A NOTE ON ALGEBRAIC SUPERPOSITIONS OF MEROMORPHIC FUNCTIONS

by

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September 1983

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1. INTRODUCTION

Understanding the structure and size of spaces of functions of n variables which are superpositions of various kinds of functions of less than n variables is a rich subject [4]. Here we are dealing with algebraic superpositions of meromorphic functions (see § 2 for the precise definition). As a typical example of such superpositions one may think for instance of functions ψ in 3 variables which may be represented in the form

$$\psi(z_1,z_2,z_3)=f(\varphi(z_1,z_2),\eta(z_3))$$

where φ, η are arbitrary meromorphic functions and f is an arbitrary holomorphic algebraic function (of course f,φ,η should be defined on suitable domains such that their composition makes sense). It will turn out that things are easier in the meromorphic case than in the case of r times continously differentiable functions [4] and final results on the space of superpositions are more precise. This is due to the fact that in the meromorphic case we dispose of a powerful tool which is differential algebra in the sense of Ritt and Kolchin. We hope this short note illustrates once more the fact that simple geometric ideas from differential algebra can be used to answer nontrivial questions in analysis.

We are indebted to A.G.Hovanski for introducing us to this subject.

2. STATEMENTS

Let $z_1, z_2, \ldots, z_n$ be coordinates on $\mathbb{C}^n$. For any domain $A \subset \mathbb{C}^n$ define:

$\mathcal{M}(A)$=field of meromorphic functions on A

$\mathcal{M}_i(A)$=subfield of $\mathcal{M}(A)$ of functions which do not depend on z_i.

$\mathcal{S}'(A)$=subfield of $\mathcal{M}(A)$ generated by $\mathcal{M}_1(A), \ldots, \mathcal{M}_n(A)$.

$\mathcal{S}''(A)$=algebraic closure of $\mathcal{S}'(A)$ in $\mathcal{M}(A)$.

$$\mathcal{S}(A) = \bigcup_{B \subset A} (\mathcal{M}(A) \cap \mathcal{S}''(B)),$$ B running through all subdomains of A.

It is reasonable to say that $\psi \in \mathcal{M}(A)$ is an algebraic superposition if and only if $\psi \in \mathcal{S}(A)$.

We may of course give the same definitions in the abstract setting, replacing $\mathcal{M}(A)$ by a universal field [3] of characteristic zero and putting $\mathcal{S} = \mathcal{S}''$. In fact all arguments we are going to use work in the abstract case too; we restricted to the analytic case only because of the function-theoretic interest of the problem.

The aim of this note is to prove the following:

THEOREM.- For any n there exist countably many nonzero differential ideals $\mathfrak{a}_k, \mathfrak{b}_k$ $(k \in \mathbb{N})$ in the ring of differential polynomials $\mathbb{Q}\{y\}$ such that for any domain $A \subset \mathbb{C}^n$ and any $\psi \in \mathcal{M}(A)$ the following are equivalent:

1) $\psi \in \mathcal{S}(A)$.

2) There exists an index k such that

$$\mathfrak{a}_k(\psi)=0 \quad \text{and} \quad \mathfrak{b}_k(\psi) \neq 0$$

COROLLARY.- Differentially transcendental meromorphic functions are not algebraic superpositions.

Recall that a function is called differentially transcendental if it does not satisfy any nonzero algebraic differential equation with rational coefficients. The existence of such meromorphic functions follows for instance from [7].

REMARK.- By the basis theorem [3,p.126] the ideals $\mathfrak{a}_k$ and $\mathfrak{b}_k$ in our theorem may be supposed finitely generated in the differential sense. It would be interesting to determine explicitely for any n a sequence of finite sets of generators for these ideals.

In what follows we prove the Theorem. We will use the standard notations from [3] and [1],[2].

3. COUNTABLY MANY CORRESPONDENCES

For any integer $m \geqslant 1$ we will construct a certain correspondence as follows. Let y, y_{ijk}, w_{ijk} be differential indeterminates with respect to derivations $\delta_1, \ldots, \delta_n$ where $i \in I = \{1, \ldots, n\}$, $j \in J = \{0, \ldots, m\}$ and $k \in K = \{1, \ldots, m\}$. Put

$$X = \mathrm{Spec}_D \mathbb{Q}\{y_{ijk}, w_{ijk};\ i \in I,\ j \in J,\ k \in K\}$$
$$Y = \mathrm{Spec}_D \mathbb{Q}\{y\}$$
$$Z = \mathrm{Spec}_D \mathbb{Q}\{y, y_{ijk}, w_{ijk};\ i \in I,\ j \in J,\ k \in K\} = X \times_{\mathbb{Q}} Y$$

Consider the following differential polynomials:

$$F_j = \sum_k y_{1jk} y_{2jk} \ldots y_{njk}, \quad G_j = \sum_k w_{1jk} w_{2jk} \ldots w_{njk}$$
$$M_j = G_0 G_1 \ldots G_m (F_j / G_j)$$
$$F = M_0 y^m + M_1 y^{m-1} + \ldots + M_m \quad \text{and} \quad S = \partial F / \partial y .$$

Let $C \subset X$ be the closed subset given by the ideal

$$\zeta = [\delta_i y_{ijk}, \delta_i w_{ijk};\ i \in I,\ j \in J,\ k \in K]$$

and $T, V \subset Z$ be the closed subsets given by $[F]$ and $[S]$

respectively. The set $p_Y(p_X^{-1}(C)\cap(T\setminus V))$ (where $p_X:Z\longrightarrow X$ and $p_Y:Z\longrightarrow Y$ are the natural projections) clearly depends on m and will be denoted by Y_m.

From § 4 it will follow that Y_m are constructible sets in the differential Zariski topology of Y and from § 5 we will get that no Y_m is dense in Y. So, modulo these two facts our Theorem will be proved if we prove the following:

CLAIM.- For any domain $A\subset\mathbb{C}^n$ we have

$$\mathcal{S}(A)=\bigcup_m\{u(y);\ u:\mathbb{Q}\{y\}\longrightarrow\mathcal{M}(A),\ \ker(u)\in Y_m\}$$

To prove the claim consider first a differential specialization u as in the right member of the above equality. By construction of Y_m there exists $P\in Z$ such that $P\cap\mathbb{Q}\{y\}=\ker(u)$, $P\supset[\Sigma, F]$, $P\not\ni S$. By Seidenberg's embedding theorem [7] there exists a subdomain $B\subset A$ and an embedding v such that the following diagram commutes:

$$\begin{array}{ccc} qf(\mathbb{Q}\{y\}/\ker(u)) & \longrightarrow & qf(R/P) \\ \downarrow & & \downarrow v \\ \mathcal{M}(A) & \longrightarrow & \mathcal{M}(B) \end{array}$$

where $R=\mathbb{Q}\{y, y_{ijk}, w_{ijk};\ i\in I,\ j\in J,\ k\in K\}$. This shows that $v(y)=\psi|_B$ satisfies a non-trivial algebraic equation with coefficients in $\mathcal{S}'(B)$ hence $\psi \in \mathcal{S}(A)$. Conversely, if $\psi \in \mathcal{M}(A) \cap \mathcal{S}''(B)$ for some B then there exists by separability a polynomial $g \in \mathcal{S}'(B)[y]$ such that $g(\psi)=0$ and $(dg/dy)(\psi) \neq 0$. Writing coefficients of g as quotients of sums of products of elements in $\mathcal{M}_1(B),\dots,\mathcal{M}_n(B)$ we see that a specialization $v:R \longrightarrow \mathcal{M}(B)$ is induced for some m, extending the specialization $y \mapsto \psi$. Clearly $\ker(v) \in p_X^{-1}(C) \cap (T \setminus V)$ so $y \mapsto \psi$ gives a $\mathcal{M}(A)$-point of Y_m and we are done.

4. CONSTRUCTIBILITY

In this § we prove the following:

PROPOSITION.- Let $f:U \longrightarrow W$ be a morphism of affine Ritt schemes which is differentially of finite type and suppose W has a Noetherian underlying topological space. Then f is constructible.

Terminology is from [1]. The case of a single derivation was proved in [1].

Proof. We may write $f=\mathrm{Spec}_D u$ where $u:A \longrightarrow B=A\{b_1,\dots,b_k\}$ is a morphism of differential algebras. By the basis theorem $\mathrm{Spec}_D B$ is also Noetherian so by a standard trick it is sufficient to prove that $f(U)$ is constructible in W. By induction on k we may assume $k=1$. Now exactly as in [1] we may reduce ourselves to proving that under the hypothesis that $A \subset B = A\{b\}$ are domains, $f(\mathrm{Spec}_D B)$ contains a non-empty open subset of $\mathrm{Spec}_D A$. This is precisely the theorem on extending specializations [3,p.140]. We won't apply however this theorem and give instead a more direct argument (as we did in [1]). On the other hand note that the theorem on extending specializations clearly is a consequence of our proposition so we will obtain in fact an alternative (and we think easier) proof of this theorem (at least in characteristic zero). There is an additional reason for giving here this alternative proof: our argument may also be applied to get informations on the differential dimension polynomials of the components of fibres of morphisms between Ritt manifolds. But to explain this application is beyond the scope of this note.

Coming back to the proof of the proposition we see that by the zero characteristic assumption we have $f(\mathrm{Spec}_D B) = f(\mathrm{Spec}\, B) \cap \mathrm{Spec}_D A$ and since $\mathrm{Spec}_D A$ is dense in $\mathrm{Spec}\, A$ it is sufficient to prove that $f(\mathrm{Spec}\, B)$ contains a non-empty Zariski open subset of $\mathrm{Spec}\, A$. But to have this it is sufficient [5, proof of 6.E] to prove the following:

LEMMA.- Suppose $A \subset B = A\{b\}$ are Ritt domains. Then there exists a subring C of B (which may not be a differential subring) containing A and there exists an element $s \in B$, $s \neq 0$ such that:

1) C is isomorphic to a ring of polynomials over A in possibly infinitely many variables and

2) $B[1/s]$ is finitely generated over C as a non-differential algebra.

Proof. Some notations first. Let Θ be the set of operators of the form $\theta = \delta_1^{a_1} \ldots \delta_n^{a_n}$. If $\eta = \delta_1^{b_1} \ldots \delta_n^{b_n}$ we write $\theta < \eta$ if either $|\theta| = a_1 + \ldots + a_n < |\eta| = b_1 + \ldots + b_n$ or there exists $t \in \{1, \ldots, n\}$ such that $a_i = b_i$ for $i < t$ and $a_t < b_t$. We write $\theta \leq \eta$ if either $\theta < \eta$ or $\theta = \eta$. We write $\theta \subseteq \eta$ if $a_i \leq b_i$ for all i.

For any $\theta \in \Theta$ put

$$B^{\theta} = A[\eta b;\ \eta < \theta]$$

and construct inductively (with respect to $\leqslant$) subsets Σ^{θ} of Θ in the following way: $\Sigma^0 = \phi$ and if η is the succesor of θ,

$$\Sigma^{\eta} = \begin{cases} \Sigma^{\theta}, & \text{if } \eta b \text{ is algebraic over } B^{\theta} \\ \Sigma^{\theta} \cup \{\eta\}, & \text{if } \eta b \text{ is transcendent over } B^{\theta}. \end{cases}$$

Put $\Sigma = \bigcup_{\theta} \Sigma^{\theta}$, $\Lambda = \Theta \setminus \Sigma$ and let Λ_{min} be the set of minimal elements of Λ with respect to the order "$\subseteq$". Clearly Λ_{min} is a finite set $\{\theta_1, \ldots, \theta_M\}$. Define

$$C = A[\theta b;\ \theta \in \Sigma]$$

By construction C is a polynomial algebra over A. Now for any $i \in \{1,\dots,M\}$ let F_i be a polynomial of minimum degree in $B^{\theta_i}[T] \setminus \{0\}$ such that $F_i(\theta_i b)=0$. By the characteristic assumption $dF_i/dT \neq 0$ and so $s_i = (dF_i/dT)(\theta_i b) \in B^{\theta_i}[\theta_i b] \setminus \{0\}$ hence $s = s_1 s_2 \dots s_M \in B \setminus \{0\}$. Now it is easy to checkthat

$$\theta b \in B^{\theta}[\theta_i b, 1/s_i] \quad \text{for } 1 \leq i \leq M \text{ and } \theta_i \subseteq \theta$$

This then easily implies that

$$B[1/s] = C[\theta_1 b, \dots, \theta_M b, 1/s_1, \dots, 1/s_M]$$

and we are done.

5. COUNTING CONSTANTS

We will show that (in notations of §3) no Y_m is dense in Y. Suppose on the contrary that there is a component of $p_X^{-1}(C) \cap (T \setminus V)$ which dominates Y. Let P be its generic point. Then $P \supset [\mathcal{C}, F]$, $P \not\supset S$ and we have an extension

$$\mathcal{F} = \mathbb{Q}\langle y \rangle \longrightarrow qf(R/P) = \mathbb{Q}\langle \bar{y}, \bar{y}_{ijk}, \bar{w}_{ijk}; i \in I, j \in J, k \in K \rangle = \mathcal{G}$$

where $\bar{x}$ denotes the class of $x \in R$ modulo P. We will get a contradiction if we prove that :

$$(\text{diff. type of } \mathcal{F} \text{ over } \mathbb{Q}) > (\text{diff. type of } \mathcal{G} \text{ over } \mathbb{Q})$$

(see [3,p.118]). Now it is well known [3] that $\mathbb{Q}\langle y\rangle$ has differential type n over $\mathbb{Q}$. Since $F \in P$ and $S \notin P$ it follows easily by induction that for any $\theta \in \Theta(s)$ (i.e. $\theta \in \Theta$, $|\theta| \leq s$) we have

$$\theta\bar{y} \in \mathbb{Q}(\bar{y}, \eta\bar{y}_{ijk}, \eta\bar{w}_{ijk};\ \eta \in \Theta(s),\ i \in I,\ j \in J,\ k \in K)$$

hence

$$\operatorname{tr.deg}_{\mathbb{Q}} \mathbb{Q}(\theta\bar{y}, \theta\bar{y}_{ijk}, \theta\bar{w}_{ijk};\ \theta \in \Theta(s), i \in I, j \in J, k \in K) \leq$$

$$\leq \sum_{i,j,k} \operatorname{tr.deg}_{\mathbb{Q}} \mathbb{Q}(\theta\bar{y}_{ijk};\ \theta \in \Theta(s)) +$$

$$+ \sum_{i,j,k} \operatorname{tr.deg}_{\mathbb{Q}} \mathbb{Q}(\theta\bar{w}_{ijk};\ \theta \in \Theta(s)) +$$

$$+ \operatorname{tr.deg}_{\mathbb{Q}} \mathbb{Q}(\bar{y}) \leq (2nm(m+1)/(n-1)!)s^{n-1} + \text{lower terms}$$

so $\mathcal{G}$ has differential type at most $n-1$ over $\mathbb{Q}$ and we are done.

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