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ISSN 0250 3638

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by

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PREPRINT SERIES IN MATHEMATICS
No.67/1983

BUCUREȘTI

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December 1983

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ON ISOLATED SINGULARITIES OF COMPLETE INTERSECTIONS

by

Alexandru DIMCA

Let $(X_o,0)$ be an isolated complete intersection singularity and $G:(X,0) \longrightarrow (B,0)$ be a miniversal deformation for $(X_o,0)$.

Consider the following general problem:

For a point $s \in B$, how many (and how "bad") singular points $x_i(s)$ can occur on the fiber $X_s = G^{-1}(s)$?

The following two estimates are already at hand.

$$(0.1) \qquad \sum_i m(X_s, x_i(s)) \leqslant m(X_o,0)$$

$$(0.2) \qquad \sum_i \left[\text{embdim}(X_s, x_i(s)) - \dim(X_s, x_i(s))\right] \leqslant \tau(X_o,0)$$

where $m(X_o,0)$ (resp. $\tau(X_o,0)$) denotes the multiplicity of the discriminant (resp. the dimension of the base space B) of the deformation G.

Indeed, (0.1) follows from [16] (4.8.2) and to derive (0.2) we note that G is a stable map germ and hence we can use Lemma 1.8 in [5], p.114.

When X_o is a homogeneous hypersurface singularity Bruce [2] has obtained a better estimate, using the interplay between deformation of singularities and their intersection forms. Namely, he showed that the number of singularities on X_s can be bounded by a number $N(d,n)$ such that

$$(0.3) \qquad \lim_{d \to \infty} \frac{N(d,n)}{\mu(X_o)} = \frac{1}{2}$$

where $d=\deg X_o$, $n=\dim X_o$ and $\mu(X_o)$ is the Milnor number of X_o. Note that in this case $\mu(X_o)=m(X_o)=\tau(X_o)$.

Finer results in this case were recently obtained by Guivental [8] and Varchenko [17].

In the present note we try to show that the method used by Bruce can give sometimes better upper bounds than (0.1) and (0.2) in the case of complete intersection singularities as well.

In fact, we consider X_o as the fiber of a function $f:(X,0)\longrightarrow(\mathbb{C},0)$, where X is itself an isolated singularity of complete intersection. The main ingredient is the use of morsifications of the function f defined on the Milnor fiber of X and of their associated vanishing cycles. This construction was introduced in [3] and is briefly reveiwed here in the first section.

The second section deals with relations between deformations of a function f as above and the intersection form S on the middle homology group of the Milnor fiber of X_o.

For weighted homogeneous singularities, we relate in the next section the rank of S to the middle Betti number of the corresponding weighted projective complete intersections and to the eigenvalues of some associated monodromy operators introduced by Hamm [10].

In this way, our upper bounds can be explicitly computed in terms of the weights and degrees of the equations for X_o in many situations (e.g. homogeneous singularities, Brieskorn-Pham singularities, curve and surface singularities).

As an application, we bound in the last section the number of isolated singularities on a projective complete intersection V, in a similar way to the hypersurface case treated by Bruce.

The upper bound we get is shown by examples to have an assymptotic behaviour similar to (0.3) and hence it gives indeed sometimes an improvement of the direct estimates (0.1) and (0.2). (Recall that for a weighted homogeneous complete intersection singularity X_o one has $m(X_o) \geqslant \tau(X_o) = \mu(X_o)$ [6]).

There is one additional difficulty comparing to the hypersurface case. To bound the number of all singular points on X_s when $\dim X_o$ is even, one has to make a suspension (stabilization) construction as in [2].

But for functions f as above this suspension (with few exceptions) takes us outside the class of weighted homogeneous singularities and we are no longer able to make explicit computations.

However, Example (4.3) below suggests that the upper bound obtained using this suspension has the same nice assymptotic behaviour.

1. The construction of morsifications

In this section we mainly recall some definitions and results from [3]. We prove that the monodromy operator constructed via morsifications coincides with the monodromy operator introduced by Hamm [10].

Let $X: g_1 = \dots = g_p = 0$ be an analytic complete intersection in a neighbourhood of the origin of $\mathbb{C}^{n+p}$ ($n \geqslant 1$, $p \geqslant 0$) having an isolated singular point at 0.

Consider also an analytic function germ $f: (\mathbb{C}^{n+p}, 0) \longrightarrow (\mathbb{C}, 0)$ such that $X_o = f^{-1}(0) \cap X$ is again a complete intersection with an isolated singularity at 0 and $\dim X_o = \dim X - 1 = n-1$.

For $\varepsilon \gg \delta > 0$ chosen sufficiently small and $r \in \mathbb{C}^p$ sufficiently general with $|r| < \delta$, it is known that the Milnor fiber of X

$$\tilde{X} = \tilde{X}_r = \left\{ x \in \mathbb{C}^{n+p};\ |x| \leqslant \varepsilon,\ g(x) = r \right\}$$

is a compact C^{∞} manifold with boundary, having the homotopy type of a bouquet of n-spheres. The number of these spheres is by definition the Milnor number $\mu(X)$ of the singularity X [10].

Let P denote the vector space of polynomials in $x_1,\dots,x_{n+p}$ of degree $\leq d$, for some $d\geq 3$. For $\eta>0$ chosen sufficiently small and for a generic polynomial $q\in P$ with $|q|<\eta$, the function $f_q=(f+q)|\tilde{X}$ has the following properties.

(1.1) The inclusion $E=\tilde{X}\cap f_q^{-1}(D_\delta)\hookrightarrow \tilde{X}$ is a homotopy equivalence, where $D_\delta=\{z\in\mathbb{C};\ |z|<\delta\}$.

(1.2) The restriction $f_q|\partial E$ is a submersion.

(1.3) The function f_q is a Morse function, i.e. it has only non-degenerate critical points with distinct critical values $c_1,\dots,c_s\in D_\delta$, where $s=\mu(X)+\mu(X_0)$.

(1.4) The topological type of the map of pairs

$$f_q:(E,f_q^{-1}(C))\longrightarrow(D_\delta,C),\quad C=\{c_1,\dots,c_s\}$$

is independent of the choices made above and moreover depends only on the restriction $f|X$.

Let $b\in D_\delta$ be a real number with $b>|c_i|$ for $i=1,\dots,s$. Then the fiber $\tilde{X}_0=f_q^{-1}(b)$ is a compact C^{∞} manifold with boundary and is homeomorphic to the Milnor fiber of the complete intersection singularity X_0.

Any path $w\in\pi_1(D_\delta\setminus C,b)$ induces a monodromy operator $h_w:H_{n-1}(\tilde{X}_0)\longrightarrow H_{n-1}(\tilde{X}_0)$. (Singular homology groups with $\mathbb{Q}$-coefficients are used throughout this paper).

We denote by h the monodromy operator corresponding to the path $w_0(t)=b\cdot\exp(2\pi it)$, $t\in[0,1]$.

On the other hand, Hamm [10] has constructed two fiber-equivalent C^{∞} fibrations associated to the function $f:(X,0)\longrightarrow(\mathbb{C},0)$ in the following way.

Let $\Sigma = X \cap S_\varepsilon$, $\Sigma_o = X_o \cap S_\varepsilon$ for $\varepsilon > 0$ small enough, where S_ε denotes the sphere of radius ε centered at the origin of $\mathbb{C}^{n+p}$.

The first fibration is then $\varphi_o : \Sigma \setminus \Sigma_o \longrightarrow S^1$, $\varphi_o(x) = f(x)/|f(x)|$ and is the generalization of the Milnor fibration.

The second fibration is the map

$$\varphi_1 : \{x \in \mathbb{C}^{n+p};\ |x| < \varepsilon,\ g(x)=0,\ |f(x)| = \delta\} \longrightarrow S^1_\delta ,$$

$\varphi_1(x) = f(x)$.

Our monodromy operator h is associated to the fibration

$$\varphi_2 : \{x \in E;\ |f_q(x)| = b\} \longrightarrow S^1_b ,\ \varphi_2(x) = f_q(x)$$

The next result shows that our construction is closely related to that of Hamm.

Lemma 1.5.

The fibrations φ_1 and φ_2 are fiber homotopy equivalent. in particular their monodromy operators are conjugate.

Proof

Let E_1 be the total space of φ_1 and note that $E_1 \subset B_\varepsilon = \{x \in \mathbb{C}^{n+p};\ |x| \leq \varepsilon\}$. Let $\overline{E}_1$ denote the closure of E_1 and $\overline{\varphi}_1$ the extension of φ_1 to $\overline{E}_1$.

Then $\overline{\varphi}_1$ is a C^∞ fibration having as fiber a smooth manifold with boundary diffeomorphic to $\tilde{X}_o$.

Since the inclusion int $\tilde{X}_o \subset \tilde{X}_o$ is a homotopy equivalence, it follows that the inclusion $E_1 \subset \bar{E}_1$ gives a fiber homotopy equivalence.

Next, one can consider still another fibration

$\varphi_3 : \{x \in \tilde{X}_r ;\ |f(x)| = \delta\} \longrightarrow S^1_\delta$, $\varphi_3(x) = f(x)$ and, using deformations arguments as in ([3], §1), it is easy to show that for δ, r and b chosen conveniently the fibrations $\bar{\varphi}_1$ and φ_3 (resp. φ_2 and φ_3) are topologically fiber equivalent. $\square$

We have associated to each critical point c_k of f_q and to an elementary path w_k joining b to c_k a thimble $\Delta_k \in H_n(\tilde{X}, \tilde{X}_o)$ and a corresponding vanishing cycle $\delta_k = \partial \Delta_k \in H_{n-1}(\tilde{X}_o)$. Note that the set $\underline{\delta} = \{\delta_1, \ldots, \delta_s\}$ is a system of generators for the vector space $H_{n-1}(\tilde{X}_o)$. [3].

Since $\tilde{X}_o$ is an orientable compact manifold with boundary $\partial \tilde{X}_o \simeq \Sigma_o$, there is a nondegenerate intersection form

$$H_{n-1}(\tilde{X}_o) \times H_{n-1}(\tilde{X}_o, \Sigma_o) \longrightarrow \mathbb{Q}$$

which gives an intersection form S on $H_{n-1}(\tilde{X}_o)$ via the morphism $j : H_{n-1}(X_o) \longrightarrow H_{n-1}(X_o, \Sigma_o)$ induced by inclusion.

For a set $\underline{\delta}$ of vanishing cycles as above, we shall denote by $S(\underline{\delta})$ the s x s matrix $(S(\delta_i, \delta_j))$ i,j=1,...,s and it is a trivial fact that $\mathrm{rk}\,S(\underline{\delta}) = \mathrm{rk}\,S$.

2. Deformations of smoothings

First we shall consider μ-constant deformations in the following sense ([3], §1).

Let $(X_t, 0) \subset (\mathbb{C}^{n+p}, 0)$ be a smooth family of complete intersections with an isolated critical point at the origin such that dim X_t=n and

$\mu(X_t)$=constant, for $t \in [0,1]$.

Assume that $f_t:(X_t,0) \longrightarrow (\mathbb{C},0)$ is a smooth family of functions such that $X_{to}=f_t^{-1}(0)$ is an isolated singularity of complete intersection with dim $X_{to}=n-1$ and $\mu(X_{to})$=constant, for $t \in [0,1]$.

In this situation we say that the functions $f_o:(X_o,0) \longrightarrow (\mathbb{C},0)$ and $f_1:(X_1,0) \longrightarrow (\mathbb{C},0)$ are μ-equivalent.

We have remarked already in [3] that in this case the topological type of the map of pairs (1.4) associated to the function f_t is the same for any $t \in [0,1]$.

We get thus the following (compare to [14], § 9).

Proposition 2.1

A μ-constant family $f_t:(X_t,0) \longrightarrow (\mathbb{C},0)$ as above gives rise to a natural identification $H_{n-1}(\tilde{X}_{oo}) \simeq H_{n-1}(\tilde{X}_{10})$ which preserves the intersection forms, the sets $\underline{\delta}$ of vanishing cycles and under which the monodromy operators h_i corresponding to the functions f_i (i=0,1) are conjugate. □

The next example will be needed in the last section.

Example 2.2

Let $a_1,\dots,a_{n+p}$ be positive integers $\geqslant 2$, $d=a_1 \dots a_{n+p}$, $d_k=d/a_k$. Assume that the defining equations $g_1=\dots=g_p=0$ of X and the function $f:(X,0) \longrightarrow (\mathbb{C},0)$ are given by weighted homogeneous polynomials of degree d with resepect to the weights $wt(x_k)=d_k$.

Then the characteristic polynomial of the monodromy operator h associated to f is given by the formulas

$$\Delta(\lambda) = \prod_{\substack{n \leq r \leq n+p \\ 1 \leq i_1 < \dots < i_r \leq n+p}} \left[\Delta_{a_{i_1} \dots a_{i_r}}(\lambda) \right]^{\binom{r-1}{n-1}}$$

where $\Delta_{b_1 \dots b_r}(\lambda) = \prod_{\substack{i_1, \dots, i_r \\ 1 \leq i_k \leq b_k - 1,\ k=1,\dots,r}} \left(e^{2\pi i (i_1/b_1 + \dots + i_r/b_r)} - \lambda\right)$.

Proof

When the polynomials g_k and f are of Brieskorn-Pham type, the result is proved by Hamm [11].

Let W denote the vector space of all the polynomials of the given weighted homogeneity type. The set $G = \{(g,f) = (g_1, \dots, g_p, f) \in W^{p+1};\ X_g = g^{-1}(0)$ and $X_{gf} = X_g \cap f^{-1}(0)$ are isolated singularities of complete intersections$\}$ is a Zariski open subset in W^{p+1}.

In particular, G is path-connected.

Moreover, if $(g,f) \in G$, then $\mu(X_g)$ and $\mu(X_{gf})$ depend only on the numbers d_i and d [6].

The result follows using (2.1). □

Now we shall consider deformations under which the initial function $f:(X,0) \longrightarrow (\mathbb{C},0)$ splits up in a number of (simpler) functions $f_i:(X_i,0) \longrightarrow (\mathbb{C},0)$ (compare to [14], § 7).

The precise setting of the problem is the following. Let $g^t(x)=0$, $t \in [0,1]$ be a smooth deformation of the equation $g(x)=0$ of X, defining for each t an analytic complete intersection X^t in some ball $B_\varepsilon \subset \mathbb{C}^{n+p}$, centered at the origin. ($g^0=g$).

On X^t we fix k distinct points $a_1(t), \dots, a_k(t)$, singular or not, for $t>0$. Let f^t be a smooth deformation of the function f such that $f^t(a_i(t))=0$ for $i=1,\dots,k$ and $t>0$ and $f^0=f$.

Consider the corresponding function germs

$$f_i^t : (X^t, a_i(t)) \longrightarrow (\mathbb{C}, 0)$$

and make the following assumptions (see [2], Def.1.b).

(a) The family f_i^t for $t \in (0,1]$ is a μ-constant family, for any $i=1,\ldots,k$. In particular $X_{io}^t = (f_i^t)^{-1}(0)$ is an isolated complete intersection singularity.

We denote by $f_i : (X_i, 0) \longrightarrow (\mathbb{C}, 0)$ the μ-class of f_i^t and by X_{io} the corresponding fiber $f_i^{-1}(0)$.

(b) $\lim_{t \to 0} |a_i(t)| = 0$ for any $i=1,\ldots,k$.

Proposition 2.3

With the above notations and assumptions,

(A) $$\mu(X) + \mu(X_o) \geq \sum_{i=1,k} (\mu(X_i) + \mu(X_{io}))$$

(B) There are sets of vanishing cycles $\underline{\delta}^i$ associated to the functions f_i, $i=1,\ldots,k$ and a set $\underline{\delta}$ associated to f such that the corresponding intersection matrices $M=S(\underline{\delta})$ and $M_i=S(\underline{\delta}^i)$ satisfy the equality

$$M = \left(\begin{array}{cccc|c} M_1 & 0 & \ldots & 0 & \\ 0 & M_2 & \ldots & 0 & \\ & & & & * \\ 0 & 0 & \ldots & M_k & \\ \hline & & * & & * \end{array}\right)$$

Proof

Starting with the function $f : (X,0) \longrightarrow (\mathbb{C}, 0)$, we construct a morsification $f_q : \tilde{X}_r \longrightarrow \mathbb{C}$ as in section 1.

We consider also some disjoint discs B_i centered at $a_i(t)$ and of radius $\varepsilon_i(t)$, $i=1,2,\dots,k$.

If t, $|r|$ and $\varepsilon_i(t)$ are chosen small enough and r and q sufficiently general, one can show the following facts (using for instance the Second Isotopy Lemma of Thom-Mather as in [3], § 1).

(i) The intersections $\widetilde{X}_{ir}=\widetilde{X}_r \cap B_i$ are naturally identified with the Milnor fibers of $(X^t, a_i(t))$ for any $i=1,2,\dots,k$.

(ii) Via the identifications, $f_q | \widetilde{X}_{ir}$ are morsifications for the functions $f_i^t:(X^t,a_i(t)) \longrightarrow (\mathbb{C},0)$ for any $i=1,2,\dots,k$.

Let $s=\mu(X)+\mu(X_o)$ and $s_i=\mu(X_i)+\mu(X_{io})$ for $i=1,2,\dots,k$. We know that f_q has precisely s critical points b_j on $\widetilde{X}_r$. From (ii) we deduce that s_i of these critical points are in the ball B_i and thus we get (A).

Moreover, if the critical point b_j is in B_i, then the corresponding vanishing cycle $\delta_j \in H_{n-1}(\widetilde{X}_o)$ has support contained in $\widetilde{X}_{io}=\widetilde{X}_o \cap B_i$, where $\widetilde{X}_o=f_q^{-1}(b)$ is taken as the Milnor fiber of X_o.

Using (ii) we find out that $\widetilde{X}_{io}$ is the Milnor fiber of X_{io} and hence the vanishing cycle δ_j corresponds to a vanishing cycle $\delta_j' \in H_{n-1}(\widetilde{X}_{io})$ for X_{io}.

On the other hand, it is clear that the intersection number of two vanishing cycles which have supports in disjoint balls B_i is zero. □

The following consequence will be used in the final section.

Corollary 2.4

With the same notations and assumptions as above,

$$\operatorname{rank} S \leq 2\left[\mu(X)+\mu(X_o)\right]-2\sum_{i=1,k}\left[\mu(X_i)+\mu_{iso}(X_{io})\right]$$

where $\mu_{iso}(X_{io})$ denotes the maximal dimension of an isotropic subspace in $H_{n-1}(\tilde{X}_{io})$ with respect to the intersection form.

Proof

Each of the intersection matrices M_i is conjugate over $\mathbb{Q}$ to a matrix

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \overline{M}_i \end{array}\right) \Big\} \mu(X_i)$$

where $\overline{M}_i = S(\underline{\beta}^i)$, $\underline{\beta}^i$ being a basis of the vector space $H_{n-1}(\tilde{X}_{io})$ extracted from the system of generators $\underline{\delta}^i$.

Moreover, we can change the basis $\underline{\beta}^i$ to a different one $\underline{\gamma}^i$, such that $S(\underline{\gamma}^i)$ has the form

$$\left(\begin{array}{c|c} 0 & * \\ \hline * & * \end{array}\right) \tag{2.5}$$

where the dimensionof the nul square block is $\mu_{iso}(X_{io})$.

Via these transformations, the matrix $\underline{M}$ is seen to be equivalent to a matrix as in (2.5), with a nul square block of dimension $\sum_{i=1,k} (\mu(X_i) + \mu_{iso}(X_{io}))$.

The result then follows by easy linear algebra. $\square$

Remark 2.6

Note the following estimates for μ_{iso}.

(i) If $n-1 = \dim X_o$ is odd, then the intersection form S is skew--symmetric and one gets

$$\mu_{iso}(X_o) \geqslant 1/2 \cdot \mu(X_o)$$

(ii) If $n-1=\dim X_o$ is even, then S is a symmetric form [say of type (μ_o, μ_+, μ_-)] and one gets

$$\mu_{iso}(X_o) \geq \mu_o + \min(\mu_+, \mu_-) \quad .$$

3. The rank of the intersection form

In this section we relate the rank of the intersection form of the weighted homogeneous singularities of complete intersections to the middle Betti number of their associated weighted projective varieties and to the multiplicity of the eigenvalue 1 for a sequence of mono dromy operators.

Let (X,0) be an isolated singularity of complete intersection in $\mathbb{C}^{n+p}$ defined by the weighted homogeneous polynomials f_i with $\deg f_i = d_i$ with respect to the weights $wt(x_j)=w_j \in N^*$, for $i=1,\dots,p$; $j=1,\dots,n+p$.

Definition 3.1

We call X a good weighted homogeneous singularity if the polynomials f_i can be chosen such that the germs

$$X_k = \{x \in \mathbb{C}^{n+p};\ f_1(x)=\dots=f_k(x)=0\} \quad k=1,\dots,p$$

are isolated singularities of complete intersections.

As remarked by Giusti, not all weighted homogeneous singularities are good (e.g. $f_1=x^2(x^5+y^3)$, $f_2=y^2(2x^5+y^3)$) .

However, if $w_1=\dots=w_{n+p}$ or if $d_1=\dots=d_p$, then a Bertini type argument shows that X is good.

To the germ X_k corresponds some algebraic variety Y_k in the weighted projective space $P(w_1,\dots,w_{n+p})$ [4].

When X is a good singularity, Y_k is a quasi-smooth complete intersection and hence a V-variety [4] (3.1.6). In particular Y_k is a $\mathbb{Q}$-homology manifold and there is a Hard Lefschetz Theorem for its cohomology [15] (1.13).

Moreover, in this case, the function germ

$$f_k:(X_{k-1},0)\longrightarrow(\mathbb{C},0)$$

has an associated monodromy operator h_k acting on $H_{n+p-k}(\tilde{X}_k)$, as described in section 1. Let S denote the intersection form on $H_n(\tilde{X})$.

Proposition 3.2

(i) $\operatorname{rk} S=\mu(X)-b_{n-1}(Y)+\varepsilon$, where $Y=Y_p$, $b_k(Y)$ denotes the k-Betti number of Y, $\varepsilon=1$ for $n\geq 3$ odd and $\varepsilon=0$ otherwise.

(ii) If X is a good singularity, then

$$\operatorname{rk} S=\mu(X)+\sum_{k=1,p}(-1)^k\operatorname{corank}(h_{p+1-k}-\mathrm{id})$$

where corank T=dim coker T.

Proof

The homology exact sequence of a pair gives

$$0\longrightarrow H_n(\partial\tilde{X})\longrightarrow H_n(\tilde{X})\xrightarrow{\ j\ }H_n(\tilde{X},\partial\tilde{X})\longrightarrow\ \ldots$$

and we derive $\operatorname{rk} S=\mu(X)-b_n(\partial\tilde{X})$.

Let S_o be a small sphere centered at the origin of $\mathbb{C}^{n+p}$ and let $\Sigma=X\cap S_o$, $\Sigma'=X_{p-1}\cap S_o$ be the links corresponding to $X=X_p$ and to X_{p-1}. Note that $\Sigma\simeq\partial\tilde{X}$.

The S^1-action on S_o given by

$$t\cdot x=(t^{w_1}\cdot x_1,\ldots,t^{w_{n+p}}\cdot x_{n+p})$$

leaves invariant the links Σ and Σ' and moreover $\Sigma/S^1\simeq Y$, $\Sigma'/S^1\simeq$ $\simeq Y_{p-1}$.

Using the Smith-Gysin exact sequence associated to this action on Σ [1] and the result of Hamm on the connectivity of Σ [10], we get

$$b_n(\partial\tilde{X})=b_{n-1}(\partial\tilde{X})=b_{n-1}(\Sigma)=b_{n-1}(Y)-\varepsilon$$

where ε is defined in the statement of the result (i).

On the other hand, comparing the Smith-Gysin exact sequences associated to the S^1-actions on Σ and Σ', we find out that the morphism $H_n(\Sigma) \longrightarrow H_n(\Sigma')$ induced by inclusion is trivial. The exact sequence of the pair (Σ',Σ) then gives

$$\dim H_n(\Sigma',\Sigma)=b_{n-1}(\Sigma)+b_n(\Sigma') \quad .$$

The exact sequence in [10] (1.8) shows that

$$\dim H_n(\Sigma',\Sigma)=\text{corank}(h_p-\text{id}) .$$

The last two formulas give the result (ii). $\square$

We mention now some situations when rk S can be computed effectively using Proposition (3.2).

Note first that $\mu(X)$ can be always computed in terms of w_j, d_i [6]. In the case of homogeneous singularities (i.e. $w_1=\ldots=w_{n+p}=1$) we can use (i), since there are formulas for $b_{n-1}(Y)$ (see for instance [12]).

When $\dim X \leqslant 2$, we can compute $b_{n-1}(Y)$ using the formula for the "geometric genus" p_g given in [4] (3.4.4) and the obvious fact that a 1-dimensional V-variety is smooth.

For the singularities described in Example (2.2) we can compute rk S using (ii).

<u>Remarks 3.3</u>

(i) Since the cohomology algebra $H^*(Y,\mathbb{Q})$ has the same structure as for an usual complete intersection i.e. except the middle dimension $H^k = \mathbb{Q}$(resp. 0) for k even (resp.odd), it follows that to know $b_{n-1}(Y)$ is equivalent to knowing the Euler-Poincaré characteristic $\chi(Y)$ of Y.

(ii) If X' is a semi-weighted homogeneous singularity of complete intersection [6] corresponding to the singularity X above, then note that the Milnor fibers $\tilde{X}$ and $\tilde{X}'$ are homeomorphic and in particular X and X' have the same intersection forms.

(iii) There is a variation operator

$$Var: H_{n-1}(\tilde{X}_o, \partial \tilde{X}_o) \longrightarrow H_{n-1}(\tilde{X}_o)$$

associated in the usual way to the monodromy operator h introduced in section 1 (see for instance [9]).

The formula 3.2.ii shows that for complete intersections (as opposed to the hypersurface case!) the variation operator is not in general an isomorphism. Indeed, otherwise one would have as in [9]

$$rk\ S = rk(h_p - id)$$

and this contradicts (3.2.ii).

4. Isolated singularities on projective complete intersections

We consider a projective complete intersection $V \subset P^{n+p}$ defined by the equations $P_1(x) = \ldots = P_p(x) = 0$, where P_i is a homogeneous polynomial of degree d_i , $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_p \geqslant 2$, $p \geqslant 2$.

We assume that V has only isolated singularities p_i , $i = 1, \ldots, N$ and we try to get an upper bound for their number N (and their complexity).

The polynomials P_i can be chosen such that the complete intersections

$$V_k = \left\{ x \in P^{n+p};\ P_1(x) = \ldots = P_k(x) = 0 \right\},\ k = 1, \ldots, p$$

are singular at most at the singular points of V.

Indeed, this can be done inductively on k, using Bertini's Theorem ([7], p.137). For instance P_1 should be taken as a generic member of the linear system $I(V)^{d_1}$, the homogeneous component of degree d_1 of the ideal $I(V)$ of the projective variety V.

Then apply Bertini's Theorem to the linear system $I(V)^{d_2}$ on the smooth part of $\{P_1=0\}$ and so on.

Moreover, we can suppose that the intersections $W_k=V_k \cap H$ are smooth for any $k=1,\dots,p$, where H is the hyperplane given by $x_o=0$.

Let $g_i^t(x)=P_i(t,x)$, $f^t(x)=P_p(t,x)$ for $t \in [0,1]$, $x \in \mathbb{C}^{n+p}$ and $i=1,\dots,p-1$.

Then $X:g^o(x)=0$ (resp. $X_o:g^o(x)=f^o(x)=0$) is an isolated singularity of complete intersection, namely the cone over the smooth variety W_{p-1} (resp. W_p).

Note that the singularities of $X^t:g^t(x)=0$ (resp. $X_o^t:g^t(x)=f^t(x)=0$) for $t\neq 0$ correspond to the singularities of V_{p-1} (resp. $V_p=V$).

Moreover all the assumptions of Prop. 2.3 are fulfilled. Using (2.4) and (3.2.i) we obtain the following.

Proposition 4.1

$$\sum_{i=1,N} (\mu(X_i)+\mu_{iso}(X_{io})) \leq \mu(X)+1/2\left[\mu(X_o)+b_{n-1}(W_p)-\varepsilon\right]$$

where X_i (resp. X_{io}) denotes the singularity of the complete intersection V_{p-1} (resp. V) at the point p_i for $i=1,\dots,N$. □

Let us denote by $B(X,X_o)$ the right-hand side in the above inequality. Then the Proposition (4.1) gives a better upper bound than (0.1) and (0.2) when, roughly speaking, $B(X,X_o) < \mu(X_o)$.

The following example illustrates the (assymptotic) behaviour of the invariants involved.

Example 4.2

We consider the case $d_1 = \ldots = d_p = d$. Then using the formulas in [6] we find that $\mu(X)$ (resp. $\mu(X_o)$) is a polynomial in d of degree n+p with leading coefficient $\binom{n+p-1}{n+1}$, resp. $\binom{n+p-1}{n}$. Moreover, $b_{n-1}(W_p)$ is in this case a polynomial in d of degree n+p-1. It follows that

$$L(n,p) := \lim_{d\to\infty} \frac{B(X,X_o)}{\mu(X_o)} = \frac{n+2(p-1)+1}{2(n+1)} .$$

Note that $L(n,p) < 1$ if and only if

$$\dim V > 2 \operatorname{codim} V - 3$$

and moreover we have as in (0.3)

$$L(n,1) = \lim_{n\to\infty} L(n,p) = \frac{1}{2} \ . \ \square$$

Note that if n=dim V is odd, then $\mu_{iso}(X_{io}) \geq 1$ for any $i=1,\ldots,N$ and hence (4.1) gives an upper bound for N, the total number of singular points on V. When n is even, it is easy to see by 2.6.ii and [14] (8.10) that

$$\mu(X_i) + \mu_{iso}(X_{io}) = 0$$

if and only if X_i is smooth and X_{io} is a simple hypersurface singularity (i.e. of type A_n, D_n, E_6, E_7 or E_8 in Arnold notations).

In order to obtain an upper bound for N in this case, one needs a suspension device as in Bruce [2].

We could imagine no better approach than the following.

Consider the suspensions $\overline{X}^t : \overline{g}^t = 0$ (resp. $\overline{X}^t_o : \overline{g}^t = \overline{f}^t = 0$) where $\overline{g}^t_i(x,z) = g^t_i(x) + \lambda_i z^2$, $\overline{f}^t(x,z) = f^t(x) + \lambda z^2$ for z, λ and λ_i in $\mathbb{C}$.

If the constants λ, λ_i are chosen general enough, $\overline{X}^o$ and $\overline{X}^o_o$ are isolated singularities of complete intersections. Moreover if $a_i(t) \in X^t_o$ is a singular point, then $(a_i(t),0) \in \overline{X}^t_o$ is again a singular point.

In this way, using (2.4) we get an upper bound for N in terms of $\mu(\bar{X}^o)$, $\mu(\bar{X}^o_o)$ and the rank of the intersection form $\bar{S}$ of $\bar{X}^o_o$.

Note that the singularities $\bar{X}^o, \bar{X}^o_o$ are no longer weighted homogeneous in general and hence one has no general method for making explicit computations.

A happy exception is presented in the following example.

Example 4.3

We consider again the case $d_1 = \ldots = d_p = d$.

Using (2.2) and (3.2.ii) it is easy to show the following.

The Milnor number $\mu(\bar{X}^o)$ resp. $\mu(\bar{X}^o_o)$ is a polynomial in d of degree n+p with leading coefficient $\binom{n+p}{n+2} + \binom{n+p-1}{n+2}$ resp. $\binom{n+p}{n+1} + \binom{n+p-1}{n+1}$.

Moreover, as in [2] Lemma 5, one can show that corank $(h_k - id)$ is a polynomial in d of degree $< n+p$ for $k=1,\ldots,p$.

Let us denote by $\bar{B}$ the upper bound we get using (2.4), namely

$$\bar{B} = \mu(\bar{X}^o) + \mu(\bar{X}^o_o) - 1/2 \cdot \mathrm{rk}\, \bar{S}$$

The above computations show that

$$\bar{L}(n,p) := \lim_{d\to\infty} \frac{\bar{B}}{\mu(X_o)} = \frac{n^2 + n(4p-1) + 4p^2 - 4p + 2}{2(n+1)(n+2)} .$$

In particular, we have as in (0.3)

$$\bar{L}(n,1) = \lim_{n\to\infty} \bar{L}(n,p) = \frac{1}{2} .$$

Remarks 4.4

(i) If we used a 'homogeneous' suspension given by $\bar{g}^t_i = g^t_i + \lambda_i z^{d_i}$, $\bar{f}^t = f^t + \lambda z^{d_p}$, then $\bar{X}^o, \bar{X}^o_o$ are homogeneous singularities, but their Milnor numbers become too big to give interesting upper bounds.

(ii) It seems that there are very [few] types of complete intersections

V for which it is known the maximum number M of singularities which can really occur on V.

The only such example known to us is when $p=d_1=d_2=2$ and it is due to Knörrer [13] Chap.II. In this case, and assuming $n \geq 3$ odd, one has the following equalities:

$$M=n+3, \quad \mu(X_o)=2n+3 \quad \text{and} \quad B(X,X_o)=(3n+7)/2$$

the maximum M being obtained for a pencil of quadrics with Segre symbol $[(1,1), (1,1), \ldots, (1,1)]$.

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