ON POSITIVE HANKEL FORMS

by

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1 Preliminaries

In a previous paper ([2]) we were concerned with the structure of the positive Toeplitz forms. Although this subject can be viewed (via the classical connection between Carathéodory-Fejér problem and trigonometric moment problem) as a restatement of the contractive intertwining dilations theory ([1]), in ([2]) we approached it via an elementary and well-known result about the 2x2 positive matrices. More precisely, if A,B,C are bounded operators on a Hilbert space $\mathcal{H}$ and A is positive and invertible, then

$$T=\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geqslant 0$$

if and only if $C \geqslant B^*A^{-1}B$.

One continue as follows:

$$C \geqslant B^*A^{-1}B =(B^*A^{-\frac{1}{2}})(B^*A^{-\frac{1}{2}})^*$$

consequently, there exists a contraction

$$Z:\overline{\mathrm{Ran}C^{\frac{1}{2}}} \longrightarrow \overline{\mathrm{Ran}A^{\frac{1}{2}}}$$ (Ran denotes the range of a bounded operator) , so that

$$B=A^{\frac{1}{2}}ZC^{\frac{1}{2}} .$$

Moreover, the factorization

$$T=\begin{pmatrix} A^{\frac{1}{2}} , & 0 \\ C^{\frac{1}{2}}Z^* , & C^{\frac{1}{2}}D_Z \end{pmatrix}\begin{pmatrix} A^{\frac{1}{2}} , & ZC^{\frac{1}{2}} \\ 0 , & D_Z C^{\frac{1}{2}} \end{pmatrix} , \quad D_Z=(I-Z^*Z)^{\frac{1}{2}}$$

is obtained.

In this note we intend to use the same result for the positive Hankel forms. But a short analysis of the " dependence on parameters" shows that we cannot pursue the same line as in the Toeplitz case.

Here, instead, the situation is more simple: let us define

$$X=C-B^{*}A^{-1}B\ ,\ X\geq 0 \quad , \text{ and}$$

$$C=B^{*}A^{-1}B+X$$

and the factorization

$$T=\begin{pmatrix} A^{\frac{1}{2}} & , & 0 \\ B^{*}A^{-\frac{1}{2}} & , & X^{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} A^{\frac{1}{2}} & , & A^{-\frac{1}{2}}B \\ 0 & , & X^{\frac{1}{2}} \end{pmatrix}$$

is obtained. Consequently, the sequence of free parameters is not a sequence of contractions (the choice sequence in the contractive intertwining dilations theory), but a double sequence $\{X_n,Y_n\}_{n=1}^{\infty}$ with X_n selfadjoint operators and Y_n positive(invetible) operators.

We also mention the fact that the algorithm we describe here is hardly as explicit as the one in the Toeplitz case.

2 The algorithm

In this section we shall give the proof of the main result of the paper. A sequence of selfadjoint operators $\{S_n\}_{n=1}^{\infty}$, $S_n\in\mathcal{L}(\mathcal{H})$ for which the operators

$$H_n:\underbrace{\mathcal{H}\oplus\ldots\oplus\mathcal{H}}_{n+1}\longrightarrow\underbrace{\mathcal{H}\oplus\ldots\oplus\mathcal{H}}_{n+1}$$

$$H_n=\begin{pmatrix} S_0, S_1,\ldots, & S_n \\ S_1\ S_2,\ldots & S_{n+1} \\ \vdots & \\ S_n,\ldots & S_{2n} \end{pmatrix}$$

are positive for every $n\in\mathbb{N}$, will be called a positive Hankel form.

Moreover, in this note we suppose that H_n are invertible operators and $S_0=I$. We will call such a Hankel form, a strictly positive Hankel form.

Now, we can state and prove the result about strictly positive Hankel forms.

2.1 THEOREM There exists a one-to-one correspondence between the set of strictly positive Hankel forms and the set of double sequences $\{K_{2n-1}, Y_n\}_{n=1}^{\infty}$, $K_{2n-1} \in \mathcal{L}(\mathcal{K})$ and Y_n positive, invertible operators, given by the formulae:

$$S_1 = K_1 ,$$

$$S_{2n-1} = (S_{n-1}, \dots, S_{2n-3}) H_{n-2}^{-1} (S_n, \dots, S_{2n-2})^t + K_{2n-1} =$$
$$= (S_n, \dots, S_{2n-2}) H_{n-2}^{-1} (S_{n-1}, \dots, S_{2n-3})^t + K_{2n-1}^*$$

$$S_{2n} = (S_n, \dots, S_{2n-2}) H_{n-2}^{-1} (S_n, \dots, S_{2n-2})^t + K_{2n-1}^* Y_{n-1}^{-1} K_{2n-1} + Y_n$$

PROOF Let $\{S_n\}_{n=1}^{\infty}$ be a strictly positive Hankel form.

$H_1 \geqslant 0$ means $S_2 \geqslant S_1^2$, so, let us define

$$S_1 = K_1$$
$$S_2 = K_1^2 + Y_1 \quad , \; Y_1 \geqslant 0$$

We also obtain the factorization

$$H_1 = \begin{pmatrix} I & , 0 \\ K_1 & Y_1^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} I & K_1 \\ 0 & Y_1^{\frac{1}{2}} \end{pmatrix}$$

and we define $F_0 = I$,

$$F_1 = \begin{pmatrix} I & K_1 \\ 0 & Y_1^{\frac{1}{2}} \end{pmatrix}$$

and it results that H_1 is invetible if and only if F_1 is invertible if and only if Y_1 is invertible. We will find the operators $\{K_{2n-1}, Y_n\}_{n\geqslant 2}$ by induction, proving

$(1)_n$ $$S_{2n-1} = (S_{n-1}, \dots, S_{2n-3}) F_{n-2}^{-1} F_{n-2}^{*-1} (S_n, \dots, S_{2n-2})^t + K_{2n-1} =$$
$$= (S_n, \dots, S_{2n-2}) F_{n-2}^{-1} F_{n-2}^{*-1} (S_{n-1}, \dots, S_{2n-3})^t + K_{2n-1}^*$$

$(2)_n$ $$S_{2n} = (S_n, \dots, S_{2n-2}) F_{n-2}^{-1} F_{n-2}^{*-1} (S_n, \dots S_{2n-2})^t + K_{2n-1}^* Y_{n-1}^{-1} K_{2n-1} + Y_n$$

$(3)_n$ $$S_{2n} = (S_n, \dots, S_{2n-1}) H_{n-1}^{-1} (S_n, \dots S_{2n-1})^t + Y_n$$

$(4)_n$ $H_n = F_n^* F_n$ where

$$F_n = \begin{pmatrix} F_{n-1}, & F_{n-1}^{*-1} (S_n, \dots, S_{2n-1})^t \\ 0 , & Y_n^{\frac{1}{2}} \end{pmatrix} .$$

For the first step, we have:

$$\begin{pmatrix} S_2, S_3 \\ S_3, S_4 \end{pmatrix} \geq \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} (S_1, S_2) = \begin{pmatrix} S_1^2, & S_1 S_2 \\ S_2 S_1, & S_2^2 \end{pmatrix}$$

so
$$\begin{pmatrix} S_2, S_3 \\ S_3, S_4 \end{pmatrix} = \begin{pmatrix} S_1^2, & S_1 S_2 \\ S_2 S_1, & S_2^2 \end{pmatrix} + \begin{pmatrix} K_2, & K_3 \\ K_3^*, & K_4 \end{pmatrix}$$

where $\begin{pmatrix} K_2, & K_3 \\ K_3^*, & K_4 \end{pmatrix} \geq 0$; but, we obtain $K_2 = Y_1$, consequently,

$$S_3 = S_1 S_2 + K_3 = S_2 S_1 + K_3^*$$
$$S_4 = S_2^2 + K_4$$

where $K_4 = K_3^* Y_1^{-1} K_3 + Y_2$, $Y_2 \geq 0$, so,

$$S_4 = (K_1^2 + Y_1)^2 + K_3^* Y_1^{-1} K_3 + Y_2.$$

$$F_2 = \begin{pmatrix} F_1, & F_1^{*-1} (S_2, S_3)^t \\ 0, & Y_2^{\frac{1}{2}} \end{pmatrix}$$

then
$$F_2^* F_2 = \begin{pmatrix} F_1^* F_1, & \begin{pmatrix} S_2 \\ S_3 \end{pmatrix} \\ (S_2, S_3), & (S_2, S_3) F_1^{-1} F_1^{-1*} \begin{pmatrix} S_2 \\ S_3 \end{pmatrix} + Y_2 \end{pmatrix}$$

But,
$$(S_2, S_3) F_1^{-1} F_1^{-1*} \begin{pmatrix} S_2 \\ S_3 \end{pmatrix} + Y_2 =$$
$$= (S_2, S_3) \begin{pmatrix} I, & -K_1 Y_1^{-\frac{1}{2}} \\ 0, & Y_1^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} I, & 0 \\ -Y_1^{-\frac{1}{2}} K_1, & Y_1^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} S_2 \\ S_3 \end{pmatrix} + Y_2 =$$
$$= S_2^2 + K_3^* Y_1^{-1} K_3 + Y_2 = S_4$$

consequently,

$$F_2^* F_2 = H_2$$

and H_2 is invertible if and only if Y_2 is invertible, so , the first step is proved.

Suppose that we obtained $\{K_{2p-1}, Y_p\}_{p=1}^{n}$; we can continue as at the first step:

$$\begin{pmatrix} S_{2n}, & S_{2n+1} \\ S_{2n+1}, & S_{2n+2} \end{pmatrix} = \begin{pmatrix} S_n, \ldots, S_{2n-1} \\ S_{n+1}, \ldots, S_{2n} \end{pmatrix} H_{n-1}^{-1} \begin{pmatrix} S_n, & S_{n+1} \\ \vdots & \vdots \\ S_{2n-1}, & S_{2n} \end{pmatrix} + \begin{pmatrix} K_{2n}, & K_{2n+1} \\ K_{2n+1}^*, & K_{2n+2} \end{pmatrix}$$

where $\begin{pmatrix} K_{2n}, & K_{2n+1} \\ K_{2n+1}^*, & K_{2n+2} \end{pmatrix} \geq 0$ .From this relation we obtain:

$$S_{2n} = (S_n, \ldots, S_{2n-1}) H_{n-1}^{-1} (S_n, \ldots S_{2n-1})^t + K_{2n}$$
$$S_{2n+1} = (S_n, \ldots, S_{2n-1}) H_{n-1}^{-1} (S_{n+1}, \ldots, S_{2n})^t + K_{2n+1} =$$
$$= (S_{n+1}, \ldots, S_{2n}) H_{n-1}^{-1} (S_n, \ldots S_{2n-1})^t + K_{2n+1}^*$$

First of all, from $(3)_n$ it results $K_{2n}=Y_n$ and

$$K_{2n+2}=K^*_{2n+1}Y_n^{-1}K_{2n+1}+Y_{n+1},\quad Y_{n+1}\geq 0,$$

so, we obtain $(1)_{n+1}$ and $(2)_{n+1}$. For $(3)_{n+1}$,

$$(S_{n+1},\dots,S_{2n+1})H_n^{-1}(S_{n+1},\dots S_{2n+1})^t+Y_{n+1}=$$

$$=(S_{n+1},\dots,S_{2n},S_{2n+1})\begin{pmatrix} F_{n-1}^{-1}, & -F_{n-1}^{-1}F_{n-1}^{-1*}(S_n,\dots,S_{2n-1})^t\cdot Y_n^{-\frac{1}{2}} \\ 0, & Y_n^{-\frac{1}{2}}\end{pmatrix}\cdot$$

$$\cdot\begin{pmatrix} F_{n-1}^{-1*} & 0 \\ -Y_n^{-\frac{1}{2}}(S_n,\dots,S_{2n-1})F_{n-1}^{-1}F_{n-1}^{-1*}, & Y_n^{-\frac{1}{2}}\end{pmatrix}(S_{n+1},\dots,S_{2n+1})^t+Y_{n+1}=$$

$$=(S_{n+1},\dots,S_{2n})F_{n-1}^{-1}F_{n-1}^{-1*}(S_{n+1},\dots,S_{2n})^t+$$

$$+(S_{2n+1}-(S_{n+1},\dots,S_{2n})F_{n-1}^{-1}F_{n-1}^{-1*}(S_n,\dots,S_{2n-1})^t)Y_n^{-1}\cdot$$

$$\cdot(S_{2n+1}-(S_n,\dots,S_{2n-1})F_{n-1}^{-1}F_{n-1}^{-1*}(S_{n+1},\dots,S_{2n})^t)+Y_{n+1}$$

and we use $(1)_{n+1}$ and $(2)_{n+1}$.

In order to obtain $(4)_{n+1}$,

$$F_{n+1}^*F_{n+1}=\begin{pmatrix} F_n^*F_n, & (S_{n+1},\dots,S_{2n+1})^t \\ (S_{n+1},\dots,S_{2n+1}), & (S_{n+1},\dots,S_{2n+1})F_n^{-1}F_n^{-1*}(S_{n+1},\dots,S_{2n+1})^t+Y_{n+1}\end{pmatrix}$$

and we use $(3)_{n+1}$.

To this end, remark that Y_{n+1} is invertible if and only if H_{n+1} is invertible. So, the theorem is proved. ∎

<u>2.2 REMARK</u> From the formula for S_{2n-1} it results that K_{2n-1} has an imposed imaginary part, so, if we write

$$K_{2n-1}=X_n+i\,\mathrm{Im}K_{2n-1},$$

X_n will be a sequance of real free parameters.

It results that there exists a one-to-one correspondence between the set of strictly positive Hankel forms and the set of double sequences $\{X_n, Y_n\}_{n=1}^{\infty}$, X_n selfadjoint operators and Y_n positive , invertible operators. ∎

2.3 REMARK As in the Toeplitz case , we can obtain a formula for computing the determinants of the matrices H_n:

if S_k are rxr matrices, $r \in \mathbb{N}$ and k=1,...n, then

$$\det H_n = \prod_{k=1}^{n} \det Y_k \quad . \blacksquare$$

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