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Abstract

In this paper the problem of stabilization of linear differential control systems with Markov perturbations and the linear-quadratic control problem for such systems are discussed.

1. Notations and Preliminaries

R^n is the real n-dimensional space. If A is a matrix (or a vector) A^* means the transpose. $H > 0$ ($H \geqslant 0$) means that H is positive (semi) definite matrix.

$\{\Omega, \mathcal{F}, \mathcal{P}\}$ is a given probability field. If x is a random variable by Ex we denote the mean value of x; $E[x|w(t)=i]$ means mean value of x conditional on the event $w(t)=i$.

Throughout this paper $w(t)$, $t \geqslant 0$ is a right continuous homogeneous Markov chain with state space the set $D=\{1,2,\dots,s\}$ and transition matrix $P(t)=[p_{ij}(t)]=e^{Qt}$. Here $Q=[q_{ij}]$ with $q_{ij} \geqslant 0$, $j \neq i$ and $\sum_{j=1}^{s} q_{ij}=0$.

Consider the system

$$\frac{dx(t)}{dt}=f(t,x(t),w(t)) \tag{1}$$

where f, x are vectors in R^n.

The solution x(.) of (1) is defined in an obvious way, joining solution arcs of (1) at jump points of w(t) (see [1]). The solution x(t) is a continuous process with probability 1. For the system (1) we define the operator U as follows:

$$(Uv)(t,x,i)=\frac{\partial v}{\partial t}(t,x,i)+f^{*}(t,x,i)\frac{\partial v}{\partial x}(t,x,i)+\sum_{j=1}^{s} v(t,x,j)q_{ij},$$

$$t\geq 0,\ x\in R^n,\ i\in D$$

where v(t,x,i) are real functions of class C^1 in (t,x) for each $i\in D$.

It is known [1]-[3] that the following relation holds:

$$E\left[v(t,x(t,t_o,x),w(t))\,|\,w(t_o)=i\right]-v(t_o,x,i)= \tag{2}$$

$$=E\left[\int_{t_o}^{t}(Uv)(u,x(u,t_o,x),w(u))du\,|\,w(t_o)=i\right],\quad t\geq t_o\geq 0,\ i\in D,\ x\in R^n$$

where $x(t,t_o,x)$, $t\geq t_o\geq 0$, $x\in R^n$ is the solution of (1) with $x(t_o,t_o,x)=x$.

2. Main results

Let us consider the control system

(3) $$\frac{dx(t)}{dt}=A(w(t))x(t)+B(w(t))u(t)$$

where A(i) are nxn matrices, B(i) are nxm-matrices, $i\in D$ and u(t) is the control vector.

Let $\mathcal{L}$ be the space of all $L=(L(1),\dots,L(s))$ where L(i) are mxn matrices.

If $L\in\mathcal{L}$ and $x\in R^n$ by $x_L(t,x)$ we denote the solution of the system (3) corresponding to $u(t)=L(w(t))x(t)$ and $x_L(0,x)=x$.

For $L\in\mathcal{L}$, $x\in R^n$, $i\in D$ and $0\leq T\leq\infty$ we define

(4) $$V_T(x,L,i)=E\left[\int_0^T x_L^*(t,x)\,(M(w(t))+L^*(w(t))N(w(t))L(w(t)))x_L(t,x)dt\,\middle|\,w(0)=i\right]$$

where $M(i)\geq 0$ and $N(i)>0$, $i\in D$ are given matrices.

<u>Definition 1</u>. $L\in\mathcal{L}$ <u>is admissible</u> (with respect to the control problem (3)-(4)) <u>if</u> $V_\infty(x,L,i)<\infty$ <u>for all</u> $x\in R^n$ and $i\in D$.

<u>Definition 2</u>. <u>The system (3) is stabilizable if there exists</u> $L\in\mathcal{L}$ <u>such that the trivial solution of (3) for</u> $u(t)=L(w(t))x(t)$ <u>is exponentially stable in mean square</u>.

<u>If the above property holds we shall say that</u> L <u>stabilizes the system</u> (3).

Now, let $\mathcal{H}$ be the space of all $H=(H(1),\dots,H(s))$ where H(i), $i\in D$ are symmetric nxn matrices.

If $H\in\mathcal{H}$ we say that H is positive definite ($H>0$) if $H(i)>0$ for all $i\in D$; $H\geq 0$ if $H(i)\geq 0$ for all $i\in D$. We shall say that the pair (A,H) is controllable if for every $i\in D$, (A(i),H(i)) is controllable.

We define the operators $F:\mathcal{L}\times\mathcal{H}\to\mathcal{H}$, $G:\mathcal{H}\to\mathcal{L}$ as follows:

$$F(L,H)(i)=(A(i)+B(i)L(i))^{*}H(i)+H(i)(A(i)+B(i)L(i))+$$
$$+\sum_{j=1}^{s}H(j)q_{ij}+M(i)+L^{*}(i)N(i)L(i),\qquad i\in D,$$

$$G(H)(i)=-N^{-1}(i)B^{*}(i)H(i),\qquad i\in D$$

Theorem 1

(i) If $L\in\mathcal{L}$ is admissible then $F(L,H)=0$ where H is defined by $x^{*}H(i)x=V_{\infty}(x,L,i)$, $x\in R^{n}$, $i\in D$.

(ii) If L is admissible and if $V_{\infty}(x,L,i)>0$ for all $x\neq 0$, $i\in D$ then L stabilizes the system (3) and the equation $F(L,K)=0$ has a unique solution in the class of positive semidefinite elements of $\mathcal{H}$.

(iii) If (A^{*},M) is controllable then every admissible system $L\in\mathcal{L}$ has the property $V_{\infty}(x,L,i)>0$ for all $x\neq 0$, $i\in D$.

(iv) If L and $K\geq 0$ verify $F(L,K)=0$ then L is admissible.

Theorem 1 is proved in [4] (see Lemmas 5 and 3).

From (ii) and (iii) of Theorem 1 it follows

Corollary 1. If (A^{*},M) is controllable then every admissible system $L\in\mathcal{L}$ stabilizes the system (3).

Let us consider the following Riccati system

$$\frac{dK_i(t)}{dt}=A^{*}(i)K_i(t)+K_i(t)A(i)+\sum_{j=1}^{s}K_j(t)q_{ij}-\tag{5}$$
$$-K_i(t)B(i)N^{-1}(i)B^{*}(i)K_i(t)+M(i),\quad K_i(0)=0,\quad i\in D$$

From the dynamic programming approach (see [1 p.192]) it follows that

(6) $0 \leq K_i(t_1) \leq K_i(t_2)$, $V_T(x,L,i) \geq x^* K_i(T)x$ for all $t_2 > t_1 \geq 0$

and all $L \in \mathcal{L}$, $x \in R^n$, $i \in D$.

Obviously, if $K(t)=(K_1(t),\ldots,K_s(t))$ defined by (5) is bounded then it is convergent (as $t \to \infty$) and its limit $K=(K(1),\ldots,K(s))$ verifies the Riccati system

(7) $A^*(i)K(i)+K(i)A(i)+\sum_j K(j)q_{ij}+M(i)-$

$-K(i)B(i)N^{-1}(i)B^*(i)K(i)=0$, $i \in D$

The system (7) can be written in the form

(8) $F(G(K),K)=0$

From Theorem 1 (Assertion (iv)) it follows directly that the following Corollary holds

Corollary 2. If $K \geq 0$ is a solution of (8) then $G(K)$ is admissible.

The next Proposition follows directly from the inequalities (6).

Proposition 1. If there exists an admissible system $L=(L(1),\ldots,L(s))$ then:

(i) $K(t)=(K_1(t),\ldots,K_s(t))$ defined by (5) is convergent (as $t\to\infty$).

(ii) <u>The Riccati equation</u> (8) <u>has a positive semidefinite solution.</u>

<u>Theorem 2.</u> <u>If</u> $K(t)=(K_1(t),\ldots,K_s(t))$ <u>defined by</u> (5) <u>is convergent</u> (as $t\to\infty$) <u>then</u> $G(\tilde{K})$ <u>is admissible and</u>

$$\min_{L\in\mathcal{L}} V_\infty(x,L,i)=V_\infty(x,G(\tilde{K}),i)=x^*\tilde{K}(i)x \text{ for all } x\in R^n,\ i\leq s$$

<u>where</u> $\tilde{K}(i)=\lim_{t\to\infty} K_i(t)$.

<u>Proof</u>

From Corollary 2 it follows that $G(\tilde{K})$ is admissible. Let $\tilde{L}=G(\tilde{K})$.

Since $F(\tilde{L},\tilde{K})=0$, using the relation (2) for the system (3) with $u(t)=\tilde{L}x(t)$ and for $v(t,x,i)=x^*\tilde{K}(i)x$ we get

$$E[\, x_{\tilde{L}}^*(T,x)\tilde{K}(w(T))x_{\tilde{L}}(T,x) \mid w(0)=i] -x^*\tilde{K}(i)x=-V_T(x,\tilde{L},i),\ T\geq 0,\ x\in R^n,\ i\in D$$

But by (6) $V_T(x,\tilde{L},i)\geq x^*K_i(T)x\geq 0.$

Hence

$$x^*K_i(T)x\leq V_T(x,\tilde{L},i)\leq x^*\tilde{K}(i)x$$

Thus

$$V_\infty(x,\tilde{L},i)=x^*\tilde{K}(i)x$$

Now, let $L \in \mathcal{L}$. By (6) we conclude that $V_\infty(x,L,i) \geq x^*\tilde{K}(i)x$. Thus, Theorem 2 is proved.

From Corollary 2, Theorem 2 and Theorem 1 (Assertion (ii)) it follows that the next corollary holds.

Corollary 3. If $K(t)=(K_1(t),\ldots,K_s(t))$ defined by (5) is convergent (as $t\to\infty$) and if its limit $\tilde{K}$ is positive definite then $G(\tilde{K})$ stabilizes the system (3).

The next result follows easily from Theorem 2 and Theorem 1 (Assertions (iii) and (ii)).

Proposition 2. If (A^*,M) is controllable and if $K(t)=$ $=(K_1(t),\ldots,K_s(t))$ defined by (5) is convergent (as $t\to\infty$) then its limit $\tilde{K}$ is positive definite and $G(\tilde{K})$ stabilizes the system (3).

Theorem 3. The following two assertions are equivalent:

(i) $K(t)=(K_1(t),\ldots,K_s(t))$ defined by (5) is convergent (as $t\to\infty$) and its limit $\tilde{K}$ is positive definite.

(ii) The equation (8) has a positive definite solution and this solution is unique in the class of positive semidefinite elements of $\mathcal{H}$.

Proof

Suppose that (i) holds. Hence $\tilde{K}>0$ is a solution of (8). Let $K_1\geq 0$; $K_2\geq 0$ be two solutions of (8). We shall prove that $K_1=K_2$.

We denote $L_1=G(K_1)$, $L_2=G(K_2)$. By Theorem 2, $V_\infty(x,L_1,i)\geq$ $\geq x^*\tilde{K}(i)x > 0$, $x\neq 0$, $i\in D$. Hence from Corollary 2 and Theorem 1

(Assertion (ii)) it follows that L_1 stabilizes the system (3). Similarly, L_2 stabilizes the system (3).

As in the proof of Theorem 2 we can prove that

$$E[x^*_{L_1}(T,x)K_1(w(T))x_{L_1}(T,x)\,|\,w(0)=i]-x^*K_1(i)x=-V_T(x,L_1,i)$$

Using the above equality, the relations $F(L_1,K_1)=0$, $F(L_2,K_2)=0$, and the relation (2) for the system (3) with $u(t)=L_1(w(t))x(t)$ and for $v(t,x,i)=x^*K_2(i)x$, by direct computations we get

$$E[x^*_{L_1}(T,x)K_1(w(T))x_{L_1}(T,x)\,|\,w(0)=i]-x^*K_1(i)x-$$

$$-(E[x^*_{L_1}(T,x)K_2(w(T))x_{L_1}(T,x)\,|\,w(0)=i]-x^*K_2(i)x)=$$

$$=-E[\int_0^T x^*_{L_1}(s,x)(K_1(w(s))-K_2(w(s)))B(w(s))N^{-1}(w(s))B^*(w(s))\cdot$$

$$\cdot(K_1(w(s))-K_2(w(s)))\cdot x_{L_1}(s,x)ds\,|\,w(0)=i]$$

Hence

$$x^*(K_2(i)-K_1(i))x\leq E[x^*_{L_1}(t,x)(K_2(w(t))-K_1(w(t)))x_{L_1}(t,x)\,|\,w(0)=i],\ t\geq 0$$

But L_1 stabilizes the system (3). Hence $\lim_{T\to\infty} E|x_{L_1}(T,x)|^2=0$. Therefore

$$x^*(K_2(i)-K_1(i))x\leq 0,\quad x\in R^n,\quad i\in D.$$

Thus $K_2\leq K_1$. Similarly, we can prove that $K_1\leq K_2$. Hence (i) $\Rightarrow$ (ii).

The assertion (ii) $\Rightarrow$ (i) follows directly from Corollary 2, and Proposition 1. Thus Theorem 3 is proved.

Theorem 4. The system (3) is stabilizable if and only if the following Riccati system

$$(9) \qquad A^*(i)S(i)+S(i)A(i)+\sum_{j=1}^{s} S(j)q_{ij}+I_n-S(i)B(i)B^*(i)S(i)=0, \quad i\in D$$

has a solution $S(i)\geq 0$, $i\in D$. (I_n is the identity matrix in R^n).

Proof

Using Proposition 1 corresponding to the case $M=I_n$, $N=I_m$ we can conclude that if the system (3) is stabilizable then the system (9) has a positive semidefinite solution.

Applying Corollary 3 and Corollary 1 in the case $M=I_n$, $N=I_m$ we obtain easily that if the system (9) has a solution $S(i)\geq 0$, $i\in D$ then the system (3) is stabilizable.

Theorem 5

Suppose that $K(t)=(K_1(t),\ldots,K_s(t))$ defined by (5) is convergent (as $t\to\infty$) and its limit $\tilde{K}$ is positive definite.

Let $L_o=(L_o(1),\ldots,L_o(s))$ be an admissible system. Then the relations

$$(10) \qquad F(L_p,K_p)=0, \quad L_{p+1}=G(K_p), \quad p\geq 0$$

define uniquely the sequences L_p and $K_p\geq 0$, $p\geq 0$.

L_p <u>and</u> K_p <u>defined by</u> (10) <u>have the following properties</u>:

(i) L_p stabilizes the system (3) for each $p \geq 0$

(ii) $K_p \geq K_{p+1} > 0$, $p \geq 0$.

(iii) $V_\infty(x, L_p, i) = x^* K_p(i) x$, $p \geq 0$. $x \in R^n$, $i \in D$.

(iv) $\lim_{p \to \infty} K_p(i) = \tilde{K}(i)$.

<u>Proof</u>

By (6) we have $V_\infty(x, L_o, i) \geq \tilde{K}(i) > 0$, $x \neq 0$, $i \in D$. Hence, from the assertion (ii) of Theorem 1 it follows that L_o stabilizes the system (3) and the relation $F(L_o, K_o) = 0$ defines uniquely the element $K_o \in \mathcal{H}$ with $K_o \geq 0$. Using again Theorem 1 (Assertion (i)) we conclude that $x^* K_o(i) x = V_\infty(x, L_o, i)$. Hence $K_o > 0$. Consider now $L_1 = G(K_o)$. We shall prove that L_1 stabilizes the system (3).

Indeed, it is easy to prove

$$\min_u \{ x^* M(i) x + u^* N(i) u + 2x^* A^*(i) K_o(i) x + 2u^* B^*(i) K_o(i) x \} =$$

$$= x^* F(L_1, K_o)(i) x - x^* \sum_{j=1}^{s} K_o(j) q_{ij} x \quad , \qquad x \in R^n, \; i \in D.$$

From the above equality, for $u = L_o(i) x$ we get

$$x^* F(L_o, K_o)(i) x - x^* \sum_{j=1}^{s} K_o(j) q_{ij} x \geq x^* F(L_1, K_o)(i) x - x^* \sum_{j=1}^{s} K_o(j) q_{ij} x$$

Hence

$$F(L_1, K_o) \leq 0$$

Using this inequality and the relation (2) for the system (3) with $u(t)=L_1(w(t))x(t)$ and for $v(t,x,i)=x^*K_o(i)x$ we obtain

(11) $$V_T(x,L_1,i)\leq x^*K_o(i)x$$

Then L_1 is admissible. Using the same reasoning as in the case of L_o we obtain that L_1 stabilizes the system (3), the relation $F(L_1,K_1)=0$ defines uniquely $K_1>0$, and $x^*K_1(i)x=V_\infty(x,L_1,i)$; hence, by (11) $K_1\leq K_o$. Repeat the above reasoning to conclude that (i)-(iii) hold. Now, let $\hat{K}=\lim_{p\to\infty} K_p$. From (10) we get $F(G(\hat{K}),\hat{K})=0$ By Theorem 3, we get $\hat{K}=\tilde{K}$. The theorem is proved.

The results in this paper extend Theorem 6.1 in [1].

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