^{r_p} w_{pjk} h_k^{(p)} \qquad 1 \leq p \leq s,\ 1 \leq j \leq m\ .$$

Now, we choose the finitely presented A algebra D such that (1.5.2), (1.5.3) hold in $D[Z]$. Then, $E = {}^{D[Z]}/_{g.D[Z]}$ will be smooth over D.

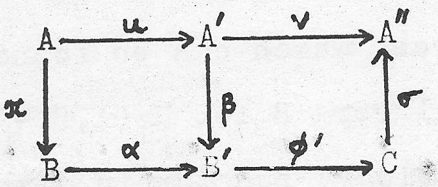
(1.6) <u>Corolary</u> i) Let $u: A \longrightarrow A'$, $v: A' \longrightarrow A''$ be desingularization morphisms. Then $v \circ u$ is too.

ii) Suppose that v has the following property: " Every polynomial system over A' has a solution in A' if it has one in A''. " (see also definition 2.1) Then, if $v\,u$ is a d-morphism, u is too.

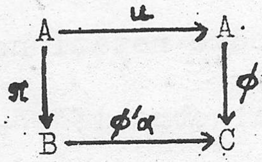
<u>Proof</u> i) Consider a commutative diagram :



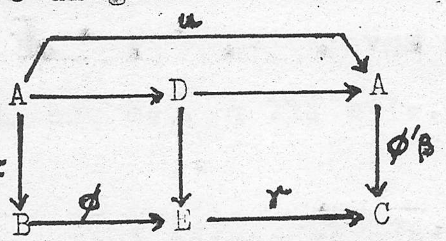
in which B is a finitely presented A algebra. Then, $B' = B \otimes_A A'$ is a finitely presented A' algebra and by hypothesis there exists a commutative diagram :



in which C is a finitely presented smooth A algebra and $\sigma\phi'\alpha = s$, $\sigma\phi\beta = v$. Applying (1.5) to the diagram :



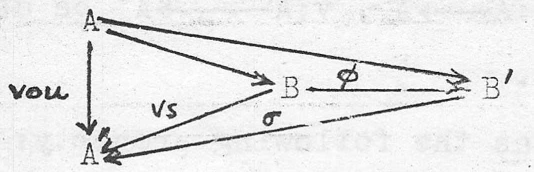
we find a commutative diagram of the form :



with: $\phi'\beta u = \gamma\phi\pi$
$\gamma\phi = \phi'\alpha$

in which D can be chosen not only finitely presented over A but just smooth over A, u being a desingularization morphism (see (1.2)). Smootheness is preserving by composition and so E is a finitely presented A algebra. Finally, $\sigma\gamma\phi=\sigma\phi\alpha'=s$.

ii) Let $B={}^{A[Y]}/_{f.A[Y]}$, $Y=(Y_1,..,Y_N)$ be a finitely presented A algebra and $s:B\longrightarrow A'$ an A-morphism. There exists a finitely presented smooth A algebra B' and a commutative diagram of the form:



because vou is a d-morphism. Suppose that $B'={}^{A[Z]}/_{g.A[Z]}$, $Z=(Z_1,..,Z_M)$. The morphism s defines in A' a solution y for the polynomial system $f(Y)=o$. Also σ defines in A'' a solution z for $g(Z)=o$. Let $h=(h_1,..,h_N)$ a system of polynomials from $A[Z]$ which defines ϕ. Then $h(z)=v(y)$. Thus, z is a solution in A'' for the system $g(Z)=o$, $h(Z)=v(y)$. By hypothesis, there exists $x\in A'^M$ such that $g(x)=o$ and $h(x)=y$. This x defines an A-morphism $s':B'\longrightarrow A'$ such that $s'\phi=s$. The proof is finished.

In the following we need some simple facts from (co)homology theory of commutative algebras, which can be found in [1].
For an arbitrarily given A algebra B let $H_1(A,B,-)$, (resp. $H^1(A,B,-)$) be its homology (resp. cohomology) functor in dimension one. The B module $H_1(A,B,B)$ is often denoted $N_{B/A}$. We recall the following well known general criterion of smootheness:

(1.7) Proposition [1] In the above notations, B is a smooth (not necessarily finitely presented) A algebra iff $N_{B/A}=o$ and $\Omega_{B/A}$ is a projective B module, which means exactly $H^1(A,B,-)=o$.

On the other hand, we have:

(1.8) Lemma [1] $H_1(A,B,-)=o$ iff $n_{B/A}=o$ and $\Omega_{B/A}$ is a flat B module.

Now, consider a finitely presented A algebra B. Then, $\Omega_{B/A}$

is a finitely presented B module and $\Omega_{B/A}$ is flat over B iff it is projective [12]. So:

(1.9) <u>Corolary</u> If B is a finitely presented A algebra, then B is smooth over A iff $H_1(A,B,-)=o$.

(1.1o) <u>Lemma</u> If $u:A \longrightarrow A'$ is a desingularization morphism then $H_1(A,A',-)=o$.

<u>Proof</u> The homology functor $H_1(A,A',-)$ commutes - with respect to A' - with filtered inductive limits and so, the lemma follows from (1.9).

Next, suppose that A,A' are noetherian rings and $u:A \longrightarrow A'$ is flat; u is called regular if for any $q \in \operatorname{Spec} A$ and $p=u^{-1}(q)$, the induced local morphism $A_p \longrightarrow A_q$ is formally smooth [11,14]. Then, the following homological criterion of regularity, due to André, holds:

(1.11) <u>Theorem</u> [1] u is regular iff $H_1(A,A',-)=o$.

From (1.1o) and (1.11) we obtain:

(1.12) <u>Corolary</u> A desingularization morphism between noetherian rings is regular.

A few comment on (1.11): It is easy to prove the sufficiency via jacobian criterion of formal smootheness [11]. The necessity is hard enough and the only known proof uses high dimensional (co)homology modules of commutative algebras.

Néron's theorem and other particular examples suggest the following:

(1.13) <u>General desingularization problem</u> Is a regular morphism of noetherian rings a desingularization morphism?

An affirmative answer would have useful applications in approximation theory as we see in the next §. On the other hand, it would offer, via (1.1o) an alternative proof for (1.11).

2. Algebraically pure morphisms, desingularization morphisms and the property of approximation.

We recall that a noetherian local ring A has the property of approximation (shortly, A is an AP-ring), if for every polynomial system $f=(f_1,..,f_m)$ from $A[X]=A[X_1,..,X_N]$, every integer $c \geqslant 1$ and every $\hat{x}=(\hat{x}_1,..,\hat{x}_N) \in \hat{A}^N$ such that $f(\hat{x})=o$, there exists $x=(x_1,..,x_N)$ from A^N such that $f(x)=o$ and $x \equiv \hat{x} \bmod \underline{m}^c.\hat{A}$, $\underline{m}$ being the maximal ideal of A [25] .

Writing congruences in equational form, A is an AP-ring iff the canonical completion morphism $A \longrightarrow \hat{A}$ has the following property:

"Every polynomial system over A has a solution in A iff it has one in $\hat{A}$".

This observation is the source for the following:

(2.1) Definition [18] A ring morphism $u:A \longrightarrow A'$ (A,A' not necessarily noetherian or local) is called algebraically pure if every polynomial system over A has a solution in A iff it has one in A'

Algebraically pure morphisms also generalize the linear case of pure module morphisms. They are stable under composition, base change, and filtered inductive limits.Other properties and some interesting applications in approximation theory can be found in [18, 16, 7] .

Let $u:A \longrightarrow A'$ be a local morphism of local rings such that the maximal ideal $\underline{m}$ of A generates the maximal ideal of A'. In this case we shall say that u is an unramified morphism. Let K and $K'=K \otimes_A A'$ be the residue fields of A and A', respectively.

A join between algebraically pure morphisms and desingularization ones introduced in the previous section is furnished by the following:

(2.2) Theorem. In the above notations, suppose additionally that u is a desingularization morphism and A is henselian. The next statements are equivalent:

i) u is algebraically pure.

ii) The residue field extension $K \subset K'$, canonically induced by u is algebraically pure.

<u>Proof</u> We have i)$\Longrightarrow$ii) because algebraically pure morphisms are stable under base change.

ii)$\Longrightarrow$i). Let $f=(f_1,..,f_m)$ a system of polynomials from $A[X]=A[X_1,..,X_N]$ and $y'=(y'_1,..,y'_N)\in A'^N$ a solution for it. We put: $B=A[X]/f.A[X]$ and let $s:B\longrightarrow A'$ be the morphism defined by y'. As u is a d-morphism, we can suppose B smooth over A at $s^{-1}(\underline{m}.A')$. Then, by jacobian criterion of smootheness, there exists a system of polynomials $g=(g_1,..,g_r)$ in the ideal $f.A[X]$, a r-minor M of the jacobian matrix$\left[\frac{\partial g}{\partial X}\right]$ such that $M(y')$ is invertible in A' and a polynomial $h\in A[X]$ with the properties: $h(y')$ is an unit of A' and $hf_i\in g.A[X]$, $1\leq i\leq m$. Clearly, y' induces a solution $(\bar{y}',\bar{u}',\bar{v}')=(\bar{y}'_1,..,\bar{y}'_N,\bar{u}',\bar{v}')$ in K' for the following system of polynomial equations over K:

$$(*)\quad \bar{g}_j(X)=o,\ 1\leq j\leq r\ ;\quad U\bar{M}(X)-1=o,\quad V\bar{h}(X)-1=o$$

where $\bar{g}_j,\bar{M},\bar{h}$ are the images of g_j,M,h in $K[X]$.

$K\subset K'$ being algebraically pure extension, ($*$) will have a solution $(\bar{y},\bar{u},\bar{v})=(\bar{y}_1,..,\bar{y}_N,\bar{u},\bar{v})$ in K. As A is henselian, the system:

$$g_j(X)=o,\ 1\leq j\leq r\ ;\ Vh(X)-1=o$$

will have a solution $(y,v)=(y_1,..,y_N,v)$ in A, by the implicit function theorem. So, we have $g(y)=o$ and $h(y)$ is an unit of A, therefore $f(y)=o$.

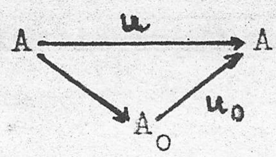
(2.3) <u>Corolary</u> Let $u:A\longrightarrow A'$ be a local unramified morphism of local rings with same residue field. Suppose A henselian. If u is a desingularization morphism, then u is algebraically pure.

(2.4) <u>Corolary</u> Let A be a local excellent and henselian ring. If general desingularization problem (1.13) has an affirmative answer then A is an AP-ring.

<u>Proof</u> The completion morphism $A\longrightarrow\hat{A}$ is regular, A being excellent [14]. Now, the statement follows directly from the previous corolary.

3. <u>Lifting smootheness.</u>

(3.1) <u>Lemma</u> Let $u:A\longrightarrow A'$ be a local formally smooth morphism of local noetherian rings. There exists a commutative diagram



with the following properties:

i) u_o is a local, flat, unramified morphism (i.e. the maximal ideal of A generates the maximal ideal of A') and $\dim A_o = \dim A'$.

ii) A_o is a localization of a polynomial A algebra – in a finite number of indeterminates– and has same residue field as A.

Proof Let $\underline{m}$, $\underline{m}'$ be the maximal ideals of A,A' respectively and $K = A/\underline{m}$ the residue field of A. The local K algebra $A'/\underline{m}A'$ is, by base change, formally smooth and consequently a regular local ring. Let $x_1, .., x_n$ be elements from $\underline{m}'$ which form, modulo $\underline{m}A'$, a regular system of parameters of $A'/\underline{m}A'$. Denote $R = A[X]$, $X = (X_1, .., X_n)$ and define $\sigma : R \longrightarrow A'$ by $\sigma(X_i) = x_i$, $1 \leq i \leq n$. Let $p = \sigma^{-1}(\underline{m}')$ and $A_o = R_p$. The A-morphism σ extends to a local morphism $u_o : A_o \longrightarrow A'$. Note that $p \cap A = \underline{m}$, therefore p coincides with the ideal generated by $\underline{m}$ and $X_1, .., X_n$ in R. If $\underline{m}_o = pR_p$ is the maximal ideal of A_o, then $\underline{m}' = \underline{m}_o A'$ and $A_o/\underline{m}_o = K$. Flatness of u_o is a consequence of the next lemma (3.2) and the equality $\dim A_o = \dim A'$ follows from the dimension formula .

(3.2) Lemma [14] Let $A \longrightarrow B \longrightarrow C$ be local morphisms of local noetherian rings and M a finite C module. Suppose B is A flat. Let k denote the residue field of A. Then, M is B flat iff M is A flat and $M \otimes_A k$ is $B \otimes_A k$ flat.

(3.3) Corolary In the above notations if the residue field of A' is a separable extension of the residue field of A then u_o is a formally smooth morphism. If A is an excellent ring, A_o is so, and u, u_o are regular morphisms.

Proof The first statement follows from [11, Th.19.7.1] and the second is a consequence of André-Radu's criterion of regularity: If $u : A \longrightarrow A'$ is a formally smooth morphism of local noetherian rings and A is excellent , then u is regular [8,2,24] (also named localisation theorem of formal smootheness)

(3.4) <u>Observation</u> The above lemma seems to be well known. It appears in different formulations in [22, 6] .

(3.5) Now, suppose that $u:A \longrightarrow A'$ is a local formally smooth morphism of local artinian rings. Using standard construction of Cohen algebras we can easy prove the existence of a filtered inductive system $\{A_i, \varphi_{ji}\}_{i \in I}$ of local artinian, essentially of finite type smooth A algebras with the following properties:

i) The transition morphisms $\varphi_{ji}:A_i \longrightarrow A_j$, $i \leq j$ are local , flat and unramified.

ii) $A' = \varinjlim A_i$

By (3.2) the canonical A-morphisms $A_i \longrightarrow A'$ are also local, flat and unramified.

(3.6) Let $u:A \longrightarrow A'$ be a local formally smooth morphism of local noetherian rings of same dimension. Then, u is unramified. Indeed, if $\underline{m}$ is the maximal ideal of A, then $A'/_{\underline{m}A'}$ is a local artinian ring. On the other hand , $A'/_{\underline{m}A'}$ being a local formally smooth algebra over a field is a regular ring and therefore a domain. Consequently, $A'/_{\underline{m}A'}$ is a field and so, $\underline{m}$ generates the maximal ideal of A'.

Let $\underline{r}$ be a proper ideal of A. Denote $\widetilde{A}=A/_{\underline{r}A}$, $\widetilde{A}'=A'/_{\underline{r}A'}$, and let $\widetilde{u}:\widetilde{A} \longrightarrow \widetilde{A}'$ be the local, formally smooth morphism induced by u. Clearly dim $\widetilde{A}$= dim $\widetilde{A}'$; therefore if $\underline{r}$ is an ideal of definition of A the local rings $\widetilde{A}$ and $\widetilde{A}'$ are both artinian.

(3.7) <u>Theorem</u> In the above notations and hypotheses consider a local, essentially of finite type, smooth $\widetilde{A}$ algebra $\widetilde{D}$ having same dimension as $\widetilde{A}$ and a local flat $\widetilde{A}$-morphism $\widetilde{v}:\widetilde{D} \longrightarrow \widetilde{A}'$ (also $\widetilde{v}$ is unramified). If A' is henselian, then there exist a local, essentially of finite type, smooth A algebra D and a local flat and unramified A-morphism $v:D \longrightarrow A'$ such that $\widetilde{D} = D \otimes_A \widetilde{A}$, $\widetilde{v}=v \otimes_A \widetilde{A}$.

<u>Proof</u> Write $\widetilde{D}=\widetilde{E}_{\widetilde{p}}$, where $\widetilde{E}$ is a finite type $\widetilde{A}$ algebra and $\widetilde{p} \in$ Spec $\widetilde{E}$. As $\widetilde{v}$ is local, the composed A-morphism $\widetilde{v}':\widetilde{E} \longrightarrow \widetilde{D} \longrightarrow \widetilde{A}'$ is such that $\widetilde{v}'^{-1}(\underline{m}\widetilde{A}')=\widetilde{p}$. Next suppose that $\widetilde{E}=\widetilde{A}[X]/_{\widetilde{J}}$, where $X=(X_1,\ldots,X_m)$ are

variables and $\tilde{J} \subset \tilde{A}[X]$ is an ideal. Then, $\tilde{p} = \tilde{P}/\tilde{J}$ with $\tilde{P} \in \operatorname{Spec} \tilde{A}[X]$ and therefore $\tilde{D} = \tilde{A}[X]_{\tilde{P}} / \tilde{J}.\tilde{A}[X]_{\tilde{P}}$. But $\tilde{A}[X] = A/_{\underline{r}}[X] = A[X]/_{\underline{r}.A[X]}$ and consequently the ideals $\tilde{J} \subset \tilde{P}$ have the form: $\tilde{J} = J/_{\underline{r}.A[X]}$, $\tilde{P} = P/_{\underline{r}.A[X]}$ with $\underline{r}.A[X] \subset J \subset P$ ideals from $A[X]$, P being prime. So $\tilde{D} = A[X]_P / J.A[X]_P$

Let $\tilde{\sigma} : A[X] \longrightarrow \tilde{A}'$ be the composed A-morphism:

$$A[X] \longrightarrow \tilde{A}[X] \longrightarrow \tilde{E} \xrightarrow{\tilde{v}'} \tilde{A}'$$

Clearly, $P = \tilde{\sigma}^{-1}(\underline{m}.\tilde{A}')$. Put $\tilde{x} = \tilde{\sigma}(X)$, $\tilde{x} = (\tilde{x}_1, .., \tilde{x}_m)$.

By hypothesis, $\tilde{E}$ is a smooth $\tilde{A}$ algebra at $\tilde{p}$: via jacobian criterion of smootheness, there exist a system of polynomials $\tilde{g} = (\tilde{g}_1, .., \tilde{g}_s)$ from $\tilde{J}$, $s = \operatorname{ht} \tilde{J}$ and a s-minor $\tilde{M}$ of the jacobian matrix $\left[\frac{\partial \tilde{g}}{\partial X}\right]$ with:

1) $\tilde{M} \notin \tilde{P}$

2) $\tilde{g}.\tilde{A}[X]_{\tilde{P}} = \tilde{J}.\tilde{A}[X]_{\tilde{P}}$

Let $g = (g_1, .., g_s)$ be some liftings in J of the polynomials $\tilde{g}_1, .., \tilde{g}_s$ and M the s-minor of the jacobian matrix $\left[\frac{\partial g}{\partial X}\right]$ corresponding to $\tilde{M}$. Then:

1') $M \notin P$

2') $g.A[X]_P + \underline{r}.A[X]_P = J.A[X]_P$

Let x be a lifting of $\tilde{x}$ in A'. As $\tilde{g}(\tilde{x}) = o$, we have $g(x) \equiv o \mod \underline{r}.A'$. But the couple $(A', \underline{r}.A')$ is henselian too, and therefore exists $x^{(o)} \in A'^m$ such that $g(x^{(o)}) = o$ and $x \equiv x^{(o)} \mod \underline{r}.A'$

Define $\sigma : A[X] \longrightarrow A'$ by $\sigma(X) = x^{(o)}$. If $h \in A[X]$ and $h \notin P$, then $f(x^{(o)})$ is invertible in A', because $x \equiv x^{(o)} \mod \underline{m}.A'$. Denote $D = A[X]_P / g.A[X]_P$ and let $v : D \longrightarrow A'$ be the local A-morphism induced by σ. By construction D is a local, essentially of finite type smooth A algebra and $D \otimes_A \tilde{A} = A[X]_P / g.A[X]_P + \underline{r}.A[X]_P = A[X]_P / J.A[X]_P = \tilde{D}$.

Moreover the structure morphism $A \longrightarrow D$ is unramified because $D/_{\underline{m}.D} \simeq \tilde{D}/_{\underline{m}.\tilde{D}} \simeq$ residue field of the local $\tilde{A}$ algebra $\tilde{D}$. Flatness of v is a consequence of (3.2).

(3.8) <u>Observation</u> We will use theorem (3.7) in the particular case when $\underline{r}$ is an ideal of definition for A; then, the local rings $\tilde{A}$, $\tilde{A}'$ are both artinian. By (3.5), the hypothesis of (3.7) holds effectively (except A' henselian, indeed).

It should be mentioned that similar lifting results where given by D. Popescu in [21,22] from which we have been inspired.

(3.9) <u>Corolary</u> Let $u:A \longrightarrow A'$ be a local, formally smooth inclusion of local noetherian rings of same dimension, A being henselian. Then, for every finite system $x'=(x_1',..,x_n')$ of elements from A' and $\underline{r}$ an ideal of definition of A there exist a local, essentially of finite smooth sub A algebra D of A' such that the inclusion $D \subset A'$ is a local, unramified and flat morphism, and a system $x=(x_1,..,x_n)$ from D such that $x \equiv x' \bmod \underline{r}.A'$.

(3.10) <u>Corolary</u> The theorem (3.7) and its corolary (3.9) still work if we assume that $u:A \longrightarrow A'$ is a local, formally smooth morphism of local noetherian rings such that the residue field of A' is a separable extension of the residue field of A' .

<u>Proof</u> Use (3.1) to obtain a decomposition $u:A \longrightarrow A_o \xrightarrow{u_o} A'$ in which A_o is a local, essentially of finite type smooth A algebra and u_o is a local, formally smooth morphism of local noetherian rings of same dimension. Then apply (3.7) or (3.9) to u_o .

Generally speaking, the local A algebra D constructed in (3.7) is not unique, and therefore a filtered inductive system of local, essentially of finite type smooth $\tilde{A}$ algebras, whose limit is $\tilde{A}'$ can not be lifted to a similar system of local A algebras. However, the following partial result holds:

(3.11) <u>Theorem</u> Let $u:A \longrightarrow A'$ be a local, formally smooth morphism of local noetherian rings containing Q (the rational number field), A' being henselian. There exists the factorization:

$$u:A \xrightarrow{v} A_o \xrightarrow{u_o} A'$$

in which A_o is limit of a filtered inductive system of local essentially of finite type smooth A algebras, whose transition morphisms are local, flat and unramified (particularly, v is a desingularization morphism) and u_o is a local, formally smooth and unramified morphism of local henselian rings with same residue field.

Proof The construction of A_o, v and u_o will be made in several steps, using the stability of desingularization morphisms under composition.

First, by (3.1) we can suppose that u is unramified. Let $\underline{m}$ be, as usual, the maximal ideal of A. Let $\{x'_i\}_{i\in I}$ be a family of elements of A' which form, modulo $\underline{m}'=\underline{m}.A'$ a transcendence basis of the residue field $k'={}^{A'}/_{\underline{m}'}$ over $k={}^{A}/_{\underline{m}}$. Denote $R=A[X_i]_{i\in I}$, $p=\underline{m}.R$ and put $A_o=R_p$. Let $\sigma:R\longrightarrow A'$ the A-morphism defined by $\sigma(X_i)=x'_i$ By choice of x_i, σ induces a local A-morphism $u_o:A_o\longrightarrow A'$. If $\underline{m}_o=p.R_p$ is the maximal ideal of A_o then, clearly we have $\underline{m}_o=\underline{m}.A_o$ and $\underline{m}_o.A'=\underline{m}'$. The residue field k_o of A_o is $k(X_i)_{i\in I}$ and therefore the extension $k_o\subset k'$ is algebraic and moreover separable, both fields having characteristic zero. We will show that the canonical morphism $A\xrightarrow{v}A_o$ is a desingularization one and A_o is a noetherian ring. For every finite subset $J\subset I$ we put $R_J=A[X_i]_{i\in J}$, $p_J=\underline{m}R_J$, $A_J=(R_J)_{p_J}$. A_J is a local noetherian ring. Let $\underline{m}_J=\underline{m}.A_J$ be its maximal ideal. If $J\subset J'$ are finite subsets of I, we have a canonical local morphism $A_J\longrightarrow A_{J'}$, which is flat and induces on residue fields a purely transcedental extension. Moreover, $\underline{m}_J.A_{J'}=\underline{m}_{J'}$. Thus, $\{A_J\}$ is a filtered inductive family of local noetherian rings, whose transition morphisms are local, flat and unramified. Clearly $A_o=\varinjlim A_J$ and by [11] A_o is noetherian. u_o is flat by (3.2) and therefore formally smooth by [11]. We relabel $u_o:A_o\longrightarrow A'$ as $u:A\longrightarrow A'$ and consequently, in the theorem (3.11) we can suppose the residue field extension $k\subset k'$ algebraic and separable.

Now, let A_o be the henselization of A. A and A_o have the same residue field and the henselization morphism $v:A\longrightarrow A_o$ is a filtered inductive limit of finite type and etale morphisms [23], i.e. a desingularization morphism. A' being henselian, there exists an unique local morphism $u_o:A_o\longrightarrow A'$ such that $u_o\circ v=u$. Clearly, v and u_o are unramified and u_o is flat by (3.2) and formally smooth by [11] .

In this way, relabeling $u_o:A_o \longrightarrow A'$ as $u:A \longrightarrow A'$ we can add at the hypotheses of (3.11) the condition: A is henselian.

We need the following auxiliary result:

(3.12) Proposition [23] Let A be a local henselian ring with maximal ideal $\underline{m}$ and residue field k and let K be a finite separable extension of k.

i) There exists a local finite and etale A algebra A_K with residue field K, unique up to an A-isomorphism.

ii) Let $u:A \longrightarrow A'$ be a local morphism of local rings, A' being henselian. Let k,k' be the residue fields of A,A' respectively and $K \subset k'$ a finite separable k-extension. There exists an unique local A-morphism $A_K \longrightarrow A'$.

We go back to the proof of (3.11). For every finite k-extension $K \subset k'$ we fix a local, finite and etale A algebra A_K and a local A-morphism $A_K \longrightarrow A'$. If $K_1 \subset K_2 \subset k'$ are finite extensions of k, there exists an unique local A-morphism $A_{K_1} \longrightarrow A_{K_2}$ which lifts the inclusion $K_1 \subset K_2$. For this, we apply (3.12),ii),the local rings A_K being henselian as finite algebras over henselian ring A. It is easy to see that the transition morphism $A_{K_1} \longrightarrow A_{K_2}$ is flat and unramified. Consequently,the family of local rings $\{A_K\}$ is inductive because the family of finite k-extensions $K \subset k'$ is too.We put $A_o = \varinjlim A_K$; A_o is a local, noetherian and henselian ring [23] and,by construction, the canonical morphism $v:A \longrightarrow A_o$ is a desingularization morphism.
The local morphisms $A_K \longrightarrow A'$ give at inductive limit a local unramified A-morphism $u_o:A_o \longrightarrow A'$, which induces an isomorphism on residue fields. u_o is flat by (3.2) and moreover formally smooth by [11] .
The proof of (3.11) is thus finished.

(3,13) Corolary In the notations of (3.11) if A is a reduced ring (domain, normal or regular ring) , A_o is too.

Proof The corolary follows from [10, 5.13].

(3.14) Corolary If A is an excellent ring, A_o is too and the morphisms u and u_o are regular.

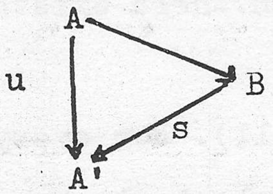
Proof It follows from [13] and from the localisation theorem of formal smootheness [8, 2] .

4. Proof of the theorem (o.1)

4.1 Preliminary observations i) By (3.1) we can suppose that A, A' have the same dimension one. Then u is unramified by (3.5).

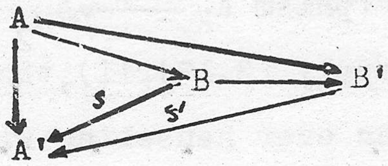
ii) Let t be a parameter of A (i.e. $t \in A$ is an element such that $A/_{t.A}$ is artinian). Then t is a parameter of A'. Recall that any power of t is still a common parameter of A and A'.

iii) Consider a commutative diagram:



(1)

in which B is a finite type A algebra. We must embed this into another of the form:



(2)

in which B' is a finite type smooth A algebra.

In (1) we can suppose s injective, i.e. B is a sub A algebra of A'. Also in (2) it is sufficient to ask that B' is smooth over A at the prime ideal $s'^{-1}(\underline{m}A')$.

(4.2) So, let $B = A[Y]/_p$ where $Y=(Y_1,\ldots,Y_N)$ are variables and $p \subset A[Y]$ is a prime ideal, the isomorphism being given by $Y \rightsquigarrow y'=(y'_1,\ldots,y'_N) \in A'^N$. The fraction field extension $Q(A) \subset Q(B)$ is separable and then, by jacobian criterion of smootheness, there exists a system of polynomials $f=(f_1,\ldots,f_r)$, $r=ht(p)$, from p and a r-minor M of the jacobian matrix $J=\left[\frac{\partial f}{\partial Y}\right]$ such that $M \notin p$. Consequently, $M(y') \neq o$. There are two possibilities:

(4.3) $\underline{M(y') \text{ is invertible in } A'}$. This does not necessarily imply that B is smooth over A at $s^{-1}(\underline{m}A') = B \cap \underline{m}A'$ because the images of $f_1,\ldots f_r$ into $A[Y]_{\sigma^{-1}(\underline{m}A')}$ could not generate the ideal $p.A[Y]_{\sigma^{-1}(\underline{m}A')}$

where $\sigma : A[Y] \longrightarrow A'$ is defined by $Y \rightsquigarrow y'$. However $A[Y]_p$ is a regular local ring of dimension r, because $A[Y]_p \cong Q(A)[Y]_p$ ($p \cap A$ being o). As $f_1, .., f_r$ form a regular system of parameters of $A[Y]_p$ we have $p.A[Y]_p = f.A[Y]_p$. Consequently, $(f):p \not\subset p$ and therefore exists $F \in (f):p$ such that $F(y') \neq o$. If $F(y')$ is an unit of A', then B is really smooth over A at $B \cap \underline{m}A'$. If not, the ideal $F(y').A'$ having height one, is $\underline{m}A'$-primary and so $\sqrt{F(y').A'} = \underline{m}A' = \sqrt{t.A'}$, t being a common fixed parameter of A and A'. Thus, there exists $e = t^{\beta} \in F(y')A'$ i.e. $e = F(y')a'$ with $a' \in A'$. By virtue of (3.9) we can suppose that A contains the elements $y = (y_1, .., y_N)$ and a such that:

$$y \equiv y' \bmod e^2.A' \quad \text{and} \quad a' \equiv a \bmod e^2.A'$$

Therefore:

$$y' = e^2.w' + y \quad \text{with} \quad w' = (w_1', .., w_N') \in A'^N$$

Taylor's expansion gives:

$$o = f(y') = f(y) + e^2\left[\frac{\partial f}{\partial Y}\right](y).w' + e^4.R(w')$$

where $R(W) = (R_1(W), .., R_r(W))$ is a system of polynomials from $A[W] = A[W_1, .., W_N]$ having no terms of degree < 2.

Thus, $f(y) \in e^2.A' \cap A = e^2.A$, u being faithfully flat and then:

$$f_i(y) = e^2.b_i \quad 1 \leq i \leq r \quad \text{with } b_i \in A .$$

So, we have:

$$b_i + \sum_{j=1}^{N} w_j' \frac{\partial f_i}{\partial Y_j}(y) + e^2.R_i(w') = o \quad . \quad 1 \leq i \leq r$$

Consider the following system of polynomials from $A[W]$:

$$h_i(W) = b_i + \sum_{j=1}^{N} W_j \frac{\partial f_i}{\partial Y_j}(y) + e^2.R_i(W) \qquad 1 \leq i \leq r$$

Clearly, w' is a solution for h and we have $f(e^2.W + y) = e^2.h(W)$. Because $\left[\frac{\partial h}{\partial W}\right](w) \equiv \left[\frac{\partial f}{\partial Y}\right](y) \bmod e^2.A'$, the matrix $\left[\frac{\partial h}{\partial W}\right](w')$ contains an invertible r-minor.

Denote $B'' = {A[W]}/{h.A[W]}$ and let $\tau : B'' \longrightarrow A'$ be the A-morphism given by $W \rightsquigarrow w'$. Clearly, B'' is smooth over A at $\tau^{-1}(\underline{m}A')$.

Next, we have $e \equiv F(y).a \bmod e^2A'$; we can suppose that $e = F(y).a$. So:

$$F(e^2.W + y) = F(y) + e^2 \sum_{j=1}^{N} W_j \frac{\partial F}{\partial Y_j}(y) + e^4.F_1(W)$$

where $F_1(W) \in A[W]$ has no terms of degree < 2.

We get:

$$a.F(e^2.W+y) = e\left[1 + e.a\sum_{j=1}^{N} W_j\frac{\partial F}{\partial Y_j}(y) + e^2.a.F_1(W)\right] = e.G(W)$$

where $G(w') \equiv 1 \mod e.A'$. Consequently, $b = G(w') \in B''$ is an invertible element of A'.

Let $P \in p$; from $F.P \in f.A[Y]$ we get:

$$a.F(e^2.W+y).P(e^2.W+y) \in f(e^2.W+y).A[W]$$

or:

$$e.G(W).P(e^2.W+y) \in e^2.h(W).A[W]$$

and finally: $G(W).P(e^2.W+y) \in h.A[W]$.

Consequently, the morphism $A[Y] \longrightarrow A[W]$ given by the substitution $Y=e^2.W+y$ induces an A-morphism $\phi : B \longrightarrow B' = B''_b$ and the proof of (o.1) is finished in this case.

(4.4) <u>$M(y')$ is not invertible in A'</u> We reduce this case to the previous one in the following way.

The ideal $M(y').A'$ has height one because $M(y') \neq o$ and A' is one dimensional; so, $M(y').A'$ is $\underline{m}A'$-primary. Therefore: $\sqrt{M(y').A'} = \underline{m}A' = \sqrt{t.A'}$,t being the common parameter for A, A' fixed in the previous case. Consequently, $t^{\alpha} \in M(y').A'$ for a suitable integer α . We put $d=t^{\alpha}$. Complete $J(y') = \left[\frac{\partial f}{\partial Y}\right](y')$ by the matrix $L' = [l'_{ij}]$, $l'_{ij} \in A'$, $r < i \leqslant N$, $1 \leqslant j \leqslant N$ such that the resulting N-matrix H' has the determinant d. By (3.9) we can suppose that A contains $y=(y_1,..,y_N)$ and $L=[l_{ij}]$, $r < i \leqslant N$, $1 \leqslant j \leqslant N$ such that

$$y' \equiv y \mod d^3.A' \quad \text{and} \quad L' \equiv L \mod d^3.A'$$

Let H be the N-matrix obtained from J(y) by bordering with L.

We have: $d=\det(H)' \equiv \det(H) \mod d^3.A'$; then $\det(H)$ and d are associated in divisibility in A' and so, in A by faithfully flatness of u.

There exists the N-matrix K over A such that:

$$H.K = K.H = d.I_N$$

where I_N is the identity N-matrix. Let:

$$y' = d^2.x+y$$

with $x=(x_1,..,x_N) \in (d.A')^N \subset (\underline{m}A')^N$

Put: $z'=H.x \in (\underline{m}A')^N$. Then: $K.z' = K.H.x = d.x$ and so:

$$y'=y+d.K.z' \qquad (3)$$

Taylor's expansion gives us:

$$o=f(y')=f(y)+d.J(y).K.z'+d^2.Q(z')$$

where $Q(Z)=(Q_1(Z),\dots,Q_r(Z))$ are polynomials from $A[Z]=A[Z_1,\dots,Z_N]$ having no terms of degree < 2.

We have: $J(y).K = [d.I_r, 0]$,where I_r is the identity r-matrix and 0 is the $r\times(N-r)$ zero matrix. So:

$$f_i(y)+d^2.(z_i'+Q_i(z'))=o \qquad 1\leq i\leq r$$

Consequently, $f_i(y)\in A\cap d^2.A' = d^2.A$ by faithfully flatness of u and then: $f_i(y)=d^2.a_i$ with $a_i\in A$ $1\leq i\leq r$.

Finally we have:

$$a_i+z_i'+Q_i(z')=o \qquad 1\leq i\leq r \qquad (4)$$

Put $B'=A[z']$ .From (3) we get: $B\subset B'\subset B_d$. Thus, $Frac(B) = Frac(B')$.

Let $\sigma: A[Z]\longrightarrow A'$ the A-morphism defined by $Z\rightsquigarrow z'$ and $q=Ker(\sigma)$.

We have $B'=A[Z]/_q$. Therefore:

$$ht(q)=N-trdeg_{Frac(A)} Frac(B')=N-trdeg_{Frac(A)} Frac(B)=ht(p)=r.$$

Consider the system of polynomials $g=(g_1,\dots,g_r)$ from $A[Z]$:

$$g_i(Z)=a_i+Z_i+Q_i(Z) \qquad 1\leq i\leq r$$

By (4) $g_i\in q$. Clearly, the jacobian matrix $\left[\frac{\partial g}{\partial Z}\right]$ has a r-minor whose image through σ is invertible in A'. In this way, we have reduced the proof to the first case.

5. A note on a Theorem of M. Artin and J. Denef.

In a recent work, M. Artin and J. Denef have obtained the following desingularization result:

Theorem 5.1 [6] Let $A\longrightarrow \bar{A}$ be a regular map of excellent local rings with same residue field k. Assume that $\bar{A}$ is normal, henselian, and of dimension two. Then $\bar{A}$ is a filtered inductive limit of smooth A algebras.

The proof uses a new extension of classical Neron's p-desingularization. When char(k)=o the condition " A and $\bar{A}$ have same residue field" can be omitted as shows the following:

Theorem 5.2 Let $A \xrightarrow{u} \bar{A}$ be a local formally smooth morphism of local noetherian rings, whose residue fields contain Q, the rational number field. Assume A excellent and $\bar{A}$ normal, henselian and of dimension two. Then, u is a desingularization morphism.

Proof. By (3.11) we have a commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{u} & \bar{A} \\ & \searrow \quad \nearrow_{u_o} & \\ & A_o & \end{array}$$

in which v is a desingularization morphism and u_o is a local, formally smooth and unramified morphism of local henselian rings with same residue field. By (3.14) A_o is still an excellent ring, so we can apply (5.1). Thus, u_o is a desingularization morphism and by (1.6) u is too.

REFERENCES

1. André,M.: Homologié des algèbres commutatives, Springer- Verlag, Berlin, 1974.

2. Andre,M.: Localisation de la lissité formelle, Manuscripta Math. 13(1974) , 297-307.

3. Artin,M.: Algebraic approximation of structures over complete local rings, Publ.Math.IHES 36(1969) 23-58.

4. Artin,M.: Construction techniques for algebraic spaces, Actes Congrès Intern.Math.,tome 1 (1970), 419-423.

5. Artin,M.: Algebraic structure of power series rings (to appear)

6. Artin,M.,Denef,J.: Smoothing of a ring morphism along a section (to appear).

7. Basarab,S.,Nica,V.,Popescu,D.: Approximation properties and existential completeness for ring morphisms,Manuscripta Math. 33(1981), 227-282.

8. Brezuleanu,A.,Radu,N.: Sur la localisation de la lissité formelle, CR Acad.Sc.Paris,Vol.276,1973,439-441.

9. Brown,M.L.: Artin's approximation property,Thesis,Cambridge, Peterhouse,1979.

10. Cipu,M.,Popescu,D.: Some extensions of Néron's p-desingularization and approximation,Rev.Roum.Math.Pures et Appl.,t.XXIV, 10(1981),1299-1304.

11. Grothendieck,A.,Dieudonné,J.:Eléments de Géometrie Algébrique,IV, Publ.Math.IHES, 20(1964).

12. Lazard,D.: Autour de la platitude,Bull.Soc.Math.France,97(1969), 81-128.

13. Marot,J.: Limites inductives d'anneaux universellement japonais (resp. excellents),CR Acad.Sc.Paris,Série A,t.285(1977), 425-429.

4. Matsumura,H.: Commutative Algebra, Benjamin,New-York,198o.

5. Néron,A. : Modèles minimaux des variétés abeliénnes sur les corps locaux et globaux, Publ.Math.IHES 21(1964).

6. Nica,V.,Popescu,D.: Approximation properties of the formally smooth morphisms,INCREST Preprint 46/1979.

7. Pfister,G.,Popescu,D.: On three dimensional local rings with the property of approximation, Rev.Roum.Math.Pures et Appl., t.XXVI 2(1981), 3o1-3o7.

8. Popescu,D.: Algebraically pure morphisms, Rev.Roum.Math.Pures et Appl.,t.XXIV, 6(1979), 947-977.

9. Popescu,D.: A remark on two dimensional local rings with the property of approximation, Math.Z. 173(198o),235-24o.

o. Popescu,D.: Global form of Neron's p-desingularization and approximation, Proceedings "Week of Algebraic Geometry", Bucharest,June 3o-July 6,198o,Teubner Texte,Band 4o, Leipzig,1981.

1. Popescu,D.: Wild desingularization and Artin approximation, INCREST Preprint 53/1981.

2. Popescu,D.: Higher dimensional Néron desingularization and approximation,INCREST Preprint 5o/1982.

3. Raynaud,M.: Anneaux locaux henseliens, Lecture Notes in Math., 169,Springer-Verlag,Berlin,197o.

4. Seydi,H.: Sur la théorie des anneaux excellents en caractéristique p,II,J.Math.Kyoto Univ.,2o-1(198o),155-167.

5. Kurke,H.,Mostowski,T.,Pfister,G.,Popescu,D.,Roczen,M.; Die Approximationseigenschaft localer Ringe,Lecture Notes in Math. 634, Springer-Verlag, Berlin, 1978.

Ghirlandei 38, Bl.Dl, ap.27,77592
Bucharest, Romania.