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ISSN 0250 3638

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Alexandru DIMCA

PREPRINT SERIES IN MATHEMATICS

No. 48/1983

BUCURESTI

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Alexandru DIMCA*)

August 1983

*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd.Pacii 220, 79622, Bucharest, Romania

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Alexandru Dimca

Let A denote the $\mathbb{C}$-algebra of germs of analytic functions defined at the origin of $\mathbb{C}^n$ and m be the maximal ideal of A.

We can think of a germ of an analytic space X at the origin of $\mathbb{C}^n$ defined by an ideal $I_X=(g_1,\dots,g_p)\subset m^2$ as the fiber of the corresponding map germ $g:(\mathbb{C}^n,0)\longrightarrow(\mathbb{C}^p,0)$. It is well-known [8] that, in the case $n\geq p$, the map germ g is finitely $\mathcal{K}$-determined if and only if X is an <u>isolated singularity of a complete intersection</u> (This will be abreviated in the sequel to ISCI).

In other words, the ISCI X is determined by the artinian $\mathbb{C}$-algebra $Q_k(X)=A/I_X+m^{k+1}$ where k is the order of $\mathcal{K}$-determinacy of the map germ g.

The purpose of this paper is to suggest that a different artinian $\mathbb{C}$-algebra, more geometrically associated to X, can play a similar role. More precisely, if X is an ISCI defined by an ideal I_X as above, then one can consider the <u>singular subspace</u> of X, which is the analytic space germ SX defined by the ideal $SI_X\subset m$ generated by the g_i and all the pxp minors in the Jacobian matrix $\left(\frac{\partial g_i}{\partial x_j}\right)$ $i=1,\dots,p$; $j=1,\dots,n$. [7]

We hope the following to be true.

<u>Conjecture</u>. Two ISCI X and X′ are isomorphic if and only if their singular subspaces SX and SX′ are isomorphic.

Here "isomorphic" means isomorphic as germs of analytic spaces. Notation: $X\sim X'$. Moreover, we shall compare throughout this paper only germs X and X′ having the same dimension and embedding dimension. In fact, when X is not a hypersurface, the embedding dimension of X is the same as that of SX.

In the hypersurface case our Conjecture is just a reformulation of the main result of Mather and Yau [6].

We prove this Conjecture for 0-dimensional ISCI (Prop.3.4) for homogeneous ISCI (Prop.5.4) and also for those ISCI whose $\mathcal{K}$-determinacy order is not too high (Prop.4.2).

This last case contains in particular most of the 1-dimensional ISCI (Cor.4.4). The proofs in section 4 are independent of the rest of the paper.

The first two cases are based on the following <u>relative</u> version of the conjecture.

Let (Y,X) be a pair of germs of analytic spaces i.e. their defining ideals satisfy $I_Y \subset I_X \subset m^2$. We say that two such pairs (Y,X) and (Y′,X′) are isomorphic if there exists a $\mathbb{C}$-algebra isomorphism $u: A \to A$ such that $u(I_Y)=I_{Y'}$ and $u(I_X)=I_{X'}$. Notation: $(Y,X)\sim(Y',X')$.

Our main result is the following.

<u>Theorem.</u> Suppose that Y,X,X′ are ISCI at the origin of $\mathbb{C}^n$ such that dim X=dim X′=dim Y-1 and $I_X=I_Y+(f)$ and $I_{X'}=I_Y+(f')$ for some $f,f'\in m$.

Then $(Y,X)\sim(Y,X')$ if and only if $(Y,SX)\sim(Y,SX')$.

In the first section we reduce this Theorem to a problem in some jet space using the finite $\mathcal{K}$-determinacy of functions defined on an ISCI, an obvious extension of our work in [1].

This problem is solved in the next section using a slightly extended version of a wellknown result of Mather on Lie groups acting on manifolds [5, Lemma 3.1].

Finally let us mention just one nice consequence of a positive answer to the Conjecture.

Suppose $X \subset Y \xrightarrow{F} B$ is a semi-universal deformation of the ISCI $X=F^{-1}(0)$. Let C denote the critical subspace of F, Δ the discriminant subspace of F and G the restriction $F/C: C \to \Delta$. Then G is a normalization map [7] and a simple computation shows that

$G^{-1}(0) \sim SX$.

In this way, the discriminant Δ determines the singular subspace SX. Thus whenever the Conjecture is true we get a much more direct and illuminating proof of the result of Wirthmüller [9] which says that the discriminant Δ determines the singularity X,

§1. Contact equivalence of function germs

Let X be an ISCI defined by an ideal $I_X=(g_1,\ldots,g_p)\subset m^2$. We denote by A_X the local ring A/I_X and by $m_X\subset A_X$ the maximal ideal.

Definition 1.1

Two functions $f,f'\in m_X$ are called $\mathcal{K}$-equivalent if $(X,X_0)\sim\sim(X,X_0')$, where the analytic space germ X_0 (resp. X_0') is defined by the ideal $I_X+(f)$ (resp. $I_X+(f')$).

Notation: $f\sim f'$. Here and in the sequel we identify a function $f\in m_X$ with some of its representatives in A.

In order to study this equivalence relation it is useful to introduce the language of group actions, jet-spaces, finite determinacy and so on from standard singularity theory (compare to the first section in [1]).

Let L be the group of germs of analytic isomorphisms $h:(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ and T the group of invertible $(p+1)\times(p+1)$ matrices $M=(m_{ij})$ over A such that the elements of the last column satisfy $m_{i,p+1}=0$ for $i=1,2,\ldots,p$.

We define a group $G=L\times T$ and an action of G on the space of map germs $B=m.A^{p+1}$ by the formula

$$(h,M).F=M(F\circ h)$$

where we think of an element $F\in B$ as a column vector with entries in m.

If X,f and f' are as in (1.1) then it is obvious that

$f\sim f'$ if and only if the column vectors $F=(g_1,\ldots,g_p,f)$ and $F'=(g_1,\ldots,g_p,f')$ are in the same G-orbit.

In analogy with the $\mathcal{K}$-tangent space of a map germ, we define the following G-tangent space for a map germ $F\in B$ associated to X and f as above

$$TGF=m.J(F)+I_X.e_1+\ldots+I_X.e_p+I_{X_0}.e_{p+1}\subset B$$

where J(F) is the A-submodule in B generated by the $\frac{\partial F}{\partial x_i}$, e_j are the elements of the standard basis for $\mathbb{C}^{p+1}$ and the ideal I_{X_0} is precisely $I_X+(f)$.

In this situation we also define

$$\text{codim } f = \dim B/TGF.$$

By passing to k-jets, the action of G on B induces actions $G^k \times J^k(n,p+1)\longrightarrow J^k(n,p+1)$ and we have the following relation between tangent spaces

$$T_{j^kF}(G^k.j^kF)=j^k(TGF).$$

Definition 1.2.

The pair (X,f) is called k-determined if the corresponding map germ F is k-G-determined (i.e. if for any other similar map germ F' with $j^kF=j^kF'$ one has $F'\in G.F$).
The pair (X,f) is called finitely determined if it is k-determined for some k.

We have the following result.

Proposition 1.3.

Let X be an ISCI with $\dim X\geq 1$ and $f\in m_X\setminus\{0\}$. Then the following are equivalent:

i. The pair (X,f) is finitely determined

ii. codim $f < \infty$

iii. The ideal $I_{X_o} = I_X + (f)$ defines an ISCI X_o.

Proof:

It is clear that i. $\Leftrightarrow$ ii. (see for instance Theorem 1.2 in [8]). If iii. does not hold then there exists a curve germ $c: (\mathbb{C},0) \to (\mathbb{C}^n,0)$ such that:

a. $f \circ c = 0$, $g_i \circ c = 0$ for $i=1,\dots,p$ where $(g_1,\dots,g_p) = I_X$.

b. $c(t) \neq 0$ for $t \neq 0$.

c. $\frac{\partial f}{\partial x_j} \circ c = \sum_{i=1}^{p} \lambda_i \cdot \frac{\partial g_i}{\partial x_j} \circ c$ for some Laurent series $\lambda_i(t)$ and any $j=1,\dots,n$. It follows that any element $a_1 e_1 + \dots + a_{p+1} e_{p+1}$ in TGF has the property $a_{p+1} \circ c = \sum_{i=1}^{p} \lambda_i (a_i \circ c)$. In particular $a_1 = \dots = a_p = 0$ implies $a_{p+1} \circ c = 0$ and this contradicts ii.

To prove iii. $\Rightarrow$ ii. we note first that it is enough to show that $TGF \supset m^N . e_{p+1}$ for some N. This follows from the fact that $\mathcal{K}$- codim $g < \infty$.

Next, for each subset $I = \{i_1 < \dots < i_{p+1}\} \subset \{1,\dots,n\}$ let M_I be the following $(p+1) \times (p+1)$ matrix

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_{i_1}} & \cdots\cdots\cdots & \frac{\partial g_1}{\partial x_{i_{p+1}}} \\ & & \\ \frac{\partial g_p}{\partial x_{i_1}} & & \frac{\partial g_p}{\partial x_{p+1}} \\ \frac{\partial f}{\partial x_{i_1}} & & \frac{\partial f}{\partial x_{p+1}} \end{pmatrix}$$

Let us denote by $a_{I,k}$ the complement of the element $\frac{\partial f}{\partial x_{i_k}}$ in the above matrix. Then we have

$$\sum_k a_{I,k} \cdot \frac{\partial g_j}{\partial x_{i_k}} = 0 \text{ for any } j=1,\dots,p$$

$$\sum_k a_{I,k} \cdot \frac{\partial f}{\partial x_{i_k}} = \det(M_I)$$

These relations show that $TGF \supset (SI_{X_o}).e_{p+1}$ and by iii. the ideal SI_{X_o} is m-primary. □

Remark 1.4.

It is perhaps interesting to note that (using the notations of the Theorem in the introduction) $X \sim X'$ does not imply $(Y,X) \sim (Y,X')$ even in very simple cases.

Take for example the plane curve singularity $Y: x^5+y^2=0$ and $f=xy$, $f'=xy+x^4$. Then it is trivial to see that $X \sim X'$. If one would have $f \sim f'$, then $f \sim xy+tx^4$ for any $t \in \mathbb{C}$ and this gives $x^4.e_2 \in TGF$, where $F=(x^5+y^2, xy)$. Explicit computations with TGF show that this is not the case.

§2. Proof of the Theorem

First we state a more general version of Mather's Lemma 3.1 in [5].

Let $\alpha: G\times U \to U$ be a C^∞-action of a Lie group G on a C^∞-manifold U. For each $u \in U$ let $\alpha_u: G \to U$ be defined by $\alpha_u(g)=\alpha(g,u)$. We denote by T_xX the tangent space to a manifold X at the point x. Then we have the following result.

Lemma 2.1.

With the above notations, a sufficient condition for a connected C^∞-submanifold V of U to be contained in a single orbit of α is the existence of a vector subspace $E \subset T_1G$ such that

i. $T_v \supset T_vV$ for any $v \in V$, where $T_v = T\alpha_v(E)$.

ii. $\dim T_v$ is independent of $v \in V$.

Proof.

Mather's Lemma 3,1 corresponds to the case $E=T_1G$.

The proof given in [5] applies also to our case just by replacing everywhere T_1G by E. □

Now we come to the proof of our Theorem. The implication $(Y,X)\sim(Y,X') \Rightarrow (Y,SX)\sim(Y,SX')$ is obvious. To prove the converse we can assume that $SI_X=SI_{X'}$. Indeed, $h^*(SI_X)=S(h^* I_X)$ for any $h\in L$.

Suppose that I_Y is generated by $g_1,\dots,g_p\in m^2$ and let $F=(g_1,\dots,g_p,f)$ and $F'=(g_1,\dots,g_p,f')$ be the map germ constructed as in the first section.

Using Prop.1.3 it is enough to show that j^kF and j^kF' are G^k-equivalent for any k.

Note that $T_1G^k=T_1L^k \oplus T_1T^k$, where $T_1L^k=J^k(n,n)$ and T_1T^k is the vector space of $(p+1)\times(p+1)$ matrices $M=(m_{ij})$ over $J^k(n,1)$ such that $m_{i,p+1}=0$ for $i=1,\dots,p$ [4].

Let $E_1\subset T_1L^k$ be the vector subspace generated by all the elements $a_I\in J^k(n,n)$, where the r-component of a_I is $j^k(a_{I,q})$ if $r=i_q$ and 0 if $r\notin I$ in the notations from the proof of (1.3).

Let $E_2\subset T_1T^k$ be the vector subspace defined by the equations $m_{ij}=0$ if $i\neq p+1$.

If we take α to be the action $G^k\times J^k(n,p+1)\longrightarrow J^k(n,p+1)$ defined above, $v=j^kF$ and $E=E_1\oplus E_2\subset T_1G^k$ then an easy computation shows that

$$T_v=T\alpha_v(E)=j^k(SI_X).e_{p+1}$$

Consider now the line W in $J^k(n,p+1)$ which joins the points v and $v'=j^kF'$ (if $v=v'$ there is nothing to prove!).

An argument similar to that in ([6] §5) shows that there is a Zariski open dense subset $V\subset W$ such that

$$V=\{w\in W;\ T_w=T_v\}.$$

Since T_wV is spanned by $j^k(f-f')_w \cdot e_{p+1} \in T_v$, one sees that all the requirements of the Lemma 2.1 are fulfilled and hence V is contained in a single G^k-orbit.

In particular v and v' are G^k - equivalent. □

§3. Zero-dimensional ISCI

In this section we shall use the Theorem to prove the Conjecture in case of 0-dimensional ISCI.

Let X be an ISCI, dim X=0 and $I_X=(g_1,\dots,g_n)\subset m^2$. Then $SI_X=I_X+(j(g))$, where $j(g)=\det(\frac{\partial g_i}{\partial x_j})$ is the jacobian of the map germ g $(i,j=1,2,\dots,n;\ n\geqslant 2)$.

It is known by Grothendieck Duality Theory that $j(g)\notin I_X$ and $a.j(g)\in I_X$ for any $a\in m$.

More precisely, one has the following.

Lemma 3.1.

$a.j(g)\in m.I_X$ for any $a\in m$.

Proof.

Note first that if $g_1',\dots,g_n'$ is a different set of generators for I_X then: $j(g')=u.j(g) \bmod mI_X$ for some unit $u\in A$. Hence the statement of (3.1) does not depend on the choice of generators for I_X.

On the other hand, if $I\subset A$ is any ideal and p is a positive integer such that I can be generated by p elements, then we introduce the following.

Definition 3.2.

We say that a systems of generators $g_1,\dots,g_p$ of I is in normal form if the following holds:

$2 \leq k_1 = \text{ord } g_1 = \ldots = \text{ord } g_{i_1} < k_2 = \text{ord } g_{i_1+1} \ldots < k_s = \text{ord } g_{i_1+\ldots+i_{s-1}+1} =$

$= \ldots = \text{ord } g_p \leq \infty$ and for any $q=1,\ldots,s$ such that $k_q < \infty$ the jets $j^{k_q} g_{i_1+\ldots+i_{q-1}+1}, \ldots, j^{k_q} g_{i_1+\ldots+i_{q-1}+i_q}$ are linearly independent mod $j^{k_q}((g_1,\ldots,g_{i_1+\ldots+i_{q-1}}))$, with $i_0=0$ and $i_s=p-i_1-\ldots-i_{s-1}$.

It is easy to see that for any ideal I and positive number p as above there exists a system of generators of I in normal form. Moreover the pairs $(k_1,i_1),\ldots,(k_s,i_s)$ depend only on I and p and not on a specific choice of a system of generators of I in normal form. (To see this just note that two such systems of generators can be related by an invertible pxp matrix over A and consider the 0-jet of this matrix!).

Now we come back to the proof of Lemma (3.1). Assume that $g_1,\ldots,g_n$ is a system of generators in normal form for the ideal I_X and for $p=n$. Note that $\text{ord } j(g) \geq \sum(\text{ord } g_i - 1) \geq k_s$. Suppose there is a relation

$a.j(g) = \sum \lambda_i . g_i$ with $a \in m$ and $\lambda_i \in A$.

If one takes the k_1-jet of this relation, it follows that $\lambda_1,\ldots,\lambda_{i_1} \in m$. By taking next the k_2-jet, it follows that $\lambda_{i_1+1},\ldots,\lambda_{i_1+i_2} \in m$ and so on. □

Corollary 3.3.

With the above notations

(i) $m.SI_X = m.I_X$

(ii) SI_X is the direct sum of the ideal $m.I_X$ with the vector space $\langle g_1,\ldots,g_n, j(g)\rangle$.

Proof

The only thing still to be proved is that the above sum is direct i.e. that $\langle g_1,\ldots,g_n, j(g)\rangle \cap mI_X = 0$. This follows from the fact that $j(g) \notin I_X$ and that $g_1,\ldots,g_n$ is an A-regular sequence in A.

The aim of this section is to prove the following.

Proposition 3.4.

The Conjecture is true for 0-dimensional ISCI.

Proof.

As in the proof of the Theorem, one can assume that $SI_X=SI_{X'}$. Let $g_1,\dots,g_n$ (resp. $g'_1,\dots,g'_n$) be a system of generators for I_X (resp. $I_{X'}$).

It follows that

$g'_i=\sum_k a_{ik}g_k+\lambda_i.j(g)$, where $a_{ik}\in A$, $\lambda_i\in\mathbb{C}$ for $i,k=1,2,\dots,n$. Using linear combinations of these equations one can take $\lambda_i=0$ for $i=1,\dots,n-1$.

In the vector space $SI_X/m.I_X=SI_{X'}/m.I_{X'}$ the classes of the elements g_i and $j(g)$ form a basis and the classes of the elements g'_i span an n-dimensional subspace by (3.3).

It follows that

$$\operatorname{rank}\begin{pmatrix} & & & 0 \\ & B & & 0 \\ & & & 0 \\ a_{n1}(0) & \dots & a_{nn}(0) & \lambda_n \end{pmatrix} = n$$

where $B=(a_{ij}(0))$ $i=1,\dots,n-1$; $j=1,\dots,n$. Then necessarily $\operatorname{rk} B=n-1$ and there exist $1\leq j_1<j_2\dots<j_{n-1}\leq n$ such that the minor in B corresponding to the columns $j_1,\dots,j_{n-1}$ is non zero.

This gives us the equality of ideals

$$(g'_1,\dots,g'_{n-1})=(g_{j_1},\dots,g_{j_{n-1}}).$$

This ideal defines a germ of an analytic space Y and, in order to apply our Theorem, we have to show that the generators g'_i

can be taken such that Y is an ISCI.

Note that $mI_X = mI_{X'} \supset m^N$ for some positive integer N. This implies that we can add to our generators g'_i any homogeneous polynomials p_i of degree N and still have

$$I_{X'} = (g'_j + p_1, \ldots, g'_{n-1} + p_{n-1}, g'_n)$$
$$I_X = (g'_1 + p_1, \ldots, g'_{n-1} + p_{n-1}, g_j)$$

where $j \neq j_k$ for $k=1,\ldots,n-1$. If the polynomials p_i are chosen general enough, it follows that $g'_1 + p_1 = \ldots = g'_{n-1} + p_{n-1} = 0$ are the equ ations of an ISCI. □

§4. Relations with the order of $\mathcal{K}$- determinacy of an ISCI

The succes of the proof of the Conjecture in the 0-dimensional case was based on the fact that the ideal SI_X was not too big compared to the ideal I_X.

In this section we present two new instances of this phenomenon. The first one is the following.

<u>Proposition 4.1.</u>

Let X and X′ be two ISCI. Assume that the defining map germ $g: (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^p, 0)$ of X is k -$\mathcal{K}$- determined and that

$$SI_X \subset I_X + m^{k+1}, \; SI_{X'} \subset I_{X'} + m^{k+1}.$$

Then $SX \sim SX'$ implies $X \sim X'$.

<u>Proof:</u>

Since $SX \sim SX'$, there exists a $\mathbb{C}$-algebra isomorphism $u: R_X \simeq$ $\simeq R_{X'}$, where $R_X = A/SI_X$ and $R_{X'} = A/SI_{X'}$. Let r(resp.r′) be the maximal ideal in R_X (resp, $R_{X'}$). Then $u(r)=r'$ and also $u(r^s)=(r')^s$ for any s. On the other hand, one obviously has

$$R_X/r^{k+1} \simeq A/SI_X+m^{k+1} \simeq A/I_X+m^{k+1}$$

and similarly for X'. Hence one gets

$$A/I_X+m^{k+1} \simeq A/I_{X'}+m^{k+1}$$

Since g is k-$\mathcal{K}$-determined, this gives $X\sim X'$. □

To state the second result, we need some more notations.

Assume that the defining map germ g of the ISCI X is put in normal form as in (3.2) with p=n-dim X= the minimal number of generators for $I_X \geq 2$.

Thus we get the pairs $(k_1,i_1),\ldots,(k_s,i_s)$ of positive integers, depending only on X. We introduce the following two numbers

$\alpha(X)=i_1(k_1-1)+\ldots+i_s(k_s-1)$

$\beta(X)=\alpha(X)-k_s+1$.

The result we are looking for is the following.

Proposition 4.2.

Using the above notations, assume that

i. $(k_1,p)\neq(2,2)$.

ii. The order of $\mathcal{K}$-determinacy of $g\leq\alpha(X)+\beta(X)-2$. Then $SX\sim SX'$ implies $X\sim X'$, for any ISCI X'.

Proof:

Assume that $SI_X=SI_{X'}$ and take $g_1',\ldots,g_p'$ a system of generators for $I_{X'}$ in normal form. Hence we get some pairs $(k_1',i_1'),\ldots,(k_t',i_t')$. Using i. we get easily $(k_1',p)\neq(2,2)$. Let M denote the matrix

$$\left(\frac{\partial g_i}{\partial x_j}\right) i=1,\dots,p;\ j=1,\dots,n .$$

Note that $k_s < \alpha(X) \leqslant$ min.ord.(pxp)-minors in M and similarly for X'. Using this and computing the pairs (k,i) corresponding to the ideal $SI_X = SI_{X'}$ and to a number of generators equal with $p+\binom{n}{p}$, one obtains that s=t and $k_r = k'_r$, $i_r = i'_r$ for r=1,...,s.

Moreover we have an equality $g' = N.g + a$, where N is a pxp matrix over A and the components a_i $(i=1,\dots,p)$ of the column vector a belong to the ideal generated by the pxp minors of M in A.

A simple investigation based on the normal form of g and g' shows that N(0) is a matrix over $\mathbb{C}$ of the following type

$$\begin{bmatrix} N_1 & & * \\ & N_2 & \\ & & \ddots \\ 0 & & & N_s \end{bmatrix}$$

where the $i_r x i_r$ matrix N_r is nondegenerate for r=1,...,s. In particular, the matrix N is invertible and hence $g' \sim \bar{g}$, where $\bar{g} = g + \bar{a}$ with $\bar{a} = N^{-1}.a$.

To finish the proof we have to show that $\bar{g} \sim g$.

For this note that

1. $j^{\alpha-1}\bar{g} = j^{\alpha-1}g$, where $\alpha = \alpha(X)$
2. $\bar{a} \in m^{\beta}.J(g) \subset TK_g$, where $\beta = \beta(X)$.

We derive from these as in ([2], Lemma (1.3)) that $j^{\alpha+\beta-2}\bar{g} \sim$ $\sim j^{\alpha+\beta-2}g$, which is precisely what we need by ii. □

We end this section with two examples which throw some light on the question of how often is the condition ii. fulfilled.

We shall consider first the case of 1-dimensional ISCI i.e. $p=n-1$.

Let $H^d(n,n-1)$ be the kernel of the natural projection

$$J^d(n,n-1) \longrightarrow J^{d-1}(n,n-1).$$

Our aim is to prove the following.

Lemma 4.3.

For $(n,d)\neq(3,2)$ there is a Zariski open and dense subset $U\subset H^d(n,n-1)$ such that $j^d g\in U$ implies the map germ g satisfies the condition 4.2 ii.

Proof.

Take U to be the set of d-jets $j^d g\in H^d(n,n-1)$ such that

$$m^{N-d+1}J(g)+(m^{N-d}.I_g+m^{N+1})\theta(g)\supset m^N.\theta(g)$$

where $N=n(d-1)-1$, $I_g=(g_1,\dots,g_{n-1})$ and $\theta(g)$ is the free A-module A^{n-1}.

It is clear that this subset U is open and since

$$(X_1^d+X_2^d,\dots,X_{n-1}^d+X_n^d)\in U$$

it follows that U is nonempty and hence dense.

We get by ([2], Corollary (1.4)) that a map germ g with $j^d g\in U$ is $(N-1)$-$\mathcal{K}$-determined.

On the other hand in the case $I_X=(g_1,\dots,g_{n-1})$ it follows $\alpha(X)\geqslant(n-1)(d-1)$, $\beta(X)\geqslant(n-2)(d-1)$ and hence the condition 4.2. ii. is fulfilled. □

Corollary 4.4.

The Conjecture is true for 1-dimensional ISCI having "generic initial parts".

More precisely, for each $n \geq 3$ and $d \geq 2$ there is a Zariski open and dense subset $U \subset H^d(n,n-1)$ such that $j^d g \in U$ for a defining map germ g for X and $SX \sim SX'$ give $X \sim X'$ for any ISCI X'.

Proof:

When $(n,d) \neq (3,2)$ the proof follows from (4.2) and (4.3). In the case $(n,d)=(3,2)$, it is known that a generic pencil of conics is equivalent to $(X_1^2-X_2^2, X_1^2-X_3^2)$ and it is 2-$\mathcal{K}$-determined. □

The following example prevents us of becoming too optimistic.

Example 4.5.

A map germ $g:(\mathbb{C}^n,0) \longrightarrow (\mathbb{C}^2,0)$ whose components are homogeneous polynomials of degree 3 cannot satisfy the condition 4.2. ii. for $n > 3$.

Proof:

In this case $\alpha = 4$, $\beta = 2$ and hence if g satisfies the condition 4.2 ii., one would obtain $TKg \supset m^5 . \theta(g)$ where $\theta(g) = A^2$. Let us denote by $(TKg)_5$ the homogeneous part of degree 5 in TKg and by P_d the vector space of homogeneous polynomials of degree d in $X_1, \ldots, X_n$. It is obvious that

$$\dim (TKg)_5 \leq n . \dim P_3 + 4 \dim P_2$$

Using this and the equality $\dim P_d = \binom{n+d-1}{d}$ one gets $\dim (TKg)_5 < 2 \dim P_5$ for $n > 3$, a contradiction. □

§ 5. Homogeneous ISCI

In this section we prove the Conjecture for ISCI which are homogeneous in the following (usual) sense.

Using a fixed coordinate system at the origin in $\mathbb{C}^n$, we identify the $\mathbb{C}$-algebra A with the $\mathbb{C}$-algebra $\mathbb{C}\{x_1,\dots,x_n\}$ of convergent power series. An ideal $I\subset A$ is homogeneous if it is generated by homogeneous polynomials $P_1,\dots,P_p$ of degree

$$2\leq d_1\leq\dots\leq d_p.$$

We assume moreover as in (3.2) that $P_{k+1}\notin(P_1,\dots,P_k)$ for $k=1,\dots,p-1$. The multidegree $d(I)=(d_1,\dots,d_p)$ is called the type of the ideal. We let I^q denote the ideal generated by all the polynomials $P\in I$ with $\deg P\leq q$.

Definition 5.1.

Two homogeneous ideals $I,J\subset A$ are equivalent (resp. linearly equivalent) if there exists an analytic isomorphism $h\in L$ (resp. h linear with respect to the fixed coordinate system) such that $h^*(I)=J$.

It is easy to establish the following basic result.

Lemma 5.2.

Two homogeneous ideals $I,J\subset A$ are equivalent if and only if I^q and J^q are linearly equivalent for any q. □

Definition 5.3.

An ISCI X is called homogeneous if its ideal I_X is homogeneous in some coordinate system (which can and will be taken as our fixed coordinate system).

We shall assume in the sequel $n\geq p\geq 2$.

The main result of this section is the following.

Proposition 5.4.

Let X, X' be two homogeneous ISCI. Then $SX\sim SX'$ implies

$X \sim X'$.

The proof will follow from a sequence of Lemmas, only one of them (5.6) using our Theorem. A part from this, the difficul situation is when X is defined by a pencil of quadrics and then a careful investigation is needed.

In one special case (n=4) we describe an explicit way of getting the ISCI X from the artinian algebra $R_X=A/SI_X$.

Lemma 5.5

The type of the ideal SI_X determines the type of the ideal I_X.

Proof:

Assume $d(I_X)=(d_1,\dots,d_p)$. Then $d(SI_X)=(d_1,\dots,d_p,\alpha,\dots,\alpha)$ where $\alpha=(d_1-1)+\dots+(d_p-1)\geq d_p$ and equality holds if and only if $(p,d_1)=(2,2)$. In this last case $d(SI_X)$ is just $(2,d_2,\dots,d_2)$. $\square$

In particular, using (5.2) we find out that I_X and $I_{X'}$ have the same type $(d_1,\dots,d_p)$.

When $\alpha > d_p$ we have $I_X=(SI_X)^{\alpha-1}$ and similarly for X'. Using again (5.2) this ends the proof of Prop. (5.4) in this case.

Lemma 5.6.

$SX \sim SX'$ implies $X \sim X'$ when $p=d_1=2<d_2$.

Proof:

Assume that $SI_X=SI_{X'}$ is a homogeneous ideal of type $(2,d_2,\dots,d_2)$. There exist homogeneous polynomials g_1, g_2 and g_2' of degree respectively 2, d_2 and d_2 such that $I_X=(g_1,g_2)$ and $I_{X'}=(g_1,g_2')$.

There are then two possibilities.

Case 1. If g_1 is a nondegenerate quadratic form, we can apply directly the Theorem.

Case 2. If g_1 is degenerate, then corank $g_1=1$ and we can take $g_1=X_1^2+\ldots+X_{n-1}^2$. There exists a positive integer N such that $m.SI_X \supset m^N$. We define two ISCI Y and Y' by the ideals

$$I_Y=(g_1+x_n^N, g_2),\ I_{Y'}=(g_1+x_n^N, g_2')$$

Then by Nakayama's Lemma we get $SI_Y=SI_X$ and $SI_{Y'}=SI_{X'}$. Using our Theorem we obtain $Y\sim Y'$. But since any ISCI is finitely $\mathcal{K}$-determined, it follows that for N big enough one has $X\sim Y$ and $X'\sim Y'$. □

Lemma 5.7.

$SX\sim SX'$ implies $X\sim X'$ when $p=d_1=d_2=2$.

Proof:

We can take the defining map germs of X and X' in the following normal form ([3], ChapXIII):

$$g=(X_1^2+\ldots+X_n^2, \lambda_1 X_1^2+\ldots+\lambda_n . X_n^2)$$
$$g'=(X_1^2+\ldots+X_n^2, \mu_1 X_1^2+\ldots+\mu_n . X_n^2)$$

with $\lambda_i, \mu_j \in \mathbb{C}$ and $\lambda_i \neq \lambda_j, \mu_i \neq \mu_j$ for $i \neq j$.

There are several cases to discuss.

Case 1. n=2,3

For these values of n there is a single class of nondegenerate pencils of quadratics and hence there is nothing to prove.

Case 2. n=4

The algebra $R_X=A/SI_X$ has a natural grading

$R_o=\mathbb{C}$, $R_1=\mathbb{C}\langle x_1,\ldots,x_n\rangle$, $R_2=\mathbb{C}\langle x_i^2\rangle/\langle g_1,g_2\rangle$. In the equation of g_2 we can take $\lambda_1=0$, $\lambda_2=1$, $\lambda_3=-1$, $\lambda_4=\lambda$. Consider an element $a\in R_1$ such that $a^2=0$.

If $a=a_1x_1+\ldots+a_4x_4$, then this condition is equivalent to

$$\begin{cases} q_1=a_1^2(1-\lambda)+\lambda a_2^2-a_4^2=0 \\ q_2=-a_1^2(1+\lambda)+\lambda a_3^2+a_4^2=0 \end{cases}$$

Hence the set of elements a as above is a complete intersection in $\mathbb{C}^4=R_1$ determined by the pencil (q_1,q_2). We show that the pencil (q_1,q_2) thus obtained is equivalent to the pencil (g_1,g_2).

For this, note the following obvious facts:

$\alpha g_1+\beta g_2$ is degenerate $\Leftrightarrow (\alpha:\beta)\in\{(0:1),(1:1),(-1;1),(-\lambda;1)\}\subset P^1$.

$\gamma q_1+\delta q_2$ is degenerate $\Leftrightarrow (\gamma;\delta)\in\{(0:1),(1:0),(1:1),(1+\lambda:1-\lambda)\}\subset P^1$.

The projective transformation $\alpha=-\gamma+\delta$, $\beta=\gamma+\delta$ sends one set of 4 points in P^1 onto the other and hence the pencils (g_1,g_2) and (q_1,q_2) are equivalent ([3]).

Case 3. $n\geq 5$

Take any $h\in Gl(n)$ such that $h^*(SI_{X'})=SI_X$ and assume that h is given by a matrix $(a_{ij})=M$. Then we have

$h^*(x_ix_j)=\sum_k a_{ik}a_{jk}.x_k^2$ +something in $\langle x_s.x_t,\ s\neq t\rangle$. Let us denote by g_{ij} the initial sum above and note that $g_{ij}\in\langle g_1,g_2\rangle$, the vector space spanned by g_1,g_2.

If each column in M has precisely one nonzero element, then $h^*x_i=a_i\ x_{s(i)}$ for some $a_i\in\mathbb{C}^*$ and some permutation $s\in S_n$. In this case it is obvious that $h^*(I_{X'})=I_X$.

Assume now there is a column (say the k-th one) having at least 2 nonzero elements (say a_{ik} and a_{jk}). Then $g_{ij}\neq 0$ and since $g_{ij}\in\langle g_1,g_2\rangle$ one has $rkg_{ij}\geq n-1$. Hence on the row i there are at least (n-1) nonzero elements. Consider the linear map

$$L:\mathbb{C}^n \longrightarrow \mathbb{C}^n,\ L(u)=(u_1 a_{i1},\dots,u_n a_{in}).$$

Now $\mathrm{rk}L \geq n-1$ and hence $\dim L(V) \geq n-2 \geq 3$, where V is the vector subspace in $\mathbb{C}^n$ spanned by the rows of M different from the i-th one. Since $L(V) \subset \langle (1,\dots,1),\ (\lambda_1,\dots,\lambda_n) \rangle$ by our hypothesis, this is a contradiction. □

References

1. Dimca,A.: Function germs defined on isolated hypersurface singularities, to appear in Compositio Math.

2. Dimca, A. and Gibson, C.G.: Classification of equidimensional contact unimodular map germs, to appear in Math.Scand.

3. Hodge, W. and Pedoe, D.: Methods of algebraic geometry, Cambridge Univ.Press, 1947.

4. Martinet, J.: Singularités de fonctions et applications differentiables, Monografias de Matematica da P.U.C(1977).

5. Mather, J.N.: Stability of C^{∞} mappings: IV: Classification of stable germs by $\mathbb{R}$-algebras. Publ.Math.IHES 37, 223-248 (1970).

6. Mather, J.N. and Yau,S.S.T.: Classification of isolated hypersurface singularities by their moduli algebras, Invent.math. 69,243-251, (1982).

7. Teissier, B.: The hunting of invariants in the geometry of the discriminant. Real and Complex singularities, P.Holm, ed., Sijthoff and Noordhoff, Alphen aan den Rijn, Netherlands, (1977).

8. Wall, C.T.C.: Finite determinacy of smooth map-germs, Bull.London Math.Soc.13, 481-539, (1981).

9. Wirthmüller, K.: Singularities determined by their discriminant, Math.Ann.252, 237-245 (1980).