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by

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It is known that every first order geometric structure can be regarded as a reduction of the bundle of all linear frames on the given differentiable manifold. See /2/, /29/.

Let M be a n-dimensional C^∞ - differentiable manifold. Then a O(n)-structure on M is exactly a Riemannian structure on M ; if G is the Lie subgroup of GL(n, R) consisting of all linear transformations of R^n which leave R^k invariant, $k \leq n$, then a G-structure on M is a k-differential system on M ; if n is even, n = 2m, then a GL(m, C) - structure on M is exactly an almost complex structure on M, etc.

On the other hand, in the framework of Finsler geometry, there have been introduced and studied several geometric structures depending on directional arguments, as the metrical Finsler structures, /26/, the symplectic and hamiltonian finslerian structures, /20/, /27/, the conformal Finsler structures, /19/, /28/, the finslerian almost complex structures, /15/, /23/, the finslerian distributions, /7/. A natural question arises here : does it exist an unitary treatment of the finslerian geometric structures, in the manner of the classical theory of G - structures ?

For the present paper, a finslerian G-structure on M is a G-principal subbundle of the GL(n, R)-principal Finsler bundle $\pi^{-1} F(M)$.

Several examples of geometric structures given by finslerian tensor fields on M are shown to obey this definition. If M carries a regular connection in the induced Finsler vector bundle, two R^n-valued differentiable 1-forms θ^h and θ^v are defined, each giving a generalization of the canonical 1-form of a G-structure, none of which having trivial kernels. This obstruction is avoided by taking the fibred product of θ^h and θ^v ; the resulting lifting technics leads to the construction of the first-order structure function of the finslerian G-structure. The dependence on the choice of the regular connection is factored out as a result of the fact that given a horizontal distribution in the principal Finsler bundle, the remaining horizontal distributions are parametrised by elements of Hom(R^{2n}, L(G)).

The first-order-structure function of the finslerian G-structure is naturally related to the two torsions of the given regular connection.

1. INTRODUCTION

We need a survey of the basic constructions concerning the Finsler bundles and the regular connections. For all these we indicate /1/ as main reference.

Let M be a n-dimensional C^∞-differentiable manifold as before. Let then :

$$\pi_1 : T(M) \longrightarrow M$$

denote the tangent bundle over M and :

$$\pi : \tilde{M} \longrightarrow M$$

it's subbundle consisting of all non-zero tangent vectors on M. Let :

$$p : \pi^{-1} TM \longrightarrow \tilde{M}$$

be the reciprocal image of $T(M)$ by π. That is, the following diagram is commutative :

$$\begin{array}{ccc} \pi^{-1}\,TM & \xrightarrow{\;p\;} & \tilde{M} \\ \Big\downarrow \hat{\pi} & & \Big\downarrow \pi \\ TM & \xrightarrow[\;\pi_1\;]{} & M \end{array}$$

where :

$$p : \tilde{M} \times T(M) \longrightarrow \tilde{M}$$

$$\hat{\pi} : \tilde{M} \times T(M) \longrightarrow T(M)$$

stand for the natural projections of the product manifold $\tilde{M} \times T(M)$.

Clearly $\pi^{-1}\,TM$ is a differentiable real vector bundle over $\tilde{M}$, having $GL(n, R)$ as structure group and R^n as standard fibre. Throughout the paper $GL(n, R)$ denotes the general linear group of order n. Then $\pi^{-1}\,TM$ is called <u>the Finsler vector bundle</u>.

A non-zero tangent vector $\tilde{x}$ on M is called <u>a tangent direction on M</u>. The fibre over $\tilde{x}$ in $\pi^{-1}\,TM$ is denoted by $\pi^{-1}_{\tilde{x}}\,TM$. Let $F(M) \longrightarrow M$ be the $GL(n, R)$ - principal bundle of all linear frames on M. Let then :

$$\pi^{-1}\,F(M) \longrightarrow \tilde{M}$$

be the reciprocal image of $F(M)$ by π. The restrictions to $\pi^{-1}\,F(M)$ of the natural projections of the product manifold $\tilde{M} \times F(M)$ give the commutative diagram bellow :

$$\begin{array}{ccc} \pi^{-1} F(M) & \longrightarrow & \tilde{M} \\ \downarrow & & \downarrow \pi \\ F(M) & \longrightarrow & M \end{array}$$

The $GL(n, R)$ - bundle $\pi^{-1} F(M)$ over $\tilde{M}$ is well known in the literature as <u>the principal Finsler bundle</u>. See also /25/.

In the sequel, we use a more Finsler - like definition of the principal Finsler bundle, which is shown to be equivalent to Matsumoto's one.

Let $\tilde{x} \in \tilde{M}$ be a fixed tangent direction on M. Let $P_{\tilde{x}}$ be the space of all izomorphisms :

$$\bar{u} : R^n \longrightarrow \pi_{\tilde{x}}^{-1} TM$$

Let P be the disjoint union of all the spaces $P_{\tilde{x}}$, for all $\tilde{x} \in \tilde{M}$. Let $\pi_p : P \longrightarrow \tilde{M}$ be the natural projection, that is $\pi_p(\bar{u}) = \tilde{x}$, if $\bar{u} \in P_{\tilde{x}}$.

It is well known that :

$$P(\tilde{M}, \pi_p, GL(n, R))$$

is a principal bundle over M. See /18/. There is a naturally induced action of $GL(n, R)$ on the product manifold $P \times R^n$. See /24/. Let then :

$$W = P \times R^n / GL(n, R)$$

be the factor space. We obtain the associated bundle :

$$W(\tilde{M}, R^n, GL(n, R))$$

of the principal bundle P. See also /13/ .

Let us denote by :

$$L_{(\bar{u},\xi)} : GL(n, R) \longrightarrow P \times R^n$$

the fundamental map defined by the action of GL(n, R) on $P \times R^n$; here $\bar{u} \in P$ and $\xi \in R^n$. Then the image $L_{(\bar{u},\xi)}(GL(n, R))$ of the fundamental map $L_{(\bar{u},\xi)}$ is exactly the GL(n, R) - orbit of the pair $(\bar{u}, \xi)$. Suppose that $\bar{u} \in P_{\tilde{x}}$. Then the map :

$$W \longrightarrow \pi^{-1} TM$$

given by :

$$L_{(\bar{u},\xi)}(GL(n, R)) \longrightarrow \bar{u}(\xi) \in \pi^{-1} TM$$

is well defined and gives an obvious bundle izomorphism of W on-to π^{-1} TM. Hence the Finsler vector bundle can be regarded as the associated bundle of the GL(n, R) - principal bundle P , having R^n as standard fibre.

A cross-section :

$$\bar{X} : \tilde{M} \longrightarrow \pi^{-1} TM$$

in the Finsler vector bundle is called <u>a finslerian vector field on M</u>.

Let $\tilde{x} \in \tilde{M}$ be a fixed tangent direction on M, and $\bar{u} \in P_{\tilde{x}}$. Then $\bar{u}$ can be regarded as the synthetic object :

$$\bar{u} = (\tilde{x} , \{ \bar{X}_1 , \ldots , \bar{X}_n \})$$

where :

$$\bar{X}_i = \bar{u}(e_i) , \qquad i = 1,2,\ldots, n$$

and $\{e_1, \ldots, e_n\}$ is the natural linear basis of R^n.
Obviously $\{\bar{X}_1, \ldots, \bar{X}_n\}$ are independent and hence give a linear basis of the fibre $\pi_{\tilde{x}}^{-1}$ TM of the Finsler vector bundle.

Then $\bar{u}$ is called <u>a finslerian frame at the direction $\tilde{x}$</u> on M.

It is easily seen that the GL(n, R)-principal bundles π^{-1} F(M) and P are actually izomorphic. Indeed, let $(\tilde{x}, u) \in \pi^{-1} F(M)$ be fixed.
Then :

$$u = (x, \{X_1, \ldots, X_n\})$$

where :

$$x = \pi(\tilde{x}), \quad X_i \in T_x(M), \quad i = 1, \ldots, n .$$

Then the map :

$$\pi^{-1} F(M) \longrightarrow P$$

defined by :

$$(\tilde{x}, u) \longrightarrow \bar{u}$$

where :

$$\bar{u} = (\tilde{x}, \{\bar{X}_1, \ldots, \bar{X}_n\}) ;$$

$$\bar{X}_i = (\tilde{x}, X_i), \quad i = 1, 2, \ldots, n ,$$

gives an obvious izomorphism of π^{-1} F(M) on-to P. Hence no distinction will be made between the bundles π^{-1} F(M) and P from now on.

We define the bundle morphism :

$$L : T(\tilde{M}) \longrightarrow \pi^{-1} TM$$

by :

$$L_{\tilde{x}}\, \tilde{X} = (\tilde{x},\ (d_{\tilde{x}} \pi)\, \tilde{X})$$

where $\tilde{x} \in \tilde{M}$, $\tilde{X} \in T_{\tilde{x}} \tilde{M}$ and $d\pi$ denotes the differential of π. Clearly L is an epimorphism.

Let $X : M \longrightarrow T(M)$ be a tangent vector field on M. It's natural lift is the finslerian vector field :

$$\bar{X} : \tilde{M} \longrightarrow \pi^{-1} TM$$

defined by :

$$\bar{X}(\tilde{x}) = (\tilde{x},\ X(\pi \tilde{x})), \qquad \tilde{x} \in \tilde{M} .$$

Thus L is exactly the natural lift of $d\pi$.

A tangent vector field $\tilde{X}$ on $\tilde{M}$ is said to be vertical if :

$$(d\, \pi)\, \tilde{X} = 0$$

Let $T\tilde{M}_V \longrightarrow \tilde{M}$ denote the subbundle of $T(\tilde{M}) \longrightarrow \tilde{M}$ consisting of all vertical tangent vectors on $\tilde{M}$. Obviously :

$$\mathrm{Ker}(L) = T\tilde{M}_V .$$

The fundamental vector field is the finslerian vector field :

$$\bar{\eta} : \tilde{M} \longrightarrow \pi^{-1} TM .$$

defined by :

$$\bar{\eta}(\tilde{x}) = (\tilde{x},\ \tilde{x}), \qquad \tilde{x} \in \tilde{M} .$$

Let (U, x^i) be a local coordinate neighborhood on M. Let then $(\pi^{-1}(U), x^i, u^i)$ denote the naturally induced local coordinates on $\tilde{M}$, see /22/. It is easily seen that locally $\bar{\eta}$ is exactly the natural lift of the intrinsic tangent vector field :

$$\eta = u^i \frac{\partial}{\partial x^i}$$

on M.

Let ∇ be a connection in the Finsler vector bundle π^{-1} TM in the sense of /18/. A tangent vector field $\tilde{X}$ on $\tilde{M}$ is said to be horizontal with respect to ∇ if :

$$\nabla_{\tilde{X}} \bar{\eta} = 0$$

Let $T\tilde{M}_h \longrightarrow \tilde{M}$ denote the subbundle of $T(\tilde{M}) \longrightarrow \tilde{M}$ consisting of all horizontal tangent vectors on $\tilde{M}$.

The connection ∇ is said to be regular if we have the following direct sum decomposition :

$$T(\tilde{M}) = T\tilde{M}_h \oplus T\tilde{M}_v .$$

Let ∇ be a given regular connection in π^{-1} TM ; hence $L|_{T\tilde{M}_h}$ is an izomorphism. Let :

$$B : \pi^{-1} TM \longrightarrow T(\tilde{M})_h$$

denote it's inverse. Then B is called the horizontal lift with respect to ∇.

We define the bundle morphism :

$$\gamma : \pi^{-1} TM \longrightarrow T(\tilde{M})$$

by :

$$\gamma \bar{X} = \frac{dc}{dt}(0)$$

where :

$$\bar{X} \in \pi^{-1}_{\tilde{x}} TM , \quad \bar{X} = (\tilde{x}, X), \quad \tilde{x} \in \tilde{M} ,$$

and :

$$C : [0, 1] \longrightarrow \widetilde{M}, \qquad C(t) = \widetilde{x} + tX,$$

$t \in [0, 1]$. Here $\frac{dc}{dt}(0)$ denotes the tangent vector of C at C(0). Clearly :

$$\mathrm{Im}(\gamma) = T\widetilde{M}_V .$$

Consequently :

$$\gamma : \pi^{-1} TM \longrightarrow T\widetilde{M}_V$$

is a vector bundle izomorphism which is called <u>the vertical lift</u>. It has been several times emphasized that any regular connection ∇ in π^{-1} TM defines locally a Finsler connection $(N^i_j, F^i_{jk}, C^i_{jk})$ on M. Also the five torsions :

$$T^i_{jk} = F^i_{jk} - F^i_{jk}$$

$$A^i_{jk} = C^i_{jk}$$

$$R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x j}$$

$$P^i_{jk} = \partial_{,k} N^i_j - F^i_{kj}$$

$$S^i_{jk} = C^i_{jk} - C^i_{kj}$$

are easily seen to be (locally) fragments of the torsicn tensors :

$$\widetilde{T}(\widetilde{X}, \widetilde{Y}) = \nabla_{\widetilde{X}} L \widetilde{Y} - \nabla_{\widetilde{Y}} L \widetilde{X} - L \; \widetilde{X}, \widetilde{Y}$$

$$\widetilde{T}_1(\widetilde{X}, \widetilde{Y}) = \nabla_{\widetilde{X}} G \widetilde{Y} - \nabla_{\widetilde{Y}} G \widetilde{X} - G \; \widetilde{X}, \widetilde{Y}$$

where :

$$G : T(\widetilde{M}) \longrightarrow \pi^{-1} TM$$

is defined by :

$$G\tilde{X} = \gamma^{-1}\tilde{X}_v$$

where $\tilde{X}_v$ denotes the vertical part of $\tilde{X}$ with respect to the regular connection ∇. See /21/.

2. FINSLERIAN G-STRUCTURES

Let G be a Lie subgroup of GL(n, R). A G-principal sub-bundle

$$B_G(M) \longrightarrow \tilde{M}$$

of the principal Finsler bundle is said to be <u>a finslerian G-structure on M</u>. The following theorem gives sufficient conditions for a submanifold of the total space of the principal Finsler bundle to be a finslerian G-structure on M.

<u>Theorem 2.1.</u>

Let G be a Lie subgroup of GL(n, R).

Let B be a submanifold of $\pi^{-1} F(M)$.

If the following conditions are satisfied :

i) The restriction of the projection :

$$\pi_P : \pi^{-1} F(M) \longrightarrow \tilde{M}$$

maps B on-to $\tilde{M}$, that is :

$$\pi_P(B) = \tilde{M} .$$

ii) Let $\bar{u} \in B$ and $a \in GL(n, R)$. Then $\bar{u}\, a \in B$ if and only if $a \in G$.

iii) For every tangent direction $\tilde{x}$ on M there is a neighbourhood $\tilde{U}$ of $\tilde{x}$ in $\tilde{M}$ and a cross-section :

$$\sigma : \tilde{U} \longrightarrow \pi^{-1} F(M)$$

in the principal Finsler bundle so that :

$$\sigma(U) \subseteq B.$$

then B is a finslerian G-structure on M.

Proof

We follow the line of /2/. Let $\bar{u} \in \pi^{-1} F(M)$, $\pi_P(\bar{u}) = \tilde{x}$. There is a neighbourhood $\tilde{U}$ of $\tilde{x}$ and a cross-section σ in the Finsler bundle such that $\sigma(\tilde{U}) \leq B$. Hence $\bar{u}$ and $\sigma(\tilde{x})$ lie in the same GL(n, R)-orbit. Let $a \in GL(n, R)$ be defined by :

$$\bar{u} = \sigma(\tilde{x}) \cdot a$$

The map :

$$\psi_{\tilde{U}} : \pi_P^{-1}(\tilde{U}) \longrightarrow \tilde{U} \times GL(n, R)$$

defined by :

$$\bar{u} \longrightarrow (\tilde{x}, a)$$

is a diffeomorphism and it's restriction to $B \cap \pi_P^{-1}(\tilde{U})$ maps $B \cap \pi_P^{-1}(\tilde{U})$ one-to-one and on-to $\tilde{U} \times G$. As G is a Lie subgroup of GL(n, R), $B \cap \pi_P^{-1}(\tilde{U})$ admits a coordinate system so that the restriction of $\psi_{\tilde{U}}$ to $B \cap \pi_P^{-1}(\tilde{U})$ is still a diffeomorphism. Then B is a G-principal bundle.

Q.E.D.

We proceed by giving some examples. Suppose M is endowed with a Finsler metric tensor, that is, a symmetric non-degenerate

(0,2) - tensor :

$$g \in \pi^{-1} T^* M \otimes \pi^{-1} T^* M .$$

Here :

$$\pi^{-1} T^* M \longrightarrow \tilde{M}$$

denotes the reciprocal image of the cotangent bundle $T^*(M)$ by π. The Finsler metric tensor g gives naturally a finslerian O(n) - structure and M, where O(n) stands for the orthogonal group. Indeed, let $B_{O(n)}(M)$ consist of all finslerian frames : $\bar{u} = (\tilde{x}, \{\bar{X}_1, \ldots, \bar{X}_n\})$ on M which satisfy the condition :

$$g_{\tilde{x}} (\bar{X}_i, \bar{X}_j) = \delta_{ij} \qquad g_{\sim}$$

It is easily seen that $B_{O(n)}(M) \longrightarrow \tilde{M}$ is a O(n) - principal bundle over $\tilde{M}$. A finslerian O(n)-structure is generally known as a metrical Finsler structure on M. See /4/, /5/.

Suppose M is even dimensional, that is n = 2m. Let then :

$$J : R^{2m} \longrightarrow R^{2m}$$

be the natural complex structure on R^{2m}. Let GL(m, C) be the Lie subgroup of GL(n, R) consisting of all C - linear transformations of R^n, that is :

$$T \in GL(n, R) \quad \text{with} \quad TJ = JT$$

It is easily seen that a finslerian GL(m, C) - structure B $B_{GL(m, C)}(M) \longrightarrow \tilde{M}$ on M gives naturally a finslerian almost complex structure on M, that is a bundle morphism :

$$J : \pi^{-1} TM \longrightarrow \pi^{-1} TM$$

with :

$$J^2 = - I,$$

where I denotes the identity morphism. Indeed, let $\tilde{x}$ be a fixed tangent direction on M and let $\bar{u}$ be a finslerian frame at the direction $\tilde{x}$, adapted to the GL(m, C) - structure, that is $\bar{u} \in B_{GL(m, C)}(M)$.

We define a linear map :

$$J_{\tilde{x}} : \pi_{\tilde{x}}^{-1} TM \longrightarrow \pi_{\tilde{x}}^{-1} TM$$

as the commutative closer of the diagram bellow :

$$\begin{array}{ccc} \pi_{\tilde{x}}^{-1} TM & \xrightarrow{\;J_{\tilde{x}}\;} & \pi_{\tilde{x}}^{-1} TM \\ \bar{u}^{-1} \big\downarrow & & \big\downarrow \bar{u} \\ R^{2m} & \xrightarrow[\;J\;]{} & R^{2m} \end{array}$$

The definition of $J_{\tilde{x}}$ does not depend upon the choice of the adapted finslerian frame at the direction $\tilde{x}$. Indeed, if $\bar{v} \in B_{GL(m, C)}(M)$ is any other finslerian adapted frame at $\tilde{x}$ then $\bar{u}$ and $\bar{v}$ are GL(m, C) - equivalent. Hence there is a C - linear map T with :

$$\bar{v} = \bar{u}\, T$$

Hence :

$$\bar{v}\, J\, \bar{v}^{-1} = \bar{u}\, T\, J (\bar{u}\, T)^{-1} = \bar{u}\, T\, J\, T^{-1}\, \bar{u}^{-1} = \bar{u}\, J\, T\, T^{-1}\, \bar{u}^{-1}$$

see also /15/.

A mapping $\mathcal{D}$ which assigns to each tangent direction $\tilde{x}$ on M a real vector subspace $\mathcal{D}_{\tilde{x}}$ of $\pi_{\tilde{x}}^{-1} TM$ of constant dimension k, that is :

$$\dim_R \mathcal{D}_{\tilde{x}} = k$$

for any $\tilde{x} \in \tilde{M}$, is said to be a <u>finslerian k-distribution on M</u>.

Let G be the Lie group of $GL(n, R)$ which leaves R^k invariant. Let $B_G(M)$ be the submanifold of $\pi^{-1} F(M)$ consisting of all finslerian frames $\bar{u}$ having the property :

$$\bar{u}^{-1} (\mathcal{D}_{\tilde{x}}) = R^k$$

It is easily seen that $B_G(M) \longrightarrow \tilde{M}$ is a finslerian G-structure on M.

We have seen that the associated bundle of $\pi^{-1} F(M)$ (or equivalently P) is :

$$\pi^{-1} TM(\tilde{M}, p, R^n, GL(n, R))$$

We remark that given a finslerian G-structure $B_G(M) \longrightarrow \tilde{M}$ on M, it's associated bundle with standard fibre R^n is exactly :

$$\pi^{-1} TM(\tilde{M}, p, R^n, G)$$

Indeed, let us denote by W_G the factor space :

$$W_G = B_G(M) \times R^n / G$$

A bundle morphism :

$$\varphi : W_G \longrightarrow \pi^{-1} T M$$

is defined by :

$$\varphi(\, L_{(\bar{u},\xi)}(G)\,) \;=\; \bar{u}(\zeta)$$

for an arbitrary G-orbit $L_{(\bar{u},\zeta)}(G) \in W_G$. The bundle morphism φ defined above is an izomorphism of W_G on-to π^{-1} TM with the following inverse :

$$\Psi : \pi^{-1}\,TM \longrightarrow W_G\ ,$$

$$\Psi(\bar{X}) \;=\; L_{(\bar{u},\ \bar{u}^{-1}(\bar{X}))}(G)$$

for any $\bar{X} \in \pi^{-1}_{\tilde{x}}$ TM, $\tilde{x} \in \tilde{M}$, and any finslerian frame $\bar{u}$ at the direction $\tilde{x}$ adapted to the finslerian G-structure. Obviously the definition of Ψ does not depend upon the choice of such $\bar{u} \in B_G(M)$.

3. THE CANONICAL 1-FORM OF A FINSLERIAN G-STRUCTURE

We define a R^n-valued differentiable 1-form on a given finslerian G-structure $B_G(M)$ as follows :

$$\overset{h}{\Theta}_{\bar{u}} \;:\; T_{\bar{u}}(B_G(M)) \longrightarrow R^n\ ,$$

$$\overset{h}{\Theta}_{\bar{u}} \;=\; \bar{u}^{-1} \circ L_{\tilde{x}} \circ (d_{\bar{u}}\,\pi_P)$$

for any $\bar{u} \in B_G(M)$, $\pi_P(\bar{u}) = \tilde{x}$. Here $d_{\bar{u}}\pi_P$ denotes the differential of π_P at $\bar{u}$. The following diagram is commutative :

$$\begin{array}{ccc}
T_{\bar{u}}(B_G(M)) & \xrightarrow{\ \overset{h}{\Theta}_{\bar{u}}\ } & R^n \\
\Big\downarrow {\scriptstyle d_{\bar{u}}\pi_P} & & \Big\downarrow {\scriptstyle \bar{u}^{-1}} \\
T_{\tilde{x}}(M) & \xrightarrow[\ L_{\tilde{x}}\ ]{} & \pi^{-1}_{\tilde{x}}\ TM
\end{array}$$

Let us put as usually :

$$V_{\bar{u}} = \mathrm{Ker}(d_{\bar{u}} \pi_P)$$

Note that $\mathrm{Ker}(\theta^h_{\bar{u}})$ is not trivial.

Let ∇ be a regular connection in $\pi^{-1}TM$ and $G : T(\widetilde{M}) \to \pi^{-1}TM$ the corresponding bundle morphism. We define a R^n-valued differentiable 1-form θ^v on $B_G(M)$ by :

$$\theta^v_{\bar{u}} : T_{\bar{u}}(B_G(M)) \longrightarrow R^n ,$$

$$\theta^v_{\bar{u}} = \bar{u}^{-1} \circ G_{\tilde{x}} \circ (d_{\bar{u}} \pi_P)$$

for any $\bar{u} \in B_G(M)$, $\pi_P(\bar{u}) = \tilde{x}$.

Then the following diagram is commutative :

$$\begin{array}{ccc} T_{\bar{u}}(B_G(M)) & \xrightarrow{\ \theta^v_{\bar{u}}\ } & R^n \\ \downarrow{\scriptstyle d_{\bar{u}}\pi_P} & & \downarrow{\scriptstyle \bar{u}^{-1}} \\ T_{\tilde{x}}(\widetilde{M}) & \xrightarrow{\ G_{\tilde{x}}\ } & \pi^{-1}_{\tilde{x}}TM \end{array}$$

Obviously $\mathrm{Ker}(\theta^v_{\bar{u}})$ is not trivial. The R^n-valued differentiable 1-forms θ^h and θ^v give rise to a R^{2n}-valued differentiable 1-form θ on $B_G(M)$, defined by :

$$\theta_{\bar{u}} : T_{\bar{u}}(B_G(M)) \longrightarrow R^{2n}$$

$$\theta_{\bar{u}} Z = (\theta^h_{\bar{u}} Z , \theta^v_{\bar{u}} Z)$$

for any $Z \in T_{\bar{u}}(B_G(M))$, $\bar{u} \in B_G(M)$.

Then Θ is said to be <u>the canonical 1-form</u> of $B_G(M)$.

Consider the direct sum decomposition :

$$R^{2n} = R^n \oplus R^n$$

Let $P_j : R^{2n} \longrightarrow R^n$, j=1,2, be the natural projections of the direct sum decomposition.

There is a naturally induced action of the general linear group of order n (and consequently of G) in R^{2n} , defined by :

$$g\,\xi = (g\,P_1\,\xi) \oplus (g\,P_2\,\xi)$$

for any $g \in GL(n, R)$, $\xi \in R^{2n}$.

<u>Theorem 3.1.</u>

Let Θ be the canonical 1-form (with respect to the regular connection ∇) of the finslerian G-structure $B_G(M)$. Then :

$$(\delta\, R_g)\,\Theta = g^{-1}\,\Theta$$

for any $g \in G$.

<u>Proof</u>

Let $Z \in T_{\bar{u}}(B_G(M))$, $\bar{u} \in B_G(M)$.

Then :

$$(\delta\, R_g)\Theta_{\bar{u}}\, Z = \Theta_{R_g(\bar{u})}\,(d_{\bar{u}}\, R_g)\, Z =$$

$$= (\,\Theta^h_{R_g(\bar{u})}\,(d_{\bar{u}}\, R_g)\, Z,\ \Theta^v_{R_g(\bar{u})}\,(d_{\bar{u}}\, R_g)\, Z)$$

Note that :

1) $(R_g(\bar{u}))^{-1} = g^{-1}\,\bar{u}^{-1}$

2) π_P is constant on the fibres in $B_G(M)$.

Consequently :

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$$\overset{h}{\theta}_{R_g(\bar{u})} (d_{\bar{u}} R_g) Z = (R_g(\bar{u}))^{-1} \circ$$

$$\circ L_{\pi_P(R_g(\bar{u}))} \circ (d_{R_g(\bar{u})} \pi_P)(d_{\bar{u}} R_g) Z =$$

$$= g^{-1} \bar{u}^{-1} \circ L_{\pi_P(\bar{u})} \circ d_{\bar{u}} (\pi_P \circ R_g) Z =$$

$$= g^{-1} \bar{u}^{-1} \circ L_{\tilde{x}} \circ (d_{\bar{u}} \pi_P) Z = g^{-1} \overset{h}{\theta}_{\bar{u}} Z$$

where $\tilde{x} = \pi_P(\bar{u})$. Also :

$$\overset{v}{\theta}_{R_g(\bar{u})} (d_{\bar{u}} R_g) Z = g^{-1} \overset{v}{\theta}_{\bar{u}} Z$$

by similar computation. Finally :

$$(\delta R_g) \theta_{\bar{u}} Z = (g^{-1} \overset{h}{\theta}_{\bar{u}} Z, g^{-1} \overset{v}{\theta}_{\bar{u}} Z) =$$

$$= (g^{-1} P_1 \theta_{\bar{u}} Z) \oplus (g^{-1} P_2 \theta_{\bar{u}} Z) = g^{-1} \theta_{\bar{u}} Z .$$

Q.E.D.

Let H be the connection - distribution in $B_G(M)$ which is naturally associated to ∇, according to /8/. We recall it's construction.

A tangent vector $Z \in T_{\bar{u}}(B_G(M))$ is said to be <u>horizontal</u> is there is a differentiable curve :

$$\bar{C} : (-\varepsilon, \varepsilon) \longrightarrow B_G(M) , \quad \varepsilon > o ,$$

$$\bar{C}(t) = (c(t), \{ \bar{X}_1(C(t)), \ldots, \bar{X}_n(C(t))\}) ,$$

$t \in (-\varepsilon, \varepsilon)$, where :

$$C(t) = \pi_P(\bar{C}(t)) ,$$

so that :

1) $\bar{C}(o) = \bar{u}$, $\frac{d\bar{C}}{dt}(o) = Z$

2) $\nabla_{\frac{dC}{dt}} \bar{X}_i = o$, $i = 1,2,\dots, n$, along

The curve :

$$C : (-\varepsilon, \varepsilon) \longrightarrow \tilde{M} .$$

Let then $H_{\bar{u}}$ denote the space of all horizontal vectors Z which are tangent to $B_G(M)$ at $\bar{u}$. As shown in /8/, the distribution :

$$H : \bar{u} \longrightarrow H_{\bar{u}} \subseteq T_{\bar{u}}(B_G(M))$$

on $B_G(M)$, gives a connection in the principal bundle $B_G(M)$, that is :

1) $T_{\bar{u}}(B_G(M)) = H_{\bar{u}} \oplus V_{\bar{u}}$

2) $(d_{\bar{u}} R_g) H_{\bar{u}} = H_{R_g(\bar{u})}$

for any $\bar{u} \in B_G(M)$, $g \in G$, where :

$$R_g : B_G(M) \longrightarrow B_G(M)$$

is the right translation with $g \in G$.

We shall restrict $\Theta_{\bar{u}}$ to $H_{\bar{u}}$ in order to obtain an izomorphism. Firstly, we show that :

$$Ker(\Theta_{\bar{u}}) = V_{\bar{u}}$$

Let us denote the restriction of $d_{\bar{u}} \pi_P$ to $H_{\bar{u}}$ by :

$$t_{\bar{u}} : H_{\bar{u}} \longrightarrow T_{\tilde{x}} M, \quad t_{\bar{u}} = (d_{\bar{u}} \pi_P)\Big|_{H_{\bar{u}}} ,$$

$\bar{u} \in B_G(M)$, $\pi_P(\bar{u}) = \tilde{x}$.

Clearly $t_{\bar{u}}$ gives an izomorphism of $H_{\bar{u}}$ on-to $T_{\tilde{x}}\tilde{M}$.

Let us denote :

$$Q^h_{\bar{u}} = t^{-1}_{\bar{u}} (T_{\tilde{x}} \tilde{M}_h)$$

$$Q^v_{\bar{u}} = t^{-1}_{\bar{u}} (T_{\tilde{x}} \tilde{M}_v)$$

for $\bar{u} \in B_G(M)$, $\pi_P(\bar{u}) = \tilde{x}$.

As ∇ is regular, we obtain easily the direct sum decomposition :

$$H_{\bar{u}} = Q^h_{\bar{u}} \oplus Q^v_{\bar{u}}$$

for any $\bar{u} \in B_G(M)$.

Theorem 3.2.

1) $\mathrm{Ker}(\theta^h_{\bar{u}}) = Q^v_{\bar{u}} \oplus V_{\bar{u}}$

2) $\mathrm{Ker}(\theta^v_{\bar{u}}) = Q^h_{\bar{u}} \oplus V_{\bar{u}}$

3) $\mathrm{Ker}(\theta_{\bar{u}}) = V_{\bar{u}}$,

for any $\bar{u} \in B_G(M)$.

Proof

As :

$$\mathrm{Ker}(\theta^h_{\bar{u}}) \subseteq T_{\bar{u}}(B_G(M)) = H_{\bar{u}} \oplus V_{\bar{u}}$$

for any given $Z \in Ker(\overset{h}{\theta}_{\bar{u}})$ we obtain :

$$Z = Z' + Z''$$

where :

$$Z' \in H_{\bar{u}} \ , \quad Z'' \in V_{\bar{u}}$$

It is enough to prove that :

$$Z' \in \overset{v}{Q}_{\bar{u}}$$

we have :

$$o = \overset{h}{\theta}_{\bar{u}} Z = \bar{u}^{-1} \circ L_{\tilde{x}} \circ (d_{\bar{u}} \pi_P) Z =$$

$$= \bar{u}^{-1} \circ L_{\tilde{x}} \circ (d_{\bar{u}} \pi) Z' = \bar{u}^{-1} \circ L_{\tilde{x}} \circ t_{\bar{u}} Z' .$$

Hence :

$$L_{\tilde{x}} \circ t_{\bar{u}} Z' = 0$$

where from :

$$t_{\bar{u}} Z' \in Ker(L_{\tilde{x}}) = T_{\tilde{x}} \widetilde{M}_v$$

Thus :

$$Ker(\overset{h}{\theta}_{\bar{u}}) = \overset{v}{Q}_{\bar{u}} + V_{\bar{u}} .$$

Also :

$$\overset{v}{Q}_{\bar{u}} \cap V_{\bar{u}} \subseteq H_{\bar{u}} \cap V_{\bar{u}} = (o)$$

hence the sum is direct. Finally :

$$Ker(\theta_{\bar{u}}) = Ker(\overset{h}{\theta}_{\bar{u}}) \cap Ker(\overset{v}{\theta}_{\bar{u}}) = V_{\bar{u}}$$

as :

$$Q^{h}_{\bar{u}} \cap Q^{v}_{\bar{u}} = (0)$$

Q.E.D.

Consequently the restriction :

$$\Theta_{\bar{u}} : H_{\bar{u}} \longrightarrow R^{2n}$$

of the canonical 1-form is an izomorphism of $H_{\bar{u}}$ on-to R^{2n}.

4. THE FIRST STRUCTURE FUNCTION OF A FINSLERIAN G-STRUCTURE

Suppose ∇ and ∇' are two regular connections in π^{-1} TM. Let $H_{\bar{u}}$, $H'_{\bar{u}}$ be the corresponding horizontal subspaces of $T_{\bar{u}}(B_G(M))$. Let $\xi \in R^{2n}$ be fixed. There is an unique pair :

$$(Z_{\bar{u}}, Z'_{\bar{u}}) \in H_{\bar{u}} \times H'_{\bar{u}}$$

so that :

$$\Theta(Z)_{\bar{u}} = \Theta(Z')_{\bar{u}} = \xi$$

Hence :

$$Z_{\bar{u}} - Z'_{\bar{u}} \in V_{\bar{u}} .$$

Thus there is an unique linear map :

$$T \in \mathrm{Hom}(R^{2n}, L(G))$$

so that :

$$T(\xi)^{*}_{\bar{u}} = Z_{\bar{u}} - Z'_{\bar{u}} .$$

Here L(G) denoted the Lie algebra of all left invariant tangent vector fields on G. Also if $A \in L(G)$ is a given left invariant tangent vector field on G, then :

$$A^* : B_G(M) \longrightarrow T(B_G(M))$$

is the fundamental tangent vector field associated to A, that is :

$$A^*_{\bar{u}} = (d_e L_{\bar{u}}) A_e$$

where $e \in G$ is the identity matrix and :

$$L_{\bar{u}} : G \longrightarrow B_G(M)$$

is the fundamental map :

$$L_{\bar{u}}(g) = R_g(\bar{u}) ,$$

$\bar{u} \in B_G(M)$, $g \in G$.

We conclude that, as well as in the classical case, /2/, given a horizontal subspace $H_{\bar{u}}$ of $T_{\bar{u}}(B_G(M))$,the remaining horizontal subspaces $H'_{\bar{u}}$ can be parametrized by elements of $Hom(R^{2n}, L(G))$.

Let ∇ be a regular connection in $\pi^{-1}TM$. Let H be it's corresponding connection - distribution in $B_G(M)$. Let $\xi, \eta \in R^{2n}$ be fixed. Then there is an unique pair :

$$(Z_{\bar{u}}, W_{\bar{u}}) \in H_{\bar{u}} \times H_{\bar{u}} .$$

so that :

$$\theta(Z)_{\bar{u}} = \xi , \quad \theta(W)_{\bar{u}} = \eta .$$

We define a function :

$$C_H : B_G(M) \longrightarrow Hom(R^{2n} \wedge R^{2n}, R^{2n})$$

by :

$$C_H(\bar{u})\,(\xi \wedge \eta) \;=\; d\theta(Z, W)_{\bar{u}}\,.$$

For any $A \in L(G)$, $\xi \in R^{2n}$, we put :

$$A\,\xi \;=\; (A\,P_1\,\xi) \oplus (A\,P_2\,\xi)\,,$$

We define also the operator :

$$\partial :\; \mathrm{Hom}\,(R^{2n}, L(G)) \longrightarrow \mathrm{Hom}(R^{2n} \wedge R^{2n}, R^{2n})$$

by :

$$(\partial T)\,(\xi \wedge \eta) \;=\; T(\xi)\eta \;-\; T(\eta)\xi$$

for any $T \in \mathrm{Hom}(R^{2n}, L(G))$ and any $\xi, \eta \in R^{2n}$.

We shall prove the following :

<u>Theorem 4.1.</u>

Let $B_G(M) \longrightarrow \tilde{M}$ be a finslerian G-structure on M and ∇, ∇' two regular connections in the Finsler vector bundle. If H, H' are the corresponding connection-distributions in $B_G(M)$, then :

$$C_{H'}(\bar{u}) \;-\; C_H(\bar{u}) \;=\; -\,\frac{1}{2}\,(\partial T)$$

for any finslerian frame $\bar{u}$ adapted to the finslerian G-structure.

<u>Proof</u>

Let $\xi, \eta \in R^{2n}$ be fixed. Then there exist :

$$Z_{\bar{u}},\; W_{\bar{u}} \in H_{\bar{u}}$$

and :

$$Z'_{\bar{u}},\; W'_{\bar{u}} \in H'_{\bar{u}}$$

so that :

$$\Theta(Z)_{\bar{u}} = \Theta(Z')_{\bar{u}} = \xi$$

$$\Theta(W)_{\bar{u}} = \Theta(W')_{\bar{u}} = \eta$$

Then :

$$C_{H'}(\bar{u})\,(\xi \wedge \eta) - C_{H}(\bar{u})\,(\xi \wedge \eta) =$$

$$= d\Theta\,(Z', W')_{\bar{u}} - d\Theta(Z, W)_{\bar{u}} =$$

$$= d\Theta\,(Z' - Z, W')_{\bar{u}} + d\Theta(Z, W' - W)_{\bar{u}} =$$

$$= d\Theta(T(\xi)^{*}, W')_{\bar{u}} + d\Theta(Z, T(\eta)^{*})_{\bar{u}} =$$

$$+ \frac{1}{2}\, T(\xi)^{*}_{\bar{u}}\,(\Theta(W')) - \frac{1}{2}\, T(\eta)^{*}_{\bar{u}}\,(\Theta(Z)) .$$

We need to prove the following :

<u>Lemma</u>

Let $Z_{\bar{u}} \in H_{\bar{u}}$ so that $\Theta(Z)_{\bar{u}} = \xi$, $\bar{u} \in B_G(M)$, $\xi \in R^{2n}$.

For any left - invariant tangent vector field $A \in L(G)$ we have :

$$A^{*}_{\bar{u}}\,(\Theta(Z)) = - A\,\xi$$

<u>Proof</u>

For any $g \in G$ we have :

$$(\Theta(Z)\circ L_{\bar{u}})(g) = \Theta(Z)_{\bar{u}g} =$$

$$= \Theta_{\bar{u}g}\,(Z_{\bar{u}g}) = \Theta_{\bar{u}g}\,((d_{\bar{u}}R_g)\,Z_{\bar{u}}) =$$

$$= (\delta R_g)\, \Theta_{\bar u} Z_{\bar u} = g^{-1}\, \Theta(Z)_{\bar u} = g^{-1} \xi ,$$

by the theorem 3.1.

Consider the function :

$$\varphi_\xi : G \longrightarrow R^{2n} , \quad \varphi_\xi(g) = g^{-1} \xi ,$$

for any $g \in G$. Then :

$$A^*(\Theta(Z))_{\bar u} = A_e(\Theta(Z) \circ L_{\bar u}) = A_e(\varphi_\xi) .$$

Let (g^i_j) be a system of local coordinates on $GL(n, R)$. Since locally $A_e \in T_e G$ is given by :

$$A_e = A^m_k \left. \frac{\partial}{\partial g^m_k} \right|_e$$

we obtain :

$$A^*(\Theta(Z))_{\bar u} = A^i_j \cdot \frac{\partial \varphi_\xi}{\partial g^i_j}$$

Let us put :

$$\xi_k = P_k \xi , \qquad k = 1,2 .$$

Thus :

$$A \xi = (A \xi_1) \oplus (A \xi_2) = (A^j_i \xi^i_1 e_j) \oplus (A^j_i \xi^i_2 e_j) =$$

$$= (A^1_i \xi^i_1 , \ldots , A^u_i \xi^i_1 , A^1_i \xi^i_2 , \ldots , A^u_i \xi^i_2) =$$

$$= A^j_i (\delta^1_j \xi^i_1 , \ldots , \delta^n_j \xi^i_1 , \delta^1_j \xi^i_2 , \ldots , \delta^n_j \xi^i_2) =$$

$$= A^j_i (\xi^i_1 e_j) \oplus (\xi^i_2 e_j) ,$$

where $\{e_1, \ldots, e_n\}$ is the natural linear basis of R^n. It is also straightforward that :

$$\frac{\partial \varphi_{\xi}}{\partial g^i_j}(e) = - (\xi^j_1 e_i) \oplus (\xi^j_2 e_i)$$

which completes the proof.

The proof of the theorem 4.1 is now obtained as follows :

$$C_{H'}(\bar{u})(\xi \wedge \eta) - C_H(\bar{u})(\xi \wedge \eta) =$$

$$= \frac{1}{2} \{ T(\eta)\xi - T(\xi)\eta \} = - \frac{1}{2} (\partial T)(\xi \wedge \eta) .$$

Q.E.D.

The image $\partial \mathrm{Hom}(R^{2n}, L(G))$ of the operator ∂ (which is the coboundary operator in a certain cohomology) is a subspace of $\mathrm{Hom}(R^{2n} \wedge R^{2n}, R^{2n})$. Let us denote by :

$$F_{2n}(L(G)) = \frac{\mathrm{Hom}(R^{2n} \wedge R^{2n}, R^{2n})}{\partial \mathrm{Hom}(R^{2n}, L(G))}$$

the factor space.

We define a function :

$$C : B_G(M) \longrightarrow F_{2n}(L(G))$$

as follows; for any $\bar{u} \in B_G(M)$ we define $C(\bar{u})$ to be the coset of $C_H(\bar{u})$ modulo $\partial \mathrm{Hom}(R^{2n}, L(G))$ for an arbitrary regular connection ∇ in π^{-1} TM. According to the theorem 4.1, the definition of C does not depend upon the choice of the regular connection, simply because the dependence was factored out. The function C is said to be <u>the first-order structure function</u> of the finslerian G-structure $B_G(M)$.

Let $\tilde{X} : \tilde{M} \longrightarrow T(\tilde{M})$ be a tangent vector field on $\tilde{M}$.

We denote by :

$$\tilde{X}^H : B_G(M) \longrightarrow T(B_G(M))$$

it's H-horizontal lift, that is :

1) $\tilde{X}^H_{\bar{u}} \in H_{\bar{u}}$, $\bar{u} \in B_G(M)$

2) $(d_{\bar{u}}\pi_P)\, \tilde{X}^H_{\bar{u}} = \tilde{X}_{\tilde{x}}$

where $\pi_P(\bar{u}) = \tilde{x}$, $\tilde{x} \in \tilde{M}$.

For every $\bar{u} \in B_G(M)$ recall the direct sum decomposition :

$$H_{\bar{u}} = Q^h_{\bar{u}} \oplus Q^v_{\bar{u}}$$

from paragraph § 3. It gives

$$\tilde{X}^H = (\tilde{X}^H)_h + (\tilde{X}^H)_v$$

where :

$$(\tilde{X}^H)_h \in Q^h \;,\quad (\tilde{X}^H)_v \in Q^v$$

On the other hand, as ∇ is regular we have the decomposition :

$$\tilde{X} = \tilde{X}_h + \tilde{X}_v$$

where :

$$\tilde{X}_h \in T\tilde{M}_h \;,\quad \tilde{X}_v \in T\tilde{M}_V$$

Clearly :

$$(\tilde{X}^H)_h = (\tilde{X}_h)^H$$

$$(\tilde{X}^H)_v = (\tilde{X})^H_v$$

and hence we write simply :

$$\tilde{X}^H = \tilde{X}^H_h + \tilde{X}^H_v$$

For any finslerian vector field :

$$\overline{X} : \tilde{M} \longrightarrow \pi^{-1} TM$$

on M we have a well known equivalent formulation :

$$f_{\overline{X}} : B_G(M) \longrightarrow R^n ,$$

$$f_{\overline{X}}(\overline{u}) = \overline{u}^{-1} (\overline{X} (\pi_P (\overline{u}))) ,$$

for any $\overline{u} \in B_G(M)$.

<u>Theorem 4.2</u>

1) $\quad \Theta^h (B \overline{X}^H) = f_{\overline{X}}$

2) $\quad \Theta^v (\gamma \overline{X}^H) = f_{\overline{X}}$

for any finslerian vector field $\overline{X}$ on M.

<u>Proof</u>

$$\Theta^h_{\overline{u}} ((B \overline{X})^H_{\overline{u}}) = \overline{u}^{-1} \circ L_{\tilde{x}} \circ (d_{\overline{u}} \pi_P) (B \overline{X})^H_{\overline{u}} =$$

$$= \overline{u}^{-1} \circ L_{\tilde{x}} (B \overline{X})_{\tilde{x}} = \overline{u}^{-1} X_{\tilde{x}} = f_{\overline{X}} (\overline{u})$$

Also :

$$\overset{v}{\Theta}_{\bar{u}}((\gamma\bar{X})^{H}_{\bar{u}}) = \bar{u}^{-1} \circ G_{\tilde{x}} \circ (d_{\bar{u}}\pi_{P})(\gamma\bar{X})^{H}_{\bar{u}} =$$

$$= \bar{u}^{-1} \circ G_{\tilde{x}}(\gamma\bar{X})_{\tilde{x}} = \bar{u}^{-1}\gamma^{-1}_{\tilde{x}}(\gamma\bar{X})_{\tilde{x}} =$$

$$= \bar{u}^{-1}\bar{X}_{\tilde{x}} = f_{\bar{X}}(\bar{u}) .$$

Q.E.D.

We recall the following result, see /8/ :

Theorem 4.3

For any finslerian frame $\bar{u}$ (at the direction $\tilde{x}$) which is adapted to the finslerian G-structure we have :

$$(\nabla_{\tilde{Y}}\bar{X})(\tilde{x}) = \bar{u}((d_{\bar{u}}f_{\bar{X}})\tilde{Y}^{H}_{\bar{u}})$$

where $\tilde{Y}$ is a tangent vector field on $\tilde{M}$ and $\bar{X}$ a finslerian vector field on M.

The formula in the theorem 4.3. may be also written (making the necessary identifications) :

$$(\nabla_{\tilde{Y}}\bar{X})(\tilde{x}) = \bar{u}(\tilde{Y}^{H}_{\bar{u}}(f_{\bar{X}})) .$$

Theorem 4.4.

1) $$\tilde{T}(\tilde{X},\tilde{Y})(\tilde{x}) = 2\,\bar{u}\,\{d\overset{h}{\Theta}(\tilde{X}^{H},\tilde{Y}^{H})_{\bar{u}}\}$$

2) $$\tilde{T}_{1}(\tilde{X},\tilde{Y})(\tilde{x}) = 2\,\bar{u}\,\{d\overset{v}{\Theta}(\tilde{X}^{H},\tilde{Y}^{H})_{\bar{u}}\}$$

for any tangent vector fields $\tilde{X}$, $\tilde{Y}$ on $\tilde{M}$,

$$\tilde{x}\in\tilde{M},\quad \bar{u}\in B_{G}(M),\quad \pi_{P}(\bar{u}) = \tilde{x} .$$

<u>Proof</u>

We denote $\overline{X} = \ell \tilde{X}$, $\overline{Y} = \ell \tilde{Y}$. Then :

$B\,\overline{Y} = \tilde{Y}_h$ and $\tilde{Y}^H_v \in Q^v$. Hence :

$$\tilde{T}(\tilde{X}, \tilde{Y})(\tilde{x}) = (\nabla_{\tilde{X}} Y - \nabla_{\tilde{Y}} X - L\left[\tilde{X}, \tilde{Y}\right])_{\tilde{x}} =$$

$$= \overline{u}(\tilde{X}^H_{\overline{u}}(f_{\overline{Y}})) - \overline{u}(\tilde{Y}^H_{\overline{u}}(f_{\overline{X}})) - L_{\tilde{x}}\left[\tilde{X}, \tilde{Y}\right]_{\tilde{x}} =$$

$$= \overline{u}\{\tilde{X}^H_{\overline{u}}(\theta^h(B\tilde{Y})^H) - \tilde{Y}^H_{\overline{u}}(\theta^h(B X)^H) -$$

$$- \overline{u}^{-1} L_{\tilde{x}}\left[(d\pi_P)\tilde{X}^H, (d\pi_P)\tilde{Y}^H\right]_{\tilde{x}}\} =$$

$$= \overline{u}\{\tilde{X}^H_{\overline{u}}(\theta^h \tilde{Y}^H_h) - \tilde{Y}^H_{\overline{u}}(\theta^h \tilde{X}^H_h) -$$

$$- \overline{u}^{-1} L_{\tilde{x}}(d_{\overline{u}}\pi_P)\left[\tilde{X}^H, \tilde{Y}^H\right]_{\overline{u}}\} =$$

$$= \overline{u}\{\tilde{X}^H_{\overline{u}}(\theta^h \tilde{Y}^H) - \tilde{Y}^H_{\overline{u}}(\theta^h \tilde{Y}^h) - \theta^h_{\overline{u}}\left[\tilde{X}^H, \tilde{Y}^H\right]_{\overline{u}}\} =$$

$$= 2\,\overline{u}\{d\theta^h(\tilde{X}^H, \tilde{Y}^H)_{\overline{u}}\}$$

Q.E.D.

There is an unique tangent vector field $H(\xi)$ on $B_G(M)$ associated with a given $\xi \in R^{2n}$ so that :

1) $H(\xi)_{\overline{u}} \in H_{\overline{u}}$, $\overline{u} \in B_G(M)$.

2) $\theta(H(\xi))_{\overline{u}} = \xi$

Then clearly :

$$C_H(\bar{u})(\xi \wedge \eta) = -\frac{1}{2}\Theta(H(\xi), H(\eta))_{\bar{u}}$$

as :

$$\Theta(H(\xi)), \quad \Theta(H(\eta))$$

are constant functions on $B_G(M)$.

It is easily seen that :

$$\Theta^h(H(\xi)), \quad \Theta^v(H(\xi))$$

are also constant functions on $B_G(M)$, that is :

$$\Theta^h(H(\xi))_{\bar{u}} = P_1 \xi .$$

$$\Theta^v(H(\xi))_{\bar{u}} = P_2 \xi$$

Then :

$$d\Theta^h(H(\xi), H(\eta))_{\bar{u}} = -\frac{1}{2}\Theta^h(H(\xi), H(\eta))_{\bar{u}}$$

$$d\Theta^v(H(\xi), H(\eta))_{\bar{u}} = -\frac{1}{2}\Theta^v(H(\xi), H(\eta))_{\bar{u}}$$

where from we obtain the following :

<u>Theorem 4.5.</u>

For any $\xi, \eta \in R^{2n}$ and any

$\bar{u} \in B_G(M)$, $\pi_P(\bar{u}) = \tilde{x}$, we have :

$$C_H(\bar{u})(\xi \wedge \eta) =$$

$$= (\bar{u}^{-1}\tilde{T}((d\pi_P)H(\xi), (d\pi_P)H(\eta))_{\tilde{x}}, \bar{u}^{-1}\tilde{T}_1((d\pi_P)H(\xi), (d\pi_P)H(\eta))_{\tilde{x}}) .$$

Proof

$$C_H(\bar{u})(\xi \wedge \eta) = -\frac{1}{2}\,\Theta(\,H(\xi),\,H(\eta)\,)_{\bar{u}} =$$

$$= -\frac{1}{2}(\Theta^{h}(\,H(\xi),\,H(\eta)\,)_{\bar{u}}\,,\;\Theta^{v}(\,H(\xi),\,H(\eta)\,)_{\bar{u}}\,) =$$

$$= (d\Theta^{h}(H(\xi),\,H(\eta))_{\bar{u}}\,,\;d\Theta^{v}(H(\xi),\,H(\eta))_{\bar{u}}\,) =$$

$$= (\bar{u}^{-1}\,\widetilde{T}(\widetilde{X},\,\widetilde{Y})_{\widetilde{x}}\,,\;\bar{u}^{-1}\,\widetilde{T}_1(\widetilde{X},\,\widetilde{Y})_{\widetilde{x}})$$

where :

$$\widetilde{X} = (d\pi_p)\,H(\xi),\quad \widetilde{Y} = (d\pi_p)\,H(\eta)\,.$$

We have also made use of :

$$\widetilde{X}^{H} = H(\xi),\quad \widetilde{Y}^{H} = H(\eta)$$

Q.E.D.

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