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THE STRUCTURE OF NAIMARK DILATION AND GAUSSIAN STATIONARY PROCESSES

by

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THE STRUCTURE OF NAIMARK DILATION AND GAUSSIAN STATIONARY PROCESSES

Gr. Arsene and T.Constantinescu

1. INTRODUCTION

The well-known interplay between classical extrapolation problems, the theory of Toeplitz operators, and the theory of Gaussian stationary processes stimulate for a long time the research in all these fields. The main object which connects these subjects is the notion of positive Toeplitz form. This can be thought as determining the correlation operator of a Gaussian stationary process, or as determining the Fourier coefficients of a semispectral measure.

The study of dilation theory of contractions in Hilbert spaces asks for a generalization of the Schur sequence which appear in Carathéodory-Fejér extrapolation problem; this is the notion of choice sequence which was used in indexing all contractive intertwining dilations and in the structure of positive Toeplitz forms. As a by-product, the description of the structure of Naimark dilation of a given semispectral measure was obtained. This fact proves now to be useful in studying the regularity properties of a Gaussian stationary process.

The contents of this paper is the following. In Section 2 we collect the results concerning the structure of some matrix contractions, the structure of positive Toeplitz forms, and the structure of Naimark dilation of a semispectral measure. Section 3 describes the connection between the theory of (vectorial) Gaussian stationary processes and dilation theory. Section 4 gives the main technical result (Theorem 4.7) which shows that the parametrization by choice sequences is useful in computations (in the general vectorial case) connected with informational regularity of the processes. As a consequence, we obtain in Section 5 a criterion (Theorem 5.2) for informational regularity of matricial Gaussian stationary processes. More explicite results (Theorem 6.1) are obtained for the scalar case in Section 6. Here we insert also two remarks concerning the connections of our setting with the notion of entropy for Gaussian stationary processes and with Szegö's Limit Theorem.

2. PRELIMINARIES

In this section we will recall the structure of Naimark dilation of a semispectral measure as presented in [13]. For this, it is necessary to review some results on the stsructure of matrix contractions (see [7], [3], [11], [12], [13]), on choice sequences (see [8], [2], [12], [13]) and on the structure of positive Toeplitz forms (see [12]). Some facts are described here in a more general setting (or more completely) than in the quoted papers; moreover, those papers contain proofs which cover the cases considered in this section.

2.1. Structure of some matrix contractions

Let H and H' be (complex) Hilbert spaces and let $T \in L(H,H')$ be a contraction (i.e. $\|T\| \leq 1$). As usual D_T and $\mathcal{D}_T$ will denote the defect operator $(=(I-T^*T)^{\frac{1}{2}})$, resp. the defect space $(=\overline{D_T(H)})$ of T. A straightforward computation shows that the operator

$$(2.1)\qquad \begin{cases} J(T): H \oplus \mathcal{D}_{T^*} \longrightarrow H' \oplus \mathcal{D}_T \\[2ex] J(T) = \begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix} \end{cases}$$

is a unitary operator.

Suppose now that $H = \bigoplus_{p=1}^{n} H_p$, where $n \in \mathbb{N}$. Then $T = (T_1, T_2, \ldots, T_n) \in L(H,H')$ is a contraction if and only if (see [7], [11])

$$(2.2)_1 \qquad \|T_1\| \leq 1,$$

and for every $2 \leq p \leq n$,

$$(2.2)_p \qquad T_p = D_{\Gamma_1^*} \cdots D_{\Gamma_{p-1}^*} \Gamma_p ,$$

where $\Gamma_1 = T_1$ and $\Gamma_k \in L(H_k, \mathcal{D}_{\Gamma_{k-1}^*})$ are contractions for every $2 \leq k \leq n$. Moreover, the defect spaces of T and T^* can be identified as follows. First, the operator defined by

$$\alpha(T) = \alpha : \mathcal{D}_T \rightarrow \mathcal{D}_{\Gamma_1} \oplus \mathcal{D}_{\Gamma_2} \oplus \cdots \oplus \mathcal{D}_{\Gamma_n}$$

$$(2.3) \qquad \alpha(T)D_T := \begin{bmatrix} D_{\Gamma_1} & -\Gamma_1^*\Gamma_2 & -\Gamma_1^*D_{\Gamma_2^*}\Gamma_3 & \cdots & -\Gamma_1^*D_{\Gamma_2^*}\cdots D_{\Gamma_{n-1}^*}\Gamma_n \\ 0 & D_{\Gamma_2} & -\Gamma_2^*\Gamma_3 & \cdots & -\Gamma_2^*D_{\Gamma_3^*}\cdots D_{\Gamma_{n-1}^*}\Gamma_n \\ 0 & 0 & D_{\Gamma_3} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{\Gamma_n} \end{bmatrix}$$

is a unitary operator. For computing D_{T^*}, consider

$$(2.4) \qquad \begin{cases} \beta(T) = \beta : D_{T^*} \to D_{\Gamma_n^*} \\ \beta(T)D_{T^*} := D_{\Gamma_n^*}\cdots D_{\Gamma_1^*}, \end{cases}$$

which is a unitary operator.

The case $H = \bigoplus_{p=1}^{\infty} H_p$ is more delicate (see [13], Sections 1.3-1.6). Let $T = (T_p)_{p=1}^{\infty} \in L(H,H')$, and denote for every $n \geq 1$, $H^{[n]} := \bigoplus_{p=1}^{n} H_p \subset H$ and $T^{[n]} := T|H^{[n]} \in L(H^{[n]},H')$. From $(2.2)_{p\geq 1}$ it follows that T is a contraction if and only if T_1 is a contraction and $T_p = D_{\Gamma_1^*}\cdots D_{\Gamma_{p-1}^*}\Gamma_p$, $(p \geq 2)$, where $\Gamma_1 = T_1$ and $\Gamma_k \in L(H_k, D_{\Gamma_{k-1}^*})$ are contractions $(k \geq 2)$. For computing D_T, define

$$(2.5)_n \qquad D_n(T) := \alpha(T^{[n]})D_{T^{[n]}} : H^{[n]} \to \bigoplus_{p=1}^{n} D_{\Gamma_p} \subset \bigoplus_{p=1}^{\infty} D_{\Gamma_p}$$

and

$$(2.6) \qquad D_\infty(T) := \underset{n\to\infty}{\text{s-lim}}\, D_n(T) : H \to \bigoplus_{p=1}^{\infty} D_{\Gamma_p}.$$

Then the operator

$$(2.7) \qquad \begin{cases} \alpha(T) : D_T \to \bigoplus_{p=1}^{\infty} D_{\Gamma_p} \\ \alpha(T)D_T := D_\infty(T) \end{cases}$$

is a unitary operator.

For computing D_{T^*}, consider the operators

$$(2.8)_n \quad \begin{cases} G_n(T) = G_n : H' \to D_{\Gamma_n^*} \\ G_n := D_{\Gamma_n^*} \ldots D_{\Gamma_2^*} D_{\Gamma_1^*}, \quad (n \geq 1) \end{cases}$$

and the operator

$$(2.9) \qquad G_\infty(T) = G_\infty := \text{s-}\lim_{n \to \infty} G_n^* G_n .$$

Then the operator

$$(2.10) \quad \begin{cases} \beta(T) : D_{T^*} \to \text{Ran}\, G_\infty(T)^- \\ \beta(T) D_{T^*} := G_\infty(T)^{1/2} \end{cases}$$

is a unitary operator.

Similar results can be obtained (by transposition) for the case of a "column" operator from H into $H' = \bigoplus_{p=1}^{n} H'_p$ (or $\bigoplus_{p=1}^{\infty} H'_p$).

Finally, consider the case of a two-by-two matrix contraction $T : H (= H_1 \oplus H_2) \to H' (= H'_1 \oplus H'_2)$, $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$, see [3]. Then T is a contraction if and only if

$$(2.11)_1 \qquad \| T_{11} \| \leq 1 ;$$

$(2.11)_2$ $\quad T_{12} = D_{T_{11}^*} \Gamma_1$, where $\Gamma_1 : H_2 \to D_{T_{11}^*}$ is a contraction;

$(2.11)_3$ $\quad T_{21} = \Gamma_2 D_{T_{11}}$, where $\Gamma_2 : D_{T_{11}} \to H'_2$ is a contraction;

and

$(2.11)_4$ $\quad T_{22} = -\Gamma_2 T_{11}^* \Gamma_1 + D_{\Gamma_2^*} \Gamma D_{\Gamma_1}$, where $\Gamma : D_{\Gamma_1} \to D_{\Gamma_2^*}$ is a contraction.

The defect spaces of T and T^* can be computed using the unitary operators

$$(2.13) \quad \begin{cases} \alpha(T) : D_T \to D_{\Gamma_2} \oplus D_\Gamma \\ \alpha(T) D_T := \begin{bmatrix} D_{\Gamma_2} D_{T_{11}} & -(D_{\Gamma_2} T_{11}^* \Gamma_1 + \Gamma_2^* \Gamma D_{\Gamma_1}) \\ 0 & D_\Gamma D_{\Gamma_1} \end{bmatrix} \end{cases}$$

and

$$\beta(T): D_{T^*} \to D_{\Gamma_1^*} \oplus D_{\Gamma^*} \tag{2.14}$$

$$\beta(T) := \alpha(T^*).$$

2.2. Choice sequences

The main object used for the indexing of contractive intertwining dilations ([8], [2]) and of positive Toeplitz forms ([12]) is the notion of *choice sequence*. A sequence of contractions $\gamma = \{\Gamma_m\}_{m=1}^{\infty}$ is called a $((H,H')$-$)$ choice sequence if $\Gamma_1 : H \to H'$ and for every $n \geq 2$, $\Gamma_n : D_{\Gamma_{n-1}} \to D_{\Gamma^*_{n-1}}$. For simplifying the writing of some formulas we take $\Gamma_0 : H \to H'$, $\Gamma_o := 0$, so $D_{\Gamma_o} = I_H$, $D_{\Gamma_o^*} = I_{H'}$, $D_{\Gamma_o} = H$, $D_{\Gamma_o^*} = H'$. Fix now a choice sequence γ. We will attach to it the space

$$(2.15)_n \qquad K_n(\gamma) = K_n := \bigoplus_{p=0}^{n-1} D_{\Gamma_p}, \quad (n \geq 1)$$

and

$$(2.16) \qquad K_+(\gamma) = K_+ := \bigoplus_{p=0}^{\infty} D_{\Gamma_p}.$$

Consider the "row" operators

$$(2.17)_n \qquad \begin{cases} R_n(\gamma) = R_n : K_n \to H' \\ R_n := (\Gamma_1, D_{\Gamma_1^*}\Gamma_2, \ldots, D_{\Gamma_1^*} \ldots D_{\Gamma^*_{n-1}} \Gamma_n) \quad (n \geq 1) \end{cases}$$

and (denoting by $P_n(\gamma) = P_n$ the orthogonal projection of K_+ onto K_n, $n \geq 1$)

$$(2.18) \qquad \begin{cases} R_\infty(\gamma) = R_\infty : K_+ \to H' \\ R_\infty := \text{s-}\lim_{n\to\infty} R_n P_n. \end{cases}$$

Using the notation of Section 2.1 we have, for $n \geq 1$, $R_n = R_\infty^{[n]}$, and we will consider the operators:

$$(2.19)_n \qquad \alpha_n(\gamma) = \alpha_n := \alpha(R_n)$$

$$(2.20)_n \qquad \beta_n(\gamma) = \beta_n := \beta(R_n)$$

$$(2.21)_n \qquad D_n(\gamma) = D_n := D_n(R_\infty)$$

$$(2.21)_\infty \qquad D_\infty(\gamma) = D_\infty := D_\infty(R_\infty)$$

(2.22) $$\alpha_\infty(\gamma) = \alpha_\infty := \alpha(R_\infty)$$

(2.23)$_n$ $$G_n(\gamma) = G_n := G_n(R_\infty)$$

(2.23)$_\infty$ $$G_\infty(\gamma) = G_\infty(R_\infty)$$

(2.24) $$\beta_\infty(\gamma) = \beta_\infty := \beta(R_\infty),$$

defined by formulas (2.3), (2.4), (2.5)$_n$, (2.6), (2.7), (2.8)$_n$, (2.9), (2.10), respectively.

Let us note that for every $k \geq 2$, $\gamma^{(k)} = \{\Gamma_p\}_{p=k}^{\infty}$ is a $(D_{\Gamma_{k-1}}, D_{\Gamma_{k-1}^*})$-choice sequence; the upper index (k) will indicate the objects associated by (2.15)$_n$-(2.24) to $\gamma^{(k)}$ (e.g. $K_n^{(k)} := K_{n-k+1}(\gamma^{(k)})$, $R_n^{(k)} := R_{n-k+1}(\gamma^{(k)})$, and so on).

Similar considerations can be made for "column" operators associated to γ. The simplest way of thinking them is to take the adjoint choice sequence $\gamma^* = \{\Gamma_n^*\}_{n=1}^{\infty}$, to consider the "row" objects associated to γ^*, and to take their adjoints. We will use the symbol "$\circ$" for denoting these "column" objects. For example, for $n \geq 1$, $\mathring{R}_n(\gamma) = \mathring{R}_n : H \to \bigoplus_{p=0}^{n-1} D_{\Gamma_p^*}$, $\mathring{R}_n := (\Gamma_1, \Gamma_2 D_{\Gamma_1}, \ldots, \Gamma_n D_{\Gamma_{n-1}} \ldots D_{\Gamma_1})^t$, (t standing for matrix transpose); $\mathring{\alpha}_\infty(\gamma) = \mathring{\alpha}_\infty : D_{\mathring{R}_\infty^*} \to \bigoplus_{p=1}^{\infty} D_{\Gamma_p^*}$, $\mathring{\alpha}_\infty D_{\mathring{R}_\infty^*} := \mathring{D}_\infty$; and so on.

The use of choice sequences in indexing contractive intertwining dilations and positive Toeplitz forms need also the following notation. (For simplicity, we will consider here only the case we will use in the sequel, namely $H = H'$.) For a fixed choice sequence $\gamma = \{\Gamma_n\}_{n=1}^{\infty}$, define for every $n \geq 1$ and $1 \leq k \leq n$

$$J_{n,k}(\gamma) = J_{n,k} :$$

(2.25)$_{n,k}$
$$: (H \oplus D_{\Gamma_1} \oplus \ldots \oplus D_{\Gamma_{k-2}}) \oplus (D_{\Gamma_{k-1}} \oplus D_{\Gamma_k^*}) \oplus (D_{\Gamma_{k+1}} \oplus \ldots \oplus D_{\Gamma_n}) \to$$
$$\to (H \oplus D_{\Gamma_1} \oplus \ldots \oplus D_{\Gamma_{k-2}}) \oplus (D_{\Gamma_{k-1}^*} \oplus D_{\Gamma_k}) \oplus (D_{\Gamma_{k+1}} \oplus \ldots \oplus D_{\Gamma_n})$$

$$J_{n,k} := I \oplus J(\Gamma_k) \oplus I,$$

where I stands for the identity operator on any space, and some parantheses in the direct sums may disappear (namely, the first and the third, for k = n = 1, the first for k = 1 and $n \geq 2$, and the third for k = n).

Finally, consider the unitary operators

(2.26)$_o$ $$V_o = I : H \to H,$$

$$(2.26)_n \quad \begin{cases} V_n(\gamma) = V_n : H \oplus D_{\Gamma_1} \oplus \cdots \oplus D_{\Gamma_{n-1}} \oplus D_{\Gamma_n^*} \to H \oplus D_{\Gamma_1} \oplus \cdots \oplus D_{\Gamma_n} \\ V_n := J_{n,1} J_{n,2} \cdots J_{n,n}, \quad (n \geq 1). \end{cases}$$

In [12], the following connections were proved.

First of all, for each $n \geq 1$, with respect to the decompositions $(H \oplus D_{\Gamma_1} \oplus \cdots \oplus D_{\Gamma_{n-1}}) \oplus D_{\Gamma_n^*}$ and $H \oplus (D_{\Gamma_1} \oplus \cdots \oplus D_{\Gamma_n})$, the operator V_n has the matrix

$$(2.27)_n \qquad V_n = \begin{bmatrix} R_n & G_n^* \\ D_n & A_n \end{bmatrix},$$

where R_n, G_n and D_n are defined by $(2.17)_n$, $(2.23)_n$, $(2.21)_n$, respectively, while A_n is the operator:

$$(2.28)_n \quad \begin{cases} A_n : D_{\Gamma_n^*} \to D_{\Gamma_1} \oplus D_{\Gamma_n} \\ A_n = -(\Gamma_1^* D_{\Gamma_2^*} \cdots D_{\Gamma_n^*}, \Gamma_2^* D_{\Gamma_3^*} \cdots D_{\Gamma_n^*}, \ldots, \Gamma_{n-1}^* D_{\Gamma_n^*}, \Gamma_n^*)^t. \end{cases}$$

Using these, it follows that definig:

$$(2.29)_1 \qquad W_1 = W_1(\gamma) = \Gamma_1 : H \to H,$$

and

$$(2.29)_n \quad \begin{cases} W_n(\gamma) = W_n : K_n \to K_n \\ W_n := V_{n-1}(I \oplus \Gamma_n), \quad (n \geq 2), \end{cases}$$

then the operator

$$(2.30) \quad \begin{cases} W_+(\gamma) = W_+ : K_+ \to K_+ \\ W_+ := \text{s-}\lim_{n \to \infty} W_n P_n \end{cases}$$

is an isometry which verifies

$$(2.31) \qquad W_+ = \begin{bmatrix} R_\infty \\ D_\infty \end{bmatrix}.$$

This operator W_+ is connected with the adequate isometries considered in [9].

In connection with $(2.28)_{n=1}^{\infty}$, we will need the following remark.

LEMMA 2.1. *For every* $n \geq 1$, *we have*

$$\ker(I - A_n A_n^*) = \{0\}. \tag{2.32$_n$}$$

PROOF. The formulas $(2.28)_{n=1}^{\infty}$ imply the recurrence relations

$$A_n = (A_{n-1} D_{\Gamma_n^*}, -\Gamma_n^*)^t, \quad n \geq 2. \tag{2.33$_n$}$$

We will prove only $(2.32)_2$; an induction argument, based on $(2.33)_{n=1}^{\infty}$, settles the whole matter.

Thus, we have to prove that $\mathcal{D}_{A_2^*} = \mathcal{D}_{\Gamma_1} \oplus \mathcal{D}_{\Gamma_2}$, where $A_2^* : \mathcal{D}_{\Gamma_1} \oplus \mathcal{D}_{\Gamma_2} \to \mathcal{D}_{\Gamma_2^*}$ is given by $A_2^* = -(D_{\Gamma_2^*}\Gamma_1, \Gamma_2)$. For this, we will use the analysis of "row operators" as described Section 2.1. Denoting by $T = (\Gamma_2, D_{\Gamma_2^*}\Gamma_1) \in L(\mathcal{D}_{\Gamma_2} \oplus \mathcal{D}_{\Gamma_1}, \mathcal{D}_{\Gamma_2^*})$, our goal is equivalent with proving that $\ker D_T = \{0\}$. But $(\Gamma'_2, D_{\Gamma'^*_2}\Gamma'_1)$, where $\Gamma'_2 = \Gamma_2|\mathcal{D}_{\Gamma_2} \in L(\mathcal{D}_{\Gamma_2}, \mathcal{D}_{\Gamma_2^*})$ and $\Gamma'_1 = P_{\mathcal{D}_{\Gamma_2^*}}^{\mathcal{D}_{\Gamma_1^*}} \Gamma_1 | \mathcal{D}_{\Gamma_1} \in L(\mathcal{D}_{\Gamma_1}, \mathcal{D}_{\Gamma_2^*})$ is the canonical form of T described in $(2.2)_{p=1}^{n}$. It is clear from the definitions that $\ker D_{\Gamma'_2} = \{0\}$ and $\ker D_{\Gamma'_1} = \{0\}$. From the proof of (2.3) (see [7], or [3]) it follows that $\ker D_{\Gamma'_2} = \{0\}$ and $\ker D_{\Gamma'_1} = \{0\}$ imply that $\ker D_T = \{0\}$. Therefore $(2.32)_2$ is proved, and this finishes the proof of the lemma.

2.3. Positive Toeplitz forms

For a Hilbert space H, a (H-) positive Toeplitz form is a sequence of operators $T = \{S_n\}_{n=1}^{\infty}$, ($S_n \in L(H)$), such that for each $n \geq 1$, the operator

$$T_n : \bigoplus_{p=1}^{n+1} H_p \to \bigoplus_{p=1}^{n+1} H_p, \quad (H_p = H, \ \forall\, p \in \mathbb{N}),$$

$$T_n := \begin{bmatrix} I & S_1 & S_2 & \cdots & S_n \\ S_1^* & I & S_1 & \cdots & S_{n-1} \\ S_2^* & S_1^* & I & \cdots & \\ \vdots & \vdots & \vdots & & \vdots \\ S_n^* & S_{n-1}^* & & & I \end{bmatrix} \tag{2.34$_n$}$$

is positive. From [12], Theorem 1.9, it follows that there exists a one-to-one correspondence between the set of H-positive Toeplitz forms $T = \{S_n\}_{n=1}^{\infty}$ and the set of

(H,H)-choice sequences $\gamma = \{\Gamma_n\}_{n=1}^{\infty}$, given by the formulas

$$(2.35)_1 \qquad S_1 = \Gamma_1$$

$$(2.35)_n \qquad S_n = R_{n-1}(\gamma)U_{n-1}(\gamma)\mathring{R}_{n-1}(\gamma) + D_{\Gamma_1^*}\dots D_{\Gamma_{n-1}^*}\Gamma_n D_{\Gamma_{n-1}}\dots D_{\Gamma_1}, \quad (n \geq 2).$$

We will denote the corresponding objects by $\mathcal{T}(\gamma)$ and $\gamma(\mathcal{T})$.

2.4. Naimark dilation

A (H-) semispectral measure F on the unit circle **T** is a linear positive map

$$(2.36) \qquad F : C(\mathbf{T}) \to L(H),$$

where $C(\mathbf{T})$ denotes the set of (complex) continuous functions on **T**. We will consider only the case when $F(1) = I$; this is not an essential restriction, and it simplifies some unimportant complications. The positive Fourier coefficients of F,

$$(2.37)_n \qquad S_n(F) = S_n := F(\chi_n), \quad n \geq 1,$$

where $\chi_n(e^{it}) = e^{int}$, define a H-positive Toeplitz form $\mathcal{T}(F) = \mathcal{T} = \{S_n\}_{n=1}^{\infty}$. The choice sequence $\gamma(\mathcal{T}(F))$ will be also denoted by $\gamma(F)$, and in writing the objects associated to $\gamma(F)$ by Section 2.2 we will use sometime F instead of $\gamma(F)$.

Let us recall the structure of the Naimark dilation (see [22], Section 1.7, for the definition and the construction) of F, in terms of $\gamma(F)$, as it was described in [13]. To this end, note that

$$(2.38)_n \qquad S_n(F) = P_1 W_+^n(F) P_1$$

(see [13], Lemma 2.3), so, having in mind (2.31), the structure of the Naimark dilation of F goes as follows. (We will fix F and omit the writing of F, or $\gamma(F)$, in denoting the objects of Section 2.2 associated to $\gamma(F)$.)

Take

$$(2.39) \qquad \mathcal{D}_* := \operatorname{Ran} G_\infty^-$$

and

$$(2.40) \qquad \mathcal{K} := \dots \oplus \mathcal{D}_* \oplus \mathcal{D}_* \oplus \mathcal{K}_+ .$$

Then define

$$(2.41) \qquad \begin{cases} W : \mathcal{K}(= (\dots \oplus \mathcal{D}_*) \oplus (\mathcal{D}_* \oplus \mathcal{K}_+)) \to \mathcal{K}(=(\dots \oplus \mathcal{D}_*) \oplus \mathcal{K}_+) \\ W := I \oplus W_{red}, \end{cases}$$

where $W_{red} : D_* \oplus K_+ \to K_+$ is defined by

$$(2.42)\qquad W_{red} := \begin{bmatrix} I & 0 \\ 0 & \alpha_\infty \end{bmatrix} J(R_\infty) \begin{bmatrix} 0 & I \\ \beta_\infty^* & 0 \end{bmatrix}.$$

It follows that W is a unitary operator whose spectral measure is the Naimark dilation of F. Note that W has the matrix

$$(2.43)\qquad W = \begin{bmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & I & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & G_\infty^{1/2} & \Gamma_1 & D_{\Gamma_1^*}\Gamma_2 & D_{\Gamma_1^*}D_{\Gamma_2^*}\Gamma_3 & \\ \cdots & 0 & -Z_1 & D_{\Gamma_1} & -\Gamma_1^*\Gamma_2 & -\Gamma_1^* D_{\Gamma_2^*}\Gamma_3 & \\ \cdots & 0 & -Z_2 & 0 & D_{\Gamma_2} & -\Gamma_2^*\Gamma_3 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

where the operators Z_n, $(n \geq 1)$, are defined by

$$(2.44)_n\qquad \begin{cases} Z_n : D_* \to D_{\Gamma_n} \\ Z_n := \Gamma_n^* H_{n+1} \end{cases}$$

using the operators

$$(2.45)_n\qquad \begin{cases} H_n : D_* \to D_{\Gamma_{n-1}^*} \\ H_n := G_\infty^{(n)\frac{1}{2}} C_{n-1} \cdots C_1, \quad (n \geq 2) \end{cases}$$

which are constructed by partial isometries

$$(2.46)_n\qquad \begin{cases} C_n : \overline{\operatorname{Ran} G_\infty^{(n)}} \to \overline{\operatorname{Ran} G_\infty^{(n+1)}} \\ C_n G_\infty^{(n)\frac{1}{2}} := G_\infty^{(n+1)\frac{1}{2}} D_{\Gamma_n^*}. \end{cases}$$

It is easy to see that, for each $n \geq 1$, the compression of W to K_n is exactly W_n defined by $(2.29)_n$, and that the restriction of W to K_+ is W_+ defined by (2.30).

3. GAUSSIAN STATIONARY PROCESSES

The aim of this short section is to recall the setting of studying the regularity conditions for Gaussian stationary processes, ([16], [25], [21], [15]) and to point out the connections of this with the structure of the Naimark dilation of semispectral measures.

Let H be a Hilbert space, and $\xi = \{\xi_n(\omega)\}_{n=-\infty}^{\infty}$ a H-valued Gaussian stationary process with discrete time and zero mean value (where ω varies in a given probability space). Then ξ is completely determined by its correlation matrix $T(\xi)$, which is a H-positive Toeplitz form (see for example [21]). We will also suppose that the (1,1) entry of $T(\xi)$ is I, which means a normalization of the spectral measure of ξ. Thus, it is possible to apply to $T(\xi)$ the analysis described in Section 2. So, the process ξ is completely determined by the H-choice sequence $\gamma(T(\xi)) = \gamma(\xi)$ (see Section 2.3); moreover, the spectral mesure of ξ is the spectral measure of the unitary operator $W \in L(K)$, where K and W are given by (2.40) and (2.43). The process itself can be thought as the sequence of the compressions of W^n to H.

Our intentionis to study some regularity conditions on ξ using the previous structure involving choice sequences. For a fixed ξ, consider its correlation matrix T and the associated choice sequence γ; we will use the notation from Section 2 for these objects. The paper [16] introduces in the analysis of processes the following operators

$$B_n(\xi) = B_n = P_{-(n)}P_+P_{-(n)} \in L(K), \quad (n \geq 1) \tag{3.1$_n$}$$

where

$$P_{-(n)} = P^K_{K_{-(n)}} \quad ; \quad K_{-(n)} = \overline{\bigvee_{k=n}^{\infty} W^{*n}H}, \quad (n \geq 1) \tag{3.2$_n$}$$

and

$$P_+ = P^K_{K_+}. \tag{3.3}$$

The fact that ξ is a Gaussian stationary process implies that for each $n \geq 2$ B_n differs from B_1 only by a finite rang operator (see for example [21], Section IV.2). Therefore, our analysis will be concentrated on $B_1 = P_-P_+P_-$, (where $P_- = P_{-(1)}$ is the projection of K onto $K_- = K_{-(1)}$).

There are quite a few notions of regularity for stationary processes [21]; some of them are equivalent in the case of Gaussian processes. We remind only some final results which use B_1: A Gaussian stationary process is completely regular iff B_1 is compact; and it is informationally regular iff B_1 is trace-class ([21], Ch.IV). In these cases, the so-called the regularity coefficient ρ and the information regularity coefficient I are defined as:

(3.4) $$n \to \rho(n) = \|B_n\|,$$

(3.5) $$n \to I(n) = -\tfrac{1}{2}\sum_k \ln(1-\lambda_k^{(n)}), \quad (n \geq 1),$$

where $\{\lambda_k^{(n)}\}$ are the eigenvalues of B_n.

The above facts suggest that a triangular factorization (which involves the choice sequence $\gamma(\xi)$) of $I-B_1$ will be very useful in this respect. This will be done in next section; the applications will be considered in Sections 5–6.

4. MAIN TECHNICAL RESULT

Our analysis of the operator B_1 (attached to a fixed Gaussian stationary process ξ) will use an approximation procedure suggested by the special stucture of the space K. (The objects from Section 3 which appear here are all attached to the choice sequence $\gamma(\xi)$.) In this respect, consider first (for every $n \geq 1$)

$(4.1)_n$ $$K_n^- = \bigvee_{k=1}^{n} W^{*k} H, \quad P_n^- = P_{K_n^-}^{K}$$

and the operator

$(4.2)_n$ $$B_{1,n} = P_n^- P_+ P_n^- .$$

Then we have

$(4.3)_n$ $$B_{1,n} = P_n^- P_+ P_n^- = P_n^- B_1 P_n^-, \quad (n \geq 1).$$

This explains our interest in studying the operators $\{B_{1,n}\}_{n=1}^{\infty}$. We start with an useful result on $\{P_n^-\}_{n=1}^{\infty}$. For explaining it, let define

$(4.4)_n$ $$Q_n = P_n P_+ = P_{K_n}^{K}, \quad (n \geq 1).$$

Now, we can state:

PROPOSITION 4.1. *For every* $n \geq 1$,

$(4.5)_n$ $$P_n^- = W^{*n} Q_n W^n .$$

The proof will use the following result.

LEMMA 4.2. *For every* $n \geq 1$, *we have*

$(4.6)_n$ $$\operatorname*{s-lim}_{m\to\infty} [(I-Q_1)(I-WQ_nW^*)(I-Q_1)]^m = I - Q_{n+1} .$$

PROOF. Having in mind that for every selfadjoint contraction T one has $\operatorname{s-lim}_{m\to\infty} T^n = P_{\ker(I-T)}$, we will prove firstly that for every $n \geq 1$

$$(4.7)_n \qquad (I - Q_1(I - WQ_nW^*)(I - Q_1) = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_nA_n^* & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

where the matrix is written with respect to the decomposition of K as $(\ldots \oplus D_* \oplus D_*) \oplus \oplus H \oplus (D_{\Gamma_1} \oplus \ldots \oplus D_{\Gamma_n}) \oplus (D_{\Gamma_{n+1}}) \oplus \ldots)$ and A_n is defined in $(2.27)_n$.

The formula $(4.7)_1$ is an immediate computation.

Fix now an arbitrary $n \geq 2$. Using (2.43), $(4.4)_n$, and the remark made after the formula $(2.46)_n$, we infer that

$$(4.8)_n \qquad WQ_nW^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & W_nW_n^* & W_n\begin{bmatrix} 0_{K_{n-1}} \\ D_{\Gamma_n} \end{bmatrix} & 0 \\ 0 & (0_{K_{n-1}} \; D_{\Gamma_n})W_n^* & D^2_{\Gamma_n} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where the right hand side is written with respect to the decomposition of K as $(\ldots D_* \oplus D_*) \oplus K_n \oplus D_{\Gamma_n} \oplus (D_{\Gamma_{n+1}} \oplus \ldots)$. Using now $(2.29)_n$ and $(2.27)_{n-1}$, we obtain from $(4.8)_n$ that

$$(4.9)_n \qquad WQ_nW^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & 0 \\ 0 & * & D_{n-1}D^*_{n-1} + A_{n-1}\Gamma_n\Gamma_n^*A^*_{n-1} & A_{n-1}\Gamma_nD_{\Gamma_n} & 0 \\ 0 & * & D_{\Gamma_n}\Gamma_n^*A^*_{n-1} & D_{\Gamma_n}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the right hand side is written with respect to the decomposition of K as $(\ldots \oplus D_* \oplus D_*) \oplus H \oplus (D_{\Gamma_1} \oplus \ldots \oplus D_{\Gamma_{n-1}}) \oplus D_{\Gamma_n} \oplus (D_{\Gamma_{n+1}} \oplus \ldots)$, and the entries marked by "$*$" are not important in the subsequent computation.

From $(4.9)_n$ it follows immediately that

$$(I - Q_1)(I - WQ_n W^*)(I - Q_1) =$$

$$(4.10)_n \qquad = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I - D_{n-1}D_{n-1}^* - A_{n-1}\Gamma_n\Gamma_n^*A_{n-1}^* & -A_{n-1}\Gamma_n D_{\Gamma_n} & 0 \\ 0 & 0 & -D_{\Gamma_n}\Gamma_n^*A_{n-1}^* & \Gamma_n^*\Gamma_n & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

with respect to the decomposition of K considered in $(4.9)_n$.

The formula $(4.10)_n$ shows that $(4.7)_n$ will be implied by the equality:

$$(4.11)_n \qquad A_n A_n^* = \begin{bmatrix} I - D_{n-1}D_{n-1}^* - A_{n-1}\Gamma_n\Gamma_n^*A_{n-1}^* & -A_{n-1}\Gamma_n D_{\Gamma_n} \\ -D_{\Gamma_n}\Gamma_n^*A_{n-1}^* & \Gamma_n^*\Gamma_n \end{bmatrix},$$

where the matrix in the right hand side is written with respect to the decomposition of $\mathcal{D}_{\Gamma_1}\oplus\dots\oplus\mathcal{D}_{\Gamma_n}$ as $(\mathcal{D}_{\Gamma_1}\oplus\dots\oplus\mathcal{D}_{\Gamma_{n-1}})\oplus\mathcal{D}_{\Gamma_n}$.

From $(2.33)_n$, it follows that

$$(4.12)_n \qquad A_n A_n^* = \begin{bmatrix} A_{n-1}D_{\Gamma_n^*}^2A_{n-1}^* & -A_{n-1}D_{\Gamma_n^*}\Gamma_n^* \\ -\Gamma_n^*D_{\Gamma_n^*}A_{n-1}^* & \Gamma_n^*\Gamma_n \end{bmatrix}.$$

Comparing $(4.11)_n$ and $(4.12)_n$ one note that it remains to show that (we use the well-known fact that, for every contraction T, $TD_T = D_{T^*}T$) for every $k \geq 1$

$$(4.13)_k \qquad A_k A_k^* = I - D_k D_k^*.$$

This can be easily done by induction. For $k=1$, this is the trivial equality $\Gamma_1^*\Gamma_1 = I - D_{\Gamma_1}^2$. Suppose now that $(4.13)_k$ is true for a fixed $k \geq 1$. Then, using $(4.12)_{k+1}$ and $(4.13)_k$, we have that

$$A_{k+1}A_{k+1}^* = \begin{bmatrix} I - D_kD_k^* - A_k\Gamma_{k+1}^*\Gamma_{k+1}A_k^* & -A_kD_{\Gamma_{k+1}^*}\Gamma_{k+1}^* \\ -\Gamma_{k+1}^*D_{\Gamma_{k+1}^*}A_k^* & \Gamma_{k+1}^*\Gamma_{k+1} \end{bmatrix}.$$

This is exactly $I - D_{k+1}D_{k+1}^*$, if we note that from the definition of D_n (see $(2.21)_n$ and

(2.3)) it follows that for each $m \geq 2$

$$(4.14)_m \qquad D_m = \begin{bmatrix} D_{m-1} & A_m \Gamma_{m+1} \\ 0 & D_{\Gamma_m} \end{bmatrix},$$

where the matrix is written with respect to the decomposition of $D_{\Gamma_1} \oplus \dots \oplus D_{\Gamma_m}$ as $(D_{\Gamma_1} \oplus \dots \oplus D_{\Gamma_{m-1}}) \oplus D_{\Gamma_m}$.

This finishes the proof of $(4.7)_n$, $n \geq 1$. We have then, for every $n \geq 1$, that

$$(4.15)_n \qquad \underset{m \to \infty}{\text{s-lim}} \, [(I - Q_1)(I - WQ_nW^*)(I - Q_1)]^m = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_n' & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

where A_n' is the projection of $D_{\Gamma_1} \oplus \dots \oplus D_{\Gamma_n}$ onto $\ker(I - A_nA_n^*)$, and the matrix in the right hand sinde is written with respect to the decomposition of K indicated in $(4.7)_n$.

The proof of the lemma is now completed using Lemma 2.1.

PROOF OF PROPOSITION 4.1. First, let us remark that

$$(4.16)_n \qquad P^K_{W^{*n}H} = W^{*n}Q_1W^n, \quad (n \geq 1).$$

This follows from the fact that for each $n \geq 1$, $W^{*n}|H \in L(H,K)$ is an isometric operator.

We will prove $(4.5)_n$, $(n \geq 1)$, by induction. For $n=1$, $(4.5)_1$ is exactly $(4.16)_1$. Suppose now that $(4.5)_n$ is true for an $n \geq 1$. Then, from $(4.1)_{n+1}$, $(4.5)_n$ and $(4.16)_{n+1}$, it follows that

$$(4.17)_{n+1} \qquad P^-_{n+1} = (W^{*n}Q_nW^n) \bigvee (W^{*(n+1)}Q_1W^{n+1}),$$

where "$\bigvee$" stands for the l.u.b. of the two projections. Denote in this proof $E = W^{*n}(I - Q_n)W^n$ and $F = W^{*(n+1)}(I - Q_1)W^{n+1}$. Then, using von Neumann formula (see for example [18], Problem 122), we have

$$(4.18)_{n+1} \qquad I - P^-_{n+1} = \underset{m \to \infty}{\text{s-lim}} \, (FEF)^m.$$

An immediate computation shows that

$$(4.19)^m_{n+1} \qquad (FEF)^m = W^{*(n+1)}[(I - Q_1)W(I - Q_n)W^*(I - Q_1)]^m W^{n+1}$$

for every $m \geq 1$. From this, Lemma 4.1 and $(4.18)_{n+1}$ it follows that

$$(4.20)_{n+1} \qquad I - P^{-}_{n+1} = W^{*(n+1)}(I - Q_{n+1})W^{n+1},$$

which is $(4.5)_{n+1}$. This concludes the proof of the proposition.

We pass now to our main goal, namely the proof of some triangular factorization results. The first attempt will be done on the operators $D^2_{W^{*n}_n}$, $n \geq 1$. The operators which are really interesting for applications are those of the more general form $D^2_{W^{*m}_n P_m}$, $n \geq 1$, $1 \leq m \leq n$ (the previous case is, essentially, that where k=n). However the methods used in the proof of this particular case are strong enough to settle the whole matter.

Fix now (for the rest of this section) an $n \geq 1$. From $(2.29)_n$, $(2.26)_{n-1}$ and $(2.25)_{n-1,k}$, $(1 \leq k \leq n-1)$, it follows that

$$W^2_n = V_{n-1}(I \oplus \Gamma_n)J_{n-1,1}J_{n-1,2}\cdots J_{n-1,n-1}(I \oplus \Gamma_n) =$$
$$= V_{n-1}(J_{n-1,1}J_{n-1,2}\cdots J_{n-1,n-2})(I \oplus \Gamma_n)J_{n-1,n-1}(I \oplus \Gamma_n),$$

and therefore

$$W^n_n = V_{n-1}(J_{n-1,1}J_{n-1,2}\cdots J_{n-1,n-2})(J_{n-1,1}J_{n-1,2}\cdots J_{n-1,n-3})\ldots(J_{n-1,1})\times$$
$$(4.21)_n \qquad \times(I \oplus \Gamma_n)J_{n-1,n-1}(I \oplus \Gamma_n)J_{n-1,n-2}J_{n-1,n-1}(I \oplus \Gamma_n)\ldots\times$$
$$\times(I \oplus \Gamma_n)J_{n-1,1}\cdots J_{n-1,n-1}(I \oplus \Gamma_n).$$

The formula $(4.21)_n$ shows that

$$(4.22)_n \qquad W^n_n = U'_{n-1}K_n$$

where U'_{n-1} is a unitary operator from $H \oplus D_{\Gamma^*_n} \oplus \cdots \oplus D_{\Gamma^*_{n-1}}$ onto K_n and the operator K_n acting between K_n and $H \oplus D_{\Gamma^*_n} \oplus \cdots \oplus D_{\Gamma^*_{n-1}}$ is defined by

$$K_n = (I \oplus \Gamma_n)J_{n-1,n-1}(I \oplus \Gamma_n)J_{n-1,n-2}J_{n-1,n-1}(I \oplus \Gamma_n)\ldots\times$$
$$(4.23)_n \qquad \times(I \oplus \Gamma_n)J_{n-1,1}\cdots J_{n-1,n-1}(I \oplus \Gamma_n).$$

In $(4.21)_n$ - and everywhere in the rest of this section - there is a delicate point concerning the notation $J_{n-1,k}$, $1 \leq k \leq n-1$. Sometimes this denotes an operator which has only the non-identical part as in $(2.25)_{n-1,k}$; the spaces between it acts result from the context. For example, the last $J_{n-1,1}$ from the formula of U'_{n-1} acts between $(H\oplus D_{\Gamma_1^*}\oplus D_{\Gamma_2^*}\oplus\dots\oplus D_{\Gamma_{n-1}^*})$ and $(H\oplus D_{\Gamma_1}\oplus D_{\Gamma_2^*}\oplus\dots\oplus D_{\Gamma_{n-1}^*})$, and so on. The same problem arises about the space on which acts the identity in $I\oplus\Gamma_n$, which also results from the context. For keeping reasonable notation we will disregard this matter - a fact which will make no difficulties.)

The operator K_n has several interesting properties. For describing them, remember the meaning of an upper index, as defined after formula (2.24); for example $K_n^{(1)}=K_n$; and $K_n^{(2)}$ acts between $D_{\Gamma_1}\oplus\dots\oplus D_{\Gamma_{n-1}}$ and $D_{\Gamma_1^*}\oplus\dots\oplus D_{\Gamma_{n-1}^*}$ by

$$(4.24)_n \qquad K_n^{(2)}=(I\oplus\Gamma_n)J^{(2)}_{n-1,n-1}(I\oplus\Gamma_n)J^{(2)}_{n-1,n-2}J^{(2)}_{n-1,n-1}(I\oplus\Gamma_n)\dots\times$$

$$\times(I\oplus\Gamma_n)J^{(2)}_{n-1,2}\dots J^{(2)}_{n-1,n-1}(I\oplus\Gamma_n).$$

It is clear that $K_n^{(n)}=\Gamma_n$, and we make the convention that $K_n^{(n+1)}$ is the zero-dimensional operator. Remember also that the symbol "o" refers to the objects associated to "column" operators.

LEMMA 4.3. *The operator* K_n *has the following properties:*

(i) $K_n=(I\oplus K_n^{(2)})W_n$;

(ii) $K_n=(I\oplus\Gamma_n)J_{n-1,n-1}\dots J_{n-1,1}(I\oplus K_n^{(2)})$;

(iii) *The first column of* K_n *is* $\mathring{R}_n$;

(iv) $K_n\begin{bmatrix}0&0\\I&0\end{bmatrix}=\mathring{D}_n(K_n^{(2)}\oplus 0)$,

where the matrix in the left hand side is written with respect to the decompositions $(D_{\Gamma_1}\oplus\dots\oplus D_{\Gamma_{n-1}})\oplus D_{\Gamma_n}$ *and* $H\oplus(D_{\Gamma_1}\oplus\dots\oplus D_{\Gamma_{n-1}})$.

PROOF. The equalities in (i) and (ii) are simple consequences of the definition. The third assertion follows by induction, using that $\mathring{R}_n=(\Gamma_1,\mathring{R}_n^{(2)}D_{\Gamma_1})^t$ and that

$$(4.25)_n \qquad K_n\begin{bmatrix}I\\0_{n-1}\end{bmatrix}=(I\oplus K_n^{(2)})V_{n-1}(I\oplus\Gamma_n)\begin{bmatrix}I\\0_{n-1}\end{bmatrix}=$$

$$=(\Gamma_1,\ K_n^{(2)}\begin{bmatrix}I\\0_{n-2}\end{bmatrix}D_{\Gamma_1})^t,$$

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where we used the first assertion, and the structure of W_n and V_{n-1}.

For proving (iv), note that using $(2.27)_{n-1}$, we have

$$(4.26)_{n-1} \qquad J_{n-1,n-1}\cdots J_{n-1,1} = \begin{bmatrix} \mathring{R}_{n-1} & \mathring{D}_{n-1} \\ \mathring{G}^*_{n-1} & \mathring{A}_{n-1} \end{bmatrix},$$

similarly, from $(4.14)_n$ it follows that

$$(4.27)_n \qquad \mathring{D}_n = \begin{bmatrix} \mathring{D}_{n-1} & 0 \\ \Gamma_n \mathring{D}_{n-1} & D_{\Gamma_n^*} \end{bmatrix}.$$

The relation (IV) results now by easy matrix computations, using (ii), $(4.26)_{n-1}$ and $(4.27)_n$. The lemma is now completely proved.

By virtue of $(4.22)_n$, the factorization of $D^2_{K_n^*}$ will imply that of $D^2_{W_n^*}$. This is the reason of the next lemma.

LEMMA 4.4. *If* $n \geq 2$, *then (on* $D_{K_n^*}$*)*

$$(4.28)_n \qquad D_{K_n^*} = U''_n \begin{bmatrix} G_n & * \\ 0 & (D_{K_n^{(3)*}} \oplus I)\mathring{D}_n^{(2)*} \end{bmatrix},$$

where the matrix in the right hand side is written with respect to the decompositions $H \oplus (D_{\Gamma_1^*} \oplus \cdots \oplus D_{\Gamma_{n-1}^*})$ *and* $D_{\Gamma_n^*} \oplus (D_{\Gamma_2^*} \oplus \cdots \oplus D_{\Gamma_n^*})$, *and the operator*

$$(4.29)_n \qquad U''_n : D_{\Gamma_n^*} \oplus (D_{K_n^{(3)*}} \oplus D_{\Gamma_n^*}) \to D_{K_n^*}$$

is a unitary operator.

PROOF. We will obtain $(4.28)_n$ using the method of dealing with two-by-two matrices as described at the end of Section 2.1. For this, note that (2.43) implies:

$$(4.30)_n \qquad W_n = \begin{bmatrix} \Gamma_1 & D_{\Gamma_1^*} R_n^{(2)} \\ \begin{bmatrix} I \\ 0_{n-2} \end{bmatrix} D_{\Gamma_1} & -\begin{bmatrix} I \\ 0_{n-2} \end{bmatrix} \Gamma_1^* R_n^{(2)} + \begin{bmatrix} 0 & 0 \\ I_{n-2} & 0 \end{bmatrix} \mathring{D}_n^{(2)} \end{bmatrix},$$

where the bigger matrix in the right hand side is written with respect to the decomposition of K_n as $H\oplus(D_{\Gamma_1}\oplus\ldots\oplus D_{\Gamma_{n-1}})$. From Lemma 4.3 (i) and $(4.30)_n$ it follows that

$$(4.31)_n \quad K_n = \begin{bmatrix} \Gamma_1 & D_{\Gamma_1^*}R_n^{(2)} \\ K_n^{(2)}\begin{bmatrix} I \\ 0_{n-2}\end{bmatrix}D_{\Gamma_1} & -K_n^{(2)}\begin{bmatrix} I \\ 0_{n-2}\end{bmatrix}\Gamma_1^*R_n^{(2)} + K_n^{(2)}\begin{bmatrix} 0 & 0 \\ I_{n-2} & 0\end{bmatrix}D_n^{(2)} \end{bmatrix}.$$

Now, Lemma 4.3 (iii) and (iv) implies that $(4.31)_n$ is in fact

$$(4.32)_n \qquad K_n = \begin{bmatrix} \Gamma_1 & D_{\Gamma_1^*}R_n^{(2)} \\ \mathring{R}_n^{(2)}D_{\Gamma_1} & -\mathring{R}_n^{(2)}\Gamma_1^*R_n^{(2)} + \mathring{D}_n^{(2)}(K_n^{(3)}\oplus 0)D_n^{(2)} \end{bmatrix}.$$

This form is quite close to the canonical form described in formulas (2.11). We only have to note that $D_n^{(2)} = \alpha_n^{(2)}D_{R_n^{(2)}}$ and $\mathring{D}_n^{(2)} = D_{\mathring{R}_n^{(2)*}}\mathring{\alpha}_n^{(2)}$ (see $(2.21)_n$ and (2.3)), and to define the contraction

$$(4.33)_n \qquad \Delta_n = \mathring{\alpha}_n^{(2)}(K_n^{(3)}\oplus 0)\alpha_n^{(2)} \quad ;$$

then $(4.32)_n$ implies

$$(4.34)_n \qquad K_n = \begin{bmatrix} \Gamma_1 & D_{\Gamma_1^*}R_n^{(2)} \\ \mathring{R}_n^{(2)}D_{\Gamma_1} & -\mathring{R}_n^{(2)}\Gamma_1^*R_n^{(2)} + D_{\mathring{R}_n^{(2)*}}\Delta_n D_{R_n^{(2)}} \end{bmatrix},$$

which is the canonical form of K_n. We can use now (2.14) to obtain (on $D_{K_n^*}$) that

$$(4.35)_n \qquad D_{K_n^*} = \beta(K_n)^*\begin{bmatrix} D_{\mathring{R}_n^{(2)*}}D_{\Gamma_1^*} & * \\ 0 & D_{\Delta_n^*}D_{\mathring{R}_n^{(2)*}} \end{bmatrix}.$$

The relation $(4.33)_n$ implies that $D_{\Delta_n^*} = \mathring{\alpha}_n^{(2)}(D_{K_n^{(3)*}}\oplus I)\mathring{\alpha}_n^{(2)*}$. From $(2.20)_n$ and (2.4) it follows that $D_{R_n^{(2)*}} = \beta_n^{(2)*}D_{\Gamma_n^*}\ldots D_{\Gamma_n^*}$, so

$$(4.36)_n \qquad D_{K_n^*} = \beta(K_n)^*(\beta_n^{(2)*} \oplus \mathring{\alpha}_n^{(2)}) \begin{bmatrix} D_{\Gamma_n^*}\cdots D_{\Gamma_2^*}D_{\Gamma_1^*} & * \\ 0 & (D_{K_n^{(3)*}} \oplus I)\mathring{D}_n^{(2)*} \end{bmatrix},$$

which finishes the proof of the lemma (taking $U''_n = \beta(K_n)^*(\beta_n^{(2)*} \oplus \mathring{\alpha}_n^{(2)}) | D_{\Gamma_n^*} \oplus \oplus (D_{K_n^{(3)*}} \oplus D_{\Gamma_n^*})$.

REMARKS 4.5. 1). As $K_1 = \Gamma_1$, we have $D_{K_1^*} = D_{\Gamma_1^*}$. Moreover, it is easy to see that the convention made after $(4.24)_n$ agrees with Lemma 4.4. Indeed

$$K_2 = \begin{bmatrix} \Gamma_1 & D_{\Gamma_1^*}\Gamma_2 \\ \Gamma_2 D_{\Gamma_1} & -\Gamma_2\Gamma_1^*\Gamma_2 \end{bmatrix}$$

so $D_{K_2^*} \simeq D_{\Gamma_2^*} \oplus D_{\Gamma_2^*}$, the last matrix in $(4.36)_2$ being $\begin{bmatrix} D_{\Gamma_2^*}D_{\Gamma_1^*} & * \\ 0 & D_{\Gamma_2^*} \end{bmatrix}$.

2). The proof of Lemma 4.6 shows a nice feature of formulas (2.13) and (2.14), namely their usefulness in obtaining triangularization results.

PROPOSITION 4.6. *The operator $D^2_{W_n^{*n}} = I - W_n^n W_n^{*n}$ admits the factorization*

$$(4.37)_n \qquad I - W_n^n W_n^{*n} = U_n F_n F_n^* U_n^*,$$

where F_n is an upper triangular matrix from $H \oplus D_{\Gamma_1^*} \oplus \cdots \oplus D_{\Gamma_{n-1}^*}$ into $D_{\Gamma_n^*} \oplus \cdots \oplus D_{\Gamma_n^*}$ (n-times) with the diagonal $(G_n^{(1)}, G_n^{(2)}, \ldots, G_n^{(n)})$, and U_n is a unitary operator from $D_{\Gamma_n^*} \oplus \cdots \oplus D_{\Gamma_n^*}$ (n-times) onto $D_{W_n^{*n}}$.

PROOF. The formula $(4.22)_n$ shows that

$$(4.38)_n \qquad I - W_n^n W_n^{*n} = U'_{n-1} D^2_{K_n^*} U'^*_{n-1}.$$

If n=1 or n=2, Remark 4.5 settles the matter. If $n \geq 3$, a repeated use of Lemma 4.4

([(n-1)/2] - times) reduces the problem to operators of the form $K_n^{(n-1)}$ or $K_n^{(n)}$ which can be treated as in Remark 4.5. The structure of the diagonal of F_n follows by simple computations with upper triangular matrices.

THEOREM 4.7. *For every* $1 \leq m \leq n$, *the operator* $D^2_{W_n^{*m}P_m} = I - P_m W_n^m W_n^{*m} P_m$ *admits the factorization*

$$(4.39)_n^m \qquad I - P_m W_n^m W_n^{*m} P_m = U_{n,m} F_{n,m} F^*_{n,m} U^*_{n,m},$$

where $F_{n,m}$ *is an upper triangular matrix from* $(H \oplus \underbrace{D_{\Gamma_1^*} \oplus \dots \oplus D_{\Gamma_{m-1}^*}}_{(m-\text{times})}) \oplus (D_{\Gamma_m} \oplus \oplus D_{\Gamma_{m+1}} \oplus \dots)$ *into* $(D_{\Gamma_n^*} \oplus \dots \oplus D_{\Gamma_n^*}) \oplus D_{\Gamma_m} \oplus D_{\Gamma_{m+1}} \oplus \dots$ *with the diagonal* $(G_n^{(1)}, G_n^{(2)}, \dots, G_n^{(m)}, I, I, \dots)$, *and* $U_{n,m}$ *is a unitary operator from* $(\underbrace{D_{\Gamma_n^*} \oplus \dots \oplus D_{\Gamma_n^*}}_{(m-\text{times})}) \oplus \oplus D_{\Gamma_m} \oplus D_{\Gamma_{m+1}} \oplus \dots$ *onto* $D_{W_n^{*m}P_m}$.

PROOF. For an $1 \leq m \leq n$, we have

$$P_m W_n^m = P_m (J_{n-1,1} \dots J_{n-1,m-1})(J_{n-1,1} \dots J_{n-1,m-2}) \dots (J_{n-1,1}) \times$$

$$(4.40)_n^m \qquad \times [(J_{n-1,m} \dots J_{n-1,n-1})(I \oplus \Gamma_n)(J_{n-1,m-1}, \dots, J_{n-1,n-1})(I \oplus \Gamma_n) \dots$$

$$\dots (J_{n-1,1} \dots J_{n-1,n-1})(I \oplus \Gamma_n)] = U'_{n-1,m-1} P_m K_{n,m},$$

where $U'_{n-1,m} = (J_{n-1,1} \dots J_{n-1,m-1})(J_{n-1,1} \dots J_{n-1,m-2}) \dots (J_{n-1,1})$ is a unitary operator, and $K_{n,m}$ has the property that

$$(4.41)_n^m \qquad K_n = (I \oplus \Gamma_n) J_{n-1,n-1} (I \oplus \Gamma_n) J_{n-1,n-2} J_{n-1,n-1} (I \oplus \Gamma_n) \dots$$

$$\dots (J_{n-1,m+1} \dots J_{n-1,n-1})(I \oplus \Gamma_n) K_{n,m}.$$

From $(4.41)_n^m$ it is clear that $P_m K_n = P_m K_{n,m}$ so $(4.40)_n^m$ implies that

$$(4.42)_n^m \qquad P_m W_n^m = U'_{n-1,m-1} P_m K_n .$$

The proof can be finished now as in Proposition 4.6.

5. REGULARITY OF GAUSSIAN STATIONARY PROCESSES

In this section we will suppose that H is *finite dimensional.* In Theorem 5.2

below we give a criterion for a Gaussian stationary process to be informationally regular with $\rho(1) < 1$; this criterion involves the choice sequence attached (by Section 4) to the process.

Let ξ be a Gaussian stationary process, and $\gamma = \gamma(\xi)$ its associated choice sequence; we will use (without mentioning ξ) the objects associated to γ in Section 2 and structure of ξ as described in Section 4.

Having in mind the formulas $(4.3)_{n=1}^{\infty}$, we will firstly give a formula for computing $\det(I - B_{1,n})$.

PROPOSITION 5.1. *For every* $n \geq 1$, *the following formula holds:*

$$(5.1)_n \qquad \det(I - B_{1,n}) = \prod_{p=1}^{n} (\det D_{\Gamma_p^*})^{2p} \prod_{p=n+1}^{\infty} (\det D_{\Gamma_p^*})^{2n}.$$

PROOF. Fix an $n \geq 1$; the formula $(4.2)_n$ and Proposition 4.1 imply that

$$(5.2)_n \qquad \begin{aligned} I - B_{1,n} &= (W^*)^n (I - Q_n W^n P_+ W^{*n} Q_n) W^n = \\ &= (W^*)^n \begin{bmatrix} I & 0 \\ 0 & I - P_n W_+^n W_+^{*n} P_n \end{bmatrix} W^n, \end{aligned}$$

where the matrix in $(5.2)_n$ is written with respect to the decomposition of K as $(K \ominus K_+) \oplus K_+$. As W is a unitary operator, it follows that

$$(5.3)_n \qquad \det(I - B_{1,n}) = \det(I - P_n W_+^n W_+^{*n} P_n) .$$

This last determinant can be computed using Theorem 4.7 and an approximation argument. From the construction of W_+ (see 2.30), it follows that the sequence $\{P_n W_k^n W_k^{*n} P_n\}_{k=1}^{\infty}$ has the limit $P_n W_+^n W_+^{*n} P_n$ (the sequence and the limit are finite rank operators, so the limit here is the uniform one). Therefore

$$(5.4)_n \qquad \det(I - B_{1,n}) = \lim_{k \to \infty} \det(I - P_n W_k^n W_k^{*n} P_n).$$

From Theorem 4.7 it follows that, for sufficiently large k,

$$(5.5)_{n,p} \qquad \begin{aligned} \det(I - P_n W_k^n W_k^{*n} P_n) &= (\det F_{k,n})^2 = \prod_{p=1}^{n} (\det G_k^{(p)})^2 P_n) = \\ &= \prod_{p=1}^{n} (\det D_{\Gamma_p^*})^{2p} \prod_{p=n+1}^{k} (\det D_{\Gamma_p^*})^{2n}. \end{aligned}$$

Thus

$$\det(I-B_{1,n}) = \prod_{p=1}^{n} (\det D_{\Gamma_p^*})^{2p} \lim_{k\to\infty} \prod_{p=n+1}^{k} (\det D_{\Gamma_p^*})^{2n} = \prod_{p=1}^{n} (\det D_{\Gamma_p^*})^{2p} \prod_{p=n+1}^{\infty} (\det D_{\Gamma_p^*})^{2n}$$

which concludes the proof of the proposition.

The regularity criterion is the following.

THEOREM 5.2. *The Gaussian stationary process* ξ *in the finite dimensional Hilbert space* H *is informationally regular and* $\rho(1) < 1$ *if and only if its associated choice sequence* $\gamma(\xi)$ *verifies the condition:*

$$\prod_{p=1}^{\infty} (\det D_{\Gamma_p^*})^{2p} > 0 . \tag{5.6}$$

In this case, we have:

$$\det(I - B_1) = \prod_{p=1}^{\infty} (\det D_{\Gamma_p^*})^{2p} , \tag{5.7}$$

and

$$I(1) = (1/2) \sum_{p=1}^{\infty} p \ln (\det D^2_{\Gamma_p^*}) . \tag{5.8}$$

PROOF. By [20], Chapter IV, the condition of informationally regularity is equivalent with B_1 being trace-class. (As usual, the ideal of trace-class operators in H is denoted by $C_1(H)=C_1$.) Thus we will prove that (5.6) is equivalent to $B_1 \in C_1$ and $\rho(1) = \|B_1\| < 1$.

Suppose first that $B_1 \in C_1$ and $\|B_1\| < 1$. As $P_n^- \to P_-$ (in the strong operator topology) and $B_1 \in C_1$, the relations $(4.3)_{n=1}^{\infty}$ imply that $B_{1,n} \to B_1$ (in the topology of C_1). Then

$$\lim_{k\to\infty} \det(I - B_{1,n}) = \det(I - B_1) . \tag{5.9}$$

As $\|B_1\| < 1$ and $B_1 \geq 0$, it follows that

$$\det(I-B_1) > 0 . \tag{5.10}$$

Now using Proposition 5.1, we have for each $n \geq 1$

$$(5.11)_n \quad \det(I - B_{1,n}) = \prod_{p=1}^{n} (\det D_{\Gamma_p^*})^{2p} \prod_{p=n+1}^{\infty} (\det D_{\Gamma_p^*})^{2n} \leq \prod_{p=1}^{n} (\det D_{\Gamma_p^*})^{2p} .$$

The relations (5.10) and $(5.11)_{n=1}^{\infty}$ imply (5.6).

Note that in this situation

$$(5.12)_n \qquad \prod_{p=n+1}^{\infty} (\det D_{\Gamma_p^*})^{2p} \leq \prod_{p=n+1}^{\infty} (\det D_{\Gamma_p^*})^{2n} \leq 1.$$

As (5.6) implies that $\lim_{k\to\infty} \prod_{p=n+1}^{\infty} (\det D_{\Gamma_p^*})^{2p} = 1$, we infer from $(5.12)_{n=1}^{\infty}$ that

$$(5.13) \qquad \lim_{k\to\infty} \prod_{p=n+1}^{\infty} (\det D_{\Gamma_p^*})^{2n} = 1 .$$

The relations (5.9), $(5.1)_{n=1}^{\infty}$ and (5.13) imply (5.7). The formula (5.8) follows by the definition of $I(1)$ (see (3.5)) and (5.7).

Let us suppose now that (5.6) is fulfilled. Take $A \in L(K)$ be an arbitrary operator such that A is trace class and $A \leq B_1$. Then $\det (P_n^-(I - A)P_n^-) \geq \det (P_n^-(I-B_1)P_n^-) = \det(I - B_{1,n})$. From Proposition 5.1, the hypothesis, and the computations in $(5.12)_{n=1}^{\infty}$ and (5.13) it follows that $\lim_{k\to\infty} \det (I - B_{1,n}) = \prod_{p=1}^{\infty} (\det D_{\Gamma_p^*})^{2p} > 0$. Because A is trace class, we infer that $\lim_{k\to\infty} \det (P_n^-(I - A)P_n^-) = \det (I - A)$. We obtain then that $\det (I - A) \geq$ $\geq \prod_{p=1}^{\infty} (\det D_{\Gamma_p^*})^{2p} > 0$. Thus B_1 is trace class and $\|B_1\| < 1$, and the theorem is completely proved.

6. SOME CONSEQUENCES AND REMARKS

A) The condition $\|B_1\| < 1$ in Theorem 5.2 might be considered as a serious restriction. That this is not the case it is proved by the following. In the classical case of $\dim H = 1$, the same obstruction appears in the study of informationally regular processes. The thourough analysis presented in [21], Chapters 4 and 5 (using results from [22] and [19]) gives detalied information in terms of the spectral density of the process (i.e. the Radon-Nikodym derivative of the spectral measure of the process). Those results have also a counterpart in our setting.

Suppose that $\dim H = 1$ and that the process ξ has the spectral density $f(\xi) = f$. Then we have the following.

THEOREM 6.1. *The scalar Gaussian stationary process ξ is informationally regular iff its spectral density f has a factorization*

(6.1) $$f(t) = |P(e^{it})|^2 g(t)$$

where P is a polynomial with roots on the unit circle and the choice sequence $\gamma(g)$ *verifies the condition*

(6.2) $$-(1/2)\sum_{p=1}^{\infty} p \ln(1 - |\gamma_k(g)|^2) < \infty .$$

PROOF. Note first that if P is a polynomial of degree m with roots on the unit circle, then

(6.3) $$\overline{P}(e^{it})/P(e^{it}) = c e^{-imt}$$

where c is a constant of modulus one.

Suppose now that ξ is informationally regular. Then $\|B_n(f)\| \to 0$ as $n \to \infty$(see (3.1)$_{n=1}^{\infty}$ for the definition of B_n). It follows that there exists an n_o such that $\|B_{n_o}\| < 1$. Then the results of [19] implies that f is representable as

$$f(t) = |P(^{it})|^2 g(t),$$

where P is a polynomial of degree n_o-1 with roots on $|z| = 1$, $\|B_1(\xi(g))\| < 1$, and $\ln g(t)$ is bounded. Then, writing $f(t) = |\alpha(e^{it})|^2$ and $g(t) = |\beta(e^{it})|^2$, where α and β are outer functions in H^2 (see for example [21], Section II.2, Theorem 1), it follows that $\alpha = P\beta$. Denoting by $\{\alpha_n\}$ and $\{\beta_n\}$ the Fourier coefficients of $\overline{\alpha}/\alpha$ and $\overline{\beta}/\beta$, respectively, and using the remark made in the beginning of the proof, we have that

(6.3) $$\sum_{p=-\infty}^{0} |p|\,|\beta_p|^2 = \sum_{p=-\infty}^{1-m} |p|\,|\beta_p|^2 + m\sum_{p=-\infty}^{1-m} |\alpha_p|^2 .$$

From [22], (see also [21], Section IV.4, Lemma 6) it follows ($\xi(t)$ being informationally regular) that $\sum_{p=-\infty}^{0} |p|\,|\alpha_p|^2 < \infty$. Now (6.3) implies that $\sum_{p=-\infty}^{0} |p|\,|\beta_p|^2 < \infty$, then (using the quoted result) $\xi(g)$ is informationally regular and so $B_1(\xi(g))$ is trace class. Applying Theorem 5.2 to $\xi(g)$ we obtain (6.2).

Conversely, suppose that f has a factorization as in (6.1). Using again Theorem 5.2 it follows that $B_1(\xi(g))$ is trace class and $\|B_1(\xi(g))\| < 1$. Note that the relation (6.3) can be easily reversed, so $\sum_{p=-\infty}^{0} |p|\,|\beta_p|^2 < \infty$ implies $\sum_{p=-\infty}^{0} |p|\,|\alpha_p|^2 < \infty$, and thus f is informationally regular.

The theorem is now completely proved.

Let us remark that the presence of choice sequences in this topic is not a surprise: in [22] and [21], Section IV.4 some orthogonal polynomials are intesively used while in [13] it is pointed out the connections between orthogonal polynomials and choice sequences (see also [5]).

B) Let us make now two remarks concerning the connection of the present setting with the notion of entropy and with Szegö Limit Theorem.

Starting from ideas of Kolmogorov and Gelfand-Yaglom (see [16]) a concept of entropy for scalar Gaussian stationary processes was developed in [16], [10], [6], and recently, for the matricial case in [1]. The presentation of the entropy in [15] (see Sections 6.8 and 6.11) concludes with the statement that it is something unclear in the "real" meaning of this notion. In our setting, the interpretation of the entropy is the following.

For a Gaussian stationary process ξ (in a finite dimensional Hilbert space) with semispectral measure F, let define the *entropy* of ξ by

$$(6.4)\qquad h(\xi) = (-1/4\pi)\int_0^{2\pi} \ln\det (dF/dt)(t)\,dt\,.$$

The following relation was obtained in [13]:

$$(6.5)\qquad G(F) = \exp((1/2\pi)\int_0^{2\Pi} \ln\det (dF/dt)\,dt)) = \prod_{p=1}^{\infty} \det D^2_{\Gamma_p^*},$$

where $\gamma = \{\Gamma_p\}_{p=1}^{\infty}$ is the choice sequence of F. ($G(F)$ is called the geometrical mean of F, or f, see [17]). From $(5.1)_1$ we infer that

$$(6.6)\qquad \det (I - B_{1,1}) = (\det D_{\Gamma_1^*})^2 \prod_{p=2}^{\infty} (\det D_{\Gamma_p^*})^2 = \prod_{p=1}^{\infty} \det D^2_{\Gamma_p^*}\,.$$

Comparing (6.4), (6.5), and (6.6) we have that

$$(6.7)\qquad h(F) = (-1/2)\ln\det (I - B_{1,1})\,.$$

This formula shows that the entropy is indeed a sort of "information rate" (as suggested in [15]), but "measuring" only the angle of the first step (see $(4.2)_1$).

It is possible to connect (in the matricial case) the determinants of $B_{1,n}$ with some objects which appears in generalizing Szegö Limit Theorem. (For Szegö Limit Theorem and its generalizations see for example [17], [14], [24], [3]). Using Proposition 1.4 from [12], it follows that for every $n \geq 1$

$$(6.8)_n \qquad (\det T_n(F))/G^{n+1}(F) = 1/\left(\prod_{p=1}^{n} (\det D_{C_p^*})^{2p} \prod_{p=n+1}^{\infty} (\det D_{\Gamma_p^*})^{2(n+1)}\right)$$

where T_n is defined in $(2.34)_n$. This shows (via $(5.1)_n$) that

$$(6.9)_n \qquad \det(I - B_{1,n}) = (G^{n+1}(F))/(\det T_n(F))$$

and so

$$(6.10) \qquad \det(I - B_1) = 1/[\lim_{n\to\infty} (\det T_n(F))/(G^{n+1}(F))] .$$

This formula is a possible interpretation of the number $\lim_{n\to\infty} (\det T_n(F))/(G^{n+1}(F))$ which appears in Szego's Theorem.

Let us mention that in [20] Hankel operators are intensively used in studying Gaussian stationary processes. Some aspects from the theory of Hankel operators are connected with dilation theory and choice sequences, and it would be interesting to explicite these at the level of stationary processes.

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