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STEADY CONVECTION IN POROUS MEDIA-I

The Solutions and their Regularity

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Abstract - This work deals with basic aspects of the study of the steady convection in fluid saturated porous media. In its first part we prove the existence of the weak solutions, via a theorem of Gossez's type, and we give some regularity properties, inter alia a natural maximum principle.

1. INTRODUCTION

The description of the motion which appear in a porous medium, caused by gravitational forces and which arises from density differences due to temperature gradients within a viscous fluid passing through a porous, rigid body, have always come up against serious difficulties. Supposing that the skeleton is fixed, we accept that the velocity of the fluid is far smaller than the acoustic velocity and thus the motion induces little changes of the pressure. That is why we will neglect the variations of the thermodynamic quantities owing to pressure changes. Moreover, we assume that the temperature differences are small enough, permitting to take advantage of the Boussinesq approximation, that is the density of the massic gravitational force is varying affinely with the temperature. So we adopt the model proposed in [1] and obtained by a homogeneization process for which we have proved the convergence [2].

Let Ω be an open connected bounded set in $\mathbb{R}^n$ (n=2 or 3) locally located on one side of the boundary $\partial\Omega$ -a manifold of class C^2, composed of a finite number of connex components. If $\underset{\sim}{u}$, p and T stand, respectively, for the Darcy's velocity, the pressure and the temperature, then they have to satisfy in some way the following system:

$$\operatorname{div} \underset{\sim}{u}=0 \quad \text{in } \Omega \tag{1.1}$$

$$\underset{\sim}{B}\underset{\sim}{u}+\nabla p=(1-\alpha(T-T_o))\underset{\sim}{g} \quad \text{in } \Omega \tag{1.2}$$

$$-\operatorname{div}(\underset{\sim}{A}\nabla T)+\underset{\sim}{u}\nabla T=0 \quad \text{in } \Omega \tag{1.3}$$

with the boundary conditions:

$$\underset{\sim}{u}\cdot\underset{\sim}{\nu}=0 \quad \text{on } \partial\Omega \tag{1.4}$$

$$T=\tau \quad \text{on } \partial\Omega \tag{1.5}$$

where $\underset{\sim}{B}$ ($\underset{\sim}{K}=\underset{\sim}{B}^{-1}$) is the positive symmetric constant tensor of permeability, $\alpha>0$ is the volumetric coefficient of thermal expansion, $\underset{\sim}{g}\in \hat{H}^2(\Omega)$ is the potential type gravitational acceleration, $\underset{\sim}{A}$ is the postive constant tensor of thermal diffusion, $\underset{\sim}{\nu}$ is the unit outward normal on $\partial\Omega$, $\tau\in H^{3/2}(\partial\Omega)$ is the non-uniform temperature of the boundary (the case τ uniform is not interesting) and $T_o>0$ a uniform reference temperature, by convenience $T_o=\frac{1}{2}(\sup_{x\in\partial\Omega}\tau+\inf_{x\in\partial\Omega}\tau)$.

As usual the scalar products and norms in $L^2(\Omega)$, $H^m(\Omega)$ and $H^1_o(\Omega)$ are respectively denoted by:

$$(u,v)=\int_\Omega u\cdot v\,dx \qquad |u|=(u,u)^{1/2}$$

$$((u,v))_m=\sum_{|j|\leq m}\left(\frac{\partial u}{\partial x_j},\frac{\partial v}{\partial x_j}\right) \qquad \|u\|_m=((u,v))_m^{1/2}$$

$$((u,v))=(\nabla u,\nabla v) \qquad \|u\|=((u,u))^{1/2}$$

and the norm in $L^p(\Omega)$ ($p\neq 2$) by $|\ |_p$. We agree to use the same notations for the scalar products and norms in $\underset{\sim}{L}^2(\Omega)=L^2(\Omega)^n$, $\underset{\sim}{H}^m(\Omega)$, $\underset{\sim}{H}^1_o(\Omega)$ and $\underset{\sim}{L}^p(\Omega)$.

Now, in order to pass to homogeneous boundary conditions we introduce for any $h>0$ an element $w_h \in H^2(\Omega)$ with the properties

$$w_h = \tau - T_o \quad \text{on } \partial\Omega \tag{1.6}$$

$$|s\nabla w_h| \leq h\|s\| \quad (\forall)\ s \in H_o^1(\Omega) \tag{1.7}$$

For this we use a standard method:

Let $\rho(x)=d(x,\partial\Omega)$=the distance from x to $\partial\Omega$. For any $\gamma>0$ let's consider

$$\Omega_\gamma = \left\{ x\in\Omega \mid \rho(x) < 2\ \exp(-1/\gamma) \right\}$$

and $f_\gamma \in C^2(\bar{\Omega})$-the Hopf's function which has the following properties:

1°. $f_\gamma = 1$ in some neighbourhood of $\partial\Omega$ (which depends on γ).

2°. $f_\gamma = 0$ in $\Omega \setminus \Omega_\gamma$

3°. $\left|\frac{\partial f_\gamma}{\partial x_k}(x)\right| \leq \frac{\gamma}{\rho(x)}$ in $\bar{\Omega}_\gamma$ $(\forall)$ k, $1\leq k\leq m$.

<u>Warning</u>. Whenever real numbers are concerned the ordinary modulus, $|\ |$, have not to be mixed up with the norm in $L^2(\Omega)$.

Let $l(\tau)\in H^2(\Omega)$ be an element obtained by "lifting the trace" of $\tau \in H^{3/2}(\partial\Omega)$; noting with $\varphi_\gamma = (l(\tau)-T_o)f_\gamma$, for any $s\in H_o^1(\Omega)$ we have:

$$|s\nabla\varphi_\gamma| \leq (2\int_\Omega s^2(l(\tau)-T_o)^2(\nabla f_\gamma)^2 dx + 2\int_\Omega s^2 f_\gamma^2(\nabla l(\tau))^2 dx)^{1/2} \leq$$

$$\leq \gamma\sqrt{6}\left|\frac{s}{\rho}(l(\tau)-T_o)\right| + \sqrt{2}(\int_{\Omega_\gamma} s^2(\nabla l(\tau))^2 dx)^{1/2} \leq$$

$$\leq \gamma\sqrt{6}\,|l(\tau)-T_o|_\infty \left|\frac{s}{\rho}\right| + \eta(\gamma)\cdot\sqrt{2}\,|s|_{4}$$

where $\eta(\gamma) = |\nabla l(\tau)|_{L^4(\Omega_\gamma)} \xrightarrow{(\gamma\to 0)} 0$ as $l(\tau)\in H^2(\Omega)\subseteq L^4(\Omega)$.

With the Hardy inequality [3]:

$$\left|\frac{s}{\rho}\right| \leq c(\Omega)\cdot\|s\| \qquad (\forall) \quad s\in H_o^1(\Omega)$$

it follows finally

$$|s\nabla\varphi_\gamma| \leq c(\tau,\Omega)\max\{\gamma,\eta(\gamma)\}.\|s\| \qquad (\forall)\ s\in H_o^1(\Omega).$$

Because for every $h>0$ we can choose $\gamma_h>0$ satisfying

$$c(\tau,\Omega).\max\{\gamma_h,\eta(\gamma_h)\}<h,$$

the element φ_{γ_h} has the properties (1.6) and (1.7).

Puting $s=T-(w_h+T_o)$, and keeping the right to choose later, in a proper way, the parameter $h>0$, the system (1.1)-(1.5) becomes:

$$\text{div } \underset{\sim}{u}=0 \qquad \text{in } \Omega \tag{1.8}$$

$$\underset{\sim}{B}\underset{\sim}{u}+\nabla p+\alpha(s+w_h)\underset{\sim}{g}=0 \qquad \text{in } \Omega \tag{1.9}$$

$$-\text{div}(\underset{\sim}{A}\nabla(s+w_h))+\underset{\sim}{u}\nabla(s+w_h)=0 \qquad \text{in } \Omega \tag{1.10}$$

$$\underset{\sim}{u}.\underset{\sim}{\nu}=0 \qquad \text{on } \partial\Omega \tag{1.11}$$

$$s=0 \qquad \text{on } \partial\Omega \tag{1.12}$$

Because of practical reasons we put the system (1.8)-(1.12) in a non--dimensional form, by defining:

$$\underset{\sim}{x}^*=L^{-1}\underset{\sim}{x}\ ; \quad \underset{\sim}{x}^*\in\Omega^*\Longleftrightarrow\underset{\sim}{x}\in\Omega;$$

$$s^*=\beta^{-1}s\ ; \quad w_h^*=\beta^{-1}w_h\ ; \quad T_o^*=\beta^{-1}T_o\ ;$$

$$\underset{\sim}{u}^*=La_1^{-1}\underset{\sim}{u}\ ; \quad p^*=a_1^{-1}b_1^{-1}p\ ;$$

$$\underset{\sim}{B}^*=b_1^{-1}\underset{\sim}{B}\ ; \quad \underset{\sim}{A}^*=a_1^{-1}\underset{\sim}{A}\ ; \quad \underset{\sim}{g}^*=g^{-1}\underset{\sim}{g}$$

where L is the length of the edge of the n-cube in which Ω can be included, $\beta = \sup_{x\in\partial\Omega} \tau - \inf_{x\in\partial\Omega} \tau$, $g=|g|_{\infty}$ and a_1, b_1 are respectively the smallest eigenvalues of A and B. Defining the non-dimensional number of Rayleigh by

$$a^{*}=\alpha\beta\, a_1^{-1} b_1^{-1} gL$$

the system (1.8)-(1.12) take the form

$$\operatorname{div} u=0 \quad \text{in } \Omega \tag{1.13}$$

$$Bu+\nabla p+a(s+w_h)g=0 \quad \text{in } \Omega \tag{1.14}$$

$$-\operatorname{div}(A\nabla(s+w_h))+u\nabla(s+w_h)=0 \quad \text{in } \Omega \tag{1.15}$$

$$u.\nu=0 \quad \text{on } \partial\Omega \tag{1.16}$$

$$s=0 \quad \text{on } \partial\Omega \tag{1.17}$$

From now on, in order to simplify the writing, we shall omit conventionnaly the index $*$, as we have already done in (1.13)-(1.17).

Remark 1.1. $\sup_{x\in\partial\Omega} w_h(x)=\frac{1}{2}$ and $\inf_{x\in\partial\Omega} w_h(x)=-\frac{1}{2}$.

2. EXISTENCE THEOREMS

The proof of the existence of the weak solutions of the problem (1.13)-(1.17) will be based on the following theorem, a slight generalisation of [4].

Theorem 2.1. Let Y' be the dual space of Y, which is a separated locally convex space, continuously imbedded in the reflexive Banach space X. If the mapping $G:X\longrightarrow Y'$ is weakly continuous, that is continuous between the weak topologies, and coercitive, that is $(\exists)$ $r>0$ such that

$$\langle Gy\, , y\rangle \geqslant 0 \qquad (\forall)\ y\in \bar{Y} \text{ with } |y| = r \quad ,$$

where $|\ |$ is the norm in X,

then $(\exists)\ x_o \in B_r = \{x\in X \mid |x| \leq r\}$ such that $Gx_o=0$.

Proof. For every $F\in\mathcal{F}$, where

$$\mathcal{F} = \{E \mid E \text{ finite-dimensional subspace of } Y\}$$

we put $j_F : F \to Y$ the natural injection, $j'_F : Y' \to F'$ the surjection defined by

$$\langle j'_F y', y\rangle = \langle y', y\rangle \qquad (\forall)\ y'\in Y' \text{ and } (\forall)\ y\in F \,,$$

and $G_F : F \to F'$ the continuous operator given by $G_F = j'_F \circ G \circ j_F$.

Let's suppose that $G_F y \neq 0$ $(\forall)\ y\in F\cap B_r$. As F is closed in Y and the topology induced on F is the Euclidean one, the set $F\cap B_r$ is compact in F. Identifying F' with F, we define $T: F\cap B_r \to F\cap B_r$ by $T_y = -r|G_F y|^{-1} G_F y$. As T is continuous and $F\cap B_r$ is convex, by Brouwer's fix point theorem, there exists $y_o \in F\cap B_r$ such that $Ty_o = y_o$. It follows:

$$\langle Gy_o\, , y_o\rangle = \langle G\circ j_F(y_o), y_o\rangle = \langle G_F y_o\, , y_o\rangle =$$

$$= -r^{-1} |G_F y_o| \langle Ty_o\, , y_o\rangle = -r\,|G_F y_o| < 0$$

which is in contradiction with the hypothesis, since $y_o \in Y$ and $|y_o| = |Ty_o| = r$. Hence, for any $F\in\mathcal{F}$ there exists $y_F \in F\cap B_r$ such that $G_F y_F = 0$. (It is obvious that if Y is finite-dimensional then the theorem is already proved.)

Now we can define for every $E\in\mathcal{F}$ the non-empty set

$$V_E = \bigcup_{F\in\mathcal{F} \mid F\supseteq E} \{y_F \in F\cap B_r \mid G_F y_F = 0\}$$

and $\tilde{V}_E$ - the weak closure of V_E in X. As B_r is weakly compact in X and the family $\{\tilde{V}_E\}_{E\in\mathcal{F}} \subseteq B_r$ has the property that every finite intersection of its sets is non-empty, it follows that $(\exists)\ x_o \in \bigcap_{E\in\mathcal{F}} \tilde{V}_E$.

For any $y\in Y$ we choose $E(y)\in\mathcal{F}$ with the property $y\in E(y)$. As $x_o\in\tilde{V}_{E(y)}$ there exists a (generalised) sequence $\{y_\delta\}_{\delta\in\Delta}$ (which depends on y), included in $V_{E(y)}$ and weakly converging to x_o. Using the definition of $V_{E(y)}$ results that for any $\delta\in\Delta$ there exists $F_\delta\in\mathcal{F}$ with the properties $F_\delta \supseteq E(y)$ and $G_{F_\delta}\ y_\delta=0$. Consequently

$$\langle Gy_\delta\ ,y\rangle=\langle G\circ j_{F_\delta}\ (y_\delta),j_{F_\delta}\ (y)\rangle=\langle G_{F_\delta}\ y_\delta\ ,y\rangle=0$$

and because G is weakly continuous we can pass the last relation to the limit; thus :

$$\langle Gx_o\ ,y\rangle=0 \qquad (\forall)\quad y\in Y\ . \qquad \square$$

Let's make now the correspondence between the elements of the problem (1.13)-(1.17) and those of Theorem 1. Denoting

$$\underset{\sim}{H}\ =\left\{\underset{\sim}{v}\in \underset{\sim}{L}^2(\Omega)\ \middle|\ \text{div}\ \underset{\sim}{v}=0 \text{ in } \Omega\ ,\ \ \underset{\sim}{v}.\underset{\sim}{\nu}=0 \text{ on } \partial\Omega \right\}$$

then instead of X we put the Hilbert space $\underset{\sim}{H}\times H^1_o(\Omega)$ with the scalar product

$$((\underset{\sim}{u},s),(\underset{\sim}{v},t))_X=(\underset{\sim}{u},\underset{\sim}{v})+((s,t))$$

and instead of Y we put the Banach space $\underset{\sim}{H}\times \overset{\circ}{W}{}^{(1)}_4(\Omega)$ with the usual norm on a product space; the operator G is defined here for any $(\underset{\sim}{u},s)\in\underset{\sim}{H}\times H^1_o(\Omega)$ and for any $(\underset{\sim}{v},t)\in\underset{\sim}{H}\times\overset{\circ}{W}{}^{(1)}_4(\Omega)$ by:

$$\langle G(\underset{\sim}{u},s),(\underset{\sim}{v},t)\rangle =a^2A(s+w_h,t)+B(\underset{\sim}{u},\underset{\sim}{v})+a(s+w_h,\underset{\sim}{g}.\underset{\sim}{v})+a^2b(\underset{\sim}{u},s+w_h,t) \qquad (2.1)$$

where $A(s,t)=(\underset{\sim}{A}\nabla s,\nabla t)$, $B(\underset{\sim}{u},\underset{\sim}{v})=(\underset{\sim}{B}\underset{\sim}{u},\underset{\sim}{v})$ and $b(\underset{\sim}{u},s,t)=(\underset{\sim}{u},t\nabla s)$.

So we have arrived to the weak formulation of the problem (1.13)-(1.17):

to find $(\underset{\sim}{u},s)\in \underset{\sim}{H}\times H^1_o(\Omega)$ such that $(\forall)$ $(\underset{\sim}{v},t)\in \underset{\sim}{H}\times \overset{\circ}{W}{}^{(1)}_4(\Omega)$ holds

$$\langle G(\underset{\sim}{u},s),\ (\underset{\sim}{v},t)\rangle=0 \qquad (2.2)$$

The "equivalence" between the problems (1.13)-(1.17) and (2.2) is legitimated by the following two propositions.

Proposition 2.1. If $(\underset{\sim}{u},s,p)$ are smooth functions satisfying (1.13)-(1.17), then taking the dual product of (1.14) and (1.15) with $\underset{\sim}{v}\in\underset{\sim}{H}$ and $t\in\overset{\circ}{W}{}^{(1)}_4(\Omega)$ there results obviously that $(\underset{\sim}{u},s)$ is a solution of (2.2). □

Proposition 2.2. If $(\underset{\sim}{u},s)\in\underset{\sim}{H}\times H^1_o(\Omega)$ satisfy (2.2) then choosing the test functions in a proper manner we get

$$B(\underset{\sim}{u},\underset{\sim}{v})+a(s+w_h,\underset{\sim}{g}\cdot\underset{\sim}{v})=0 \qquad (\forall)\quad \underset{\sim}{v}\in\underset{\sim}{H} \qquad (2.3)$$

$$A(s+w_h,t)+b(\underset{\sim}{u},s+w_h,t)=0 \qquad (\forall)\quad t\in\overset{\circ}{W}{}^{(1)}_4(\Omega) \qquad (2.4)$$

As $(\underset{\sim}{B}\underset{\sim}{u}+a(s+w_h)\underset{\sim}{g})\in\underset{\sim}{L}^2(\Omega)$ and the orthogonal complement of $\underset{\sim}{H}$ in $\underset{\sim}{L}^2(\Omega)$ is

$$\underset{\sim}{H}^{\perp}=\left\{\underset{\sim}{w}\in\underset{\sim}{L}^2(\Omega)\ \middle|\ (\exists)\ p\in H^1(\Omega)\ \text{such that}\ \underset{\sim}{w}=\nabla p\right\}$$

(for related properties the reader is refered to [3]) it follows from (2.3) that $(\exists)$ $p\in H^1(\Omega)$ such that (1.14) it is satisfied in $\underset{\sim}{L}^2(\Omega)$. Also because

$$\left|b(\underset{\sim}{u},s+w_h,t)\right|\leqslant c_o\,|\underset{\sim}{u}|\,\left|\nabla(s+w_h)\right|\,|t|_\infty \qquad (2.5)$$

and $\overset{\circ}{W}{}^{(1)}_4(\Omega)\subseteq L^\infty(\Omega)$ it follows from (2.4) that (1.15) is satisfied in $W^{(-1)}_{4/3}(\Omega)$. Finally, from the definitions of $\underset{\sim}{H}$ and $H^1_o(\Omega)$ results that $\underset{\sim}{u}$ and s satisfy (1.13) in the distribution sense and (1.16)-(1.17) in the trace senses of the spaces $\underset{\sim}{H}$ and $H^1_o(\Omega)$, respectively. □

Here it is the main result of this section:

Theorem 2.2. The problem (2.2) has at least one solution.

Proof. We shall check that G defined by (2.1) is: (a) weakly-continuous and (b) coercitive; thus the result is a straight consequence of Theorem 1.

(a) Let $(\underset{\sim}{u}_k, s_k) \xrightarrow{(k\to\infty)} (\underset{\sim}{u}, s)$ weakly in $\underset{\sim}{H} \times H_o^1(\Omega)$. Obviously, we have only to prove that

$$b(\underset{\sim}{u}_k, s_k, t) \to b(\underset{\sim}{u}, s, t), \qquad (\forall)\ t \in \overset{o}{W}{}_4^{(1)}(\Omega)$$

all the other convergences being trivial.

For this let's notice that

$$b(\underset{\sim}{u}_k, s_k, t) - b(\underset{\sim}{u}, s, t) = b(\underset{\sim}{u}_k, s_k - s, t) - b(\underset{\sim}{u}_k - \underset{\sim}{u}, t, s)$$

As $(s\nabla t) \in \underset{\sim}{L}^2(\Omega)$ it follows $b(\underset{\sim}{u}_k - \underset{\sim}{u}, t, s) \to 0$. For the other term we have the estimation

$$\left|b(\underset{\sim}{u}_k, s_k - s, t)\right| \leq |\underset{\sim}{u}_k| \cdot |\nabla t|_4 |s_k - s|_4 \leq C_1 \|\underset{\sim}{u}_k\| \|t\|_{\overset{o}{W}{}_4^{(1)}(\Omega)} \cdot |s_k - s|_4 \tag{2.6}$$

Since $\{\|\underset{\sim}{u}_k\|\}_k$ is bounded, as $\{\underset{\sim}{u}_k\}_k$ is weakly convergent in $\underset{\sim}{H}$, and $|s_k - s|_4 \to 0$, as the imbedding of $H_o^1(\Omega)$ in $L^4(\Omega)$ is compact, from (2.6) follows $b(\underset{\sim}{u}_k, s_k - s, t) \to 0$.

(b) From (2.1), for any $(\underset{\sim}{u}, s) \in \underset{\sim}{H} \times \overset{o}{W}{}_4^{(1)}(\Omega)$ we have:

$$\begin{aligned} \langle G(\underset{\sim}{u}, s), (\underset{\sim}{u}, s)\rangle &= a^2 A(s,s) + B(\underset{\sim}{u}, \underset{\sim}{u}) - a^2 (\operatorname{div}(A\nabla w_h), s) + \\ &+ a(s, \underset{\sim}{g}.\underset{\sim}{u}) + a(w_h, \underset{\sim}{g}.\underset{\sim}{u}) + a^2 b(\underset{\sim}{u}, w_h, s) \geq a^2 \|s\|^2 + \|\underset{\sim}{u}\|^2 - a^2 |\operatorname{div}(\underset{\sim}{A}\nabla w_h)| \cdot |s| - \\ &- a|s| \cdot |\underset{\sim}{u}| - a|w_h| \cdot |\underset{\sim}{u}| - a^2 |\underset{\sim}{u}| \, |s\nabla w_h| \end{aligned} \tag{2.6}$$

Using the property (1.7) of w_h and the Friedrichs' inequality

$$|s| \leqslant 2a_o^{-1} \|s\| \qquad (\forall)\ s\in H_o^1(\Omega) \qquad (a_o \geqslant 2\pi\sqrt{n}) \tag{2.7}$$

the relation (2.6) becomes

$$\langle G(\underset{\sim}{u},s),(\underset{\sim}{u},s)\rangle \geqslant a^2\|s\|^2+|\underset{\sim}{u}|^2-(2aa_o^{-1}+ha^2)|\underset{\sim}{u}|\cdot\|s\| -$$
$$-2a^2a_o^{-1}|\mathrm{div}\,(\underset{\sim}{A}\nabla w_h)|\cdot\|s\| -a|w_h|\cdot|\underset{\sim}{u}| \tag{2.8}$$

If $h>0$ is sufficiently small so that

$$h<2a^{-1}a_o^{-1}(a_o-1) \tag{2.9}$$

we obtain immediately from (2.8) the coercitive property of G. □

3. REGULARITY PROPERTIES

Lemma 3.1. If $(\underset{\sim}{u},s)\in \underset{\sim}{H}\times H_o^1(\Omega)$ is a solution of the problem (2.2), then $\underset{\sim}{u}\in \underset{\sim}{H}^1(\Omega)$ and

$$\|\underset{\sim}{u}\|_1 \leqslant C_o(\Omega) \tag{3.1}$$

where C_o is a "constant" (depends only on the data).

Proof. From the Proposition 2.2 $(\exists)$ $p\in H^1(\Omega)$ so that (1.14) is satisfied in $\underset{\sim}{L}^2(\Omega)$. Taking in account (1.13) and (1.16) it follows that p has to verify:

$$-\mathrm{div}(\underset{\sim}{K}\nabla p)=a\ \mathrm{div}(\underset{\sim}{K}\underset{\sim}{g}(s+w_h)) \quad \text{in } \Omega \tag{3.2}$$

$$(\underset{\sim}{K}\nabla p)\underset{\sim}{\nu}=-a(\tau-T_o)(\underset{\sim}{K}\underset{\sim}{g}). \quad \text{on } \partial\Omega \tag{3.3}$$

Let's notice that the system (3.2)-(3.3) ~~is a~~ Neumann problem; as the compatibility condition is obviously satisfied, there exists a unique p (up to an additive constant) which verify (3.2)-(3.3). Moreover, as

$$\operatorname{div}(\underset{\approx}{K}g(s+w_h))=(s+w_h)K_{ij}\frac{\partial g_j}{\partial x_i}+K_{ij}g_j\frac{\partial(s+w_h)}{\partial x_i}\in L^2(\Omega)$$

$$(\tau-T_o)(\underset{\approx}{K}g)\cdot\underset{\sim}{\nu}\in H^{3/2}(\partial\Omega)\subseteq H^{1/2}(\partial\Omega)$$

by usual regularity results for the Neumann problem [5] we get $p\in H^2(\Omega)$ and

$$\|p\|_2\leqslant c_2\|s\|+c_3 \qquad (c_i\text{-"constant"}) \tag{3.4}$$

Now, from (1.14) results $\underset{\sim}{u}\in\underset{\sim}{H}^1(\Omega)$ and we can improve the inequality (2.5) in the following way

$$\left|b(\underset{\sim}{u},s+w_h,t)\right|\leqslant\left|\underset{\sim}{u}\right|_4\cdot\left|\nabla(s+w_h)\right|\left|t\right|_4\leqslant c_4\left\|\underset{\sim}{u}\right\|_1\left|\nabla(s+w_h)\right|\cdot\left\|t\right\|,$$

$$(\forall)\ t\in H_o^1(\Omega) \tag{3.5}$$

Hence (1.15) is actually satisfied in $H^{-1}(\Omega)$ and therefore

$$\left\langle G(\underset{\sim}{u},s),(\underset{\sim}{v},t)\right\rangle=0 \qquad (\forall)\ (\underset{\sim}{v},t)\in\underset{\sim}{H}\times H_o^1(\Omega) \tag{3.6}$$

Puting $(\underset{\sim}{v},t)=(\underset{\sim}{u},s)$ in (2.6) we obtain

$$|\underset{\sim}{u}|^2+\|s\|^2\leqslant c_5|\underset{\sim}{u}|+c_6\cdot\|s\|$$

and afterwords

$$\|s\|\leqslant c_1(\Omega) \tag{3.7}$$

Then, via (3.4) we find that p is bounded in $H^2(\Omega)$. Thus the relation (3.1) can be obtained straightly from (1.14). $\square$

The following weak maximum principle is formulated in terms of inequality in the sense of $H^1(\Omega)$. That's why we start by recalling this notion and some propositions, following [6].

Let $u\in H^1(\Omega)$ and $E\subseteq\bar{\Omega}$; we say that u is nonnegative on E in the sense of $H^1(\Omega)$, or briefly, $u\geqslant 0$ on E in $H^1(\Omega)$, if there exists a sequence

$u_n \in W_\infty^{(1)}(\Omega)$ such that $u_n(x) \geq 0$ for $x \in E$ and $u_n \longrightarrow u$ in $H^1(\Omega)$. Let $v \in H^1(\Omega)$; naturally, we say that $u \leq v$ on E in $H^1(\Omega)$ if $v-u \geq 0$ on E in $H^1(\Omega)$. As v may be a constant, we define

$$\sup_E u = \inf\left\{m \in \mathbb{R} \,\middle|\, u \leq m \text{ on } E \text{ in } H^1(\Omega)\right\}$$

<u>Proposition 3.1</u>. If $u \geq 0$ on E in $H^1(\Omega)$, then $u \geq 0$ (a.e.) on E.

<u>Proposition 3.2</u>. If $\sup_{\partial\Omega} u < \infty$ then for any $M \geq \sup_{\partial\Omega} u$ we have: $\max\{u-M,0\} \in H_0^1(\Omega)$ and $\max\{u-M,0\} \geq 0$ on Ω in $H^1(\Omega)$.

<u>Proposition 3.3</u>. Let $u \in W_p^{(1)}(\Omega)$ $(p \geq 1)$; then $v = \max\{u,0\} \in W_p^{(1)}(\Omega)$; and we have is the sense of distributions:

$$\nabla v = \begin{cases} \nabla u \text{ in } \left\{x \in \Omega \,\middle|\, u > 0 \text{ on } \{x\} \text{ in } H^1(\Omega)\right\} \\ \\ 0 \text{ in } \left\{x \in \Omega \,\middle|\, u \leq 0 \text{ on } \{x\} \text{ in } H^1(\Omega)\right\}. \end{cases}$$

We can pass now to our **maximum principle.**

<u>Theorem 3.1</u>. If $(\underset{\sim}{u},s) \in \underset{\sim}{H} \times H_0^1(\Omega)$ is a solution of the problem (2.2) then $s \in L^\infty(\Omega)$ and

$$|s+w_h|_\infty \leq \frac{1}{2} \tag{3.8}$$

<u>Proof</u>

Choosing in (3.6) $\underset{\sim}{v} = \underset{\sim}{0}$ we get

$$A(s+w_h, t) = b(\underset{\sim}{u}, t, s+w_h) \qquad (\forall)\ t \in H_0^1(\Omega) \tag{3.9}$$

Via the Remark 1.1, we have from Proposition 3.2 $R = \max\left\{s+w_h - \frac{1}{2}, 0\right\} \in H_0^1(\Omega)$ and from Proposition 3.3

$$\nabla R=\begin{cases}\nabla(s+w_h) & \text{when } R\neq 0\\ 0 & \text{when } R=0\end{cases}$$

Puting t=R in (3.9) we obtain

$$\|R\|^2\leqslant A(R,R)=A(s+w_h,R)=b(\underset{\sim}{u},R,s+w_h)=b(\underset{\sim}{u},R,R)=0 \tag{3.10}$$

and hence R=0 in $H^1_o(\Omega)$, that is $(s+w_h)\leqslant\frac{1}{2}$ on Ω in $H^1(\Omega)$. According to Proposition 3.1,

$$s+w_h\leqslant\frac{1}{2}\quad (a.e)\quad \text{on } \Omega \tag{3.11}$$

Analogously, with $R=\min\left\{s+w_h+\frac{1}{2},0\right\}$ we get

$$s+w_h\geqslant-\frac{1}{2}\quad (a.e)\text{ on } \Omega \tag{3.12}$$

Thus (3.8) is proved and concomitantly the whole Theorem because $w_h\in H^2(\Omega)\subseteq$ $\subseteq L^\infty(\Omega)$. □

Lemma 3.2. If $(\underset{\sim}{u},s)\in\underset{\sim}{H}\times H^1_o(\Omega)$ is a solution of the problem (2.2) then for any subdomain $\Omega'\subseteq\overline{\Omega}'\subseteq\Omega$ we have $s\in H^2(\Omega')$ and

$$\|s\|_{H^2(\Omega')}\leqslant c(\Omega')\qquad (c(\Omega')\text{-"constant"}) \tag{3.13}$$

Proof. Let be Ω'' with the property $\overline{\Omega}'\subseteq\Omega''\subseteq\overline{\Omega}''\subseteq\Omega$ and $t\in L^2(\Omega'')$; for any $\varepsilon\in\mathbb{R}$ and i, $1\leqslant i\leqslant n$, we define

$$\tau^i_\varepsilon t(x)=t(x+\varepsilon e_i)$$

where e_i is the versor of the $0x_i$-axis. Let's notice that

$$(\tau_\varepsilon^i t, R) = (t, \tau_{-\varepsilon}^i R) \qquad (\forall) \quad t, R \in L^2(\bar{\Omega}'')$$

Finally we design

$$\Delta_\varepsilon^i t = \frac{1}{\varepsilon}(\tau_\varepsilon^i t - t)$$

Let now ε be with $|\varepsilon| < d(\bar{\Omega}'', \partial\Omega)$. Choosing successively in (2.2) the test functions $(\underset{\sim}{0}, t)$ and $(\underset{\sim}{0}, \tau_{-\varepsilon}^i t)$, with $t \in H_0^1(\Omega'')$, we obtain:

$$A(s,t) + b(\underset{\sim}{u}, s, t) = (\mathrm{div}(\underset{\sim}{A} \nabla w_h), t) - b(\underset{\sim}{u}, w_h, t) \tag{3.14}$$

$$A(\tau_\varepsilon^i s, t) + b(\tau_\varepsilon^i \underset{\sim}{u}, \tau_\varepsilon^i s, t) = (\mathrm{div}(\underset{\sim}{A} \nabla w_h), \tau_{-\varepsilon}^i t) - b(\underset{\sim}{u}, w_h, \tau_{-\varepsilon}^i t) \tag{3.15}$$

Subtracting (3.14) from (3.15) and dividing with ε, it results

$$A(\Delta_\varepsilon^i s, t) + b(\tau_\varepsilon^i \underset{\sim}{u}, \tau_\varepsilon^i s, t) + b(\Delta_\varepsilon^i \underset{\sim}{u}, s, t) = (\mathrm{div}(\underset{\sim}{A} \nabla w_h), \Delta_{-\varepsilon}^i t) - b(\underset{\sim}{u}, w_h, \Delta_{-\varepsilon}^i t) \tag{3.16}$$

With $f \in \mathcal{D}(\Omega'')$, $f = 1$ on $\bar{\Omega}'$ we define $F = fs$. Puting in (3.16) $t = \Delta_\varepsilon^i F$ and taking in account that (see [7]):

$$\left| \Delta_{-\varepsilon}^i t \right|_{L^2(\Omega'')} \leq c_1 \| t \|$$

we obtain the estimation

$$\| \Delta_\varepsilon^i F \| \leq \left| \Delta_\varepsilon^i \underset{\sim}{u} \right| \cdot |s|_\infty + c_1 \left(\left| \mathrm{div}(\underset{\sim}{A} \nabla w_h) \right| + |\underset{\sim}{u}|_4 \cdot |\nabla w_h|_4 \right)$$

and finally

$$\| \Delta_\varepsilon^i F \| \leq c_2 \left(\| \underset{\sim}{u} \|_1 |s|_\infty + \left| \mathrm{div}(\underset{\sim}{A} \nabla w_h) \right| + \| \underset{\sim}{u} \|_1 \cdot \| w_h \|_2 \right) \tag{3.17}$$

According to (3.1) and (3.8), it follows that the sequence $\{ \Delta_\varepsilon^i F \}_\varepsilon$ is bounded in $H^1(\Omega'')$ and hence we can extract a subsequence (still denoted ε) such that

$$\Delta_\varepsilon^i F \rightharpoonup R \quad \text{weakly in } H^1(\Omega'')$$

But as $\Delta_\varepsilon^i F$ converge to $\frac{\partial s}{\partial x_i}$ in $\mathcal{D}'(\Omega')$, it results that $\frac{\partial s}{\partial x_i} = R \in H^1(\Omega')$ for any i, $1 \leq i \leq n$. □

Theorem 3.2. If $(\underset{\sim}{u}, s) \in \underset{\sim}{H} \times H_0^1(\Omega)$ is a solution of the problem (2.2) then $s \in H^2(\Omega)$ and

$$\|s\|_2 \leq c_2(\Omega) \tag{3.18}$$

Moreover, $\underset{\sim}{u} \in \underset{\sim}{H}^2(\Omega)$ and

$$\|\underset{\sim}{u}\|_2 \leq c_3(\Omega) \tag{3.19}$$

Proof. With Lemma 3.2 we have been made sure of the regularity inside the domain. So we have to prove only the regularity on the boundary. For this, with the local charts and the partition of unity we can confine ourselves, by an usual step (see [7]), to the case of a n-parallelipipedon Y with supp $Y \subseteq Y \cup \Sigma$ (in Fig.3.1, $\Sigma = \{x \in \partial Y \mid x_n = 0\}$). With the technique used in Lemma 3.2 one prove that $\frac{\partial^2 s}{\partial x_i \partial x_j} \in L^2(Y)$ if at least one the indices i, j is different of n. Afterwards from (1.15) results $\frac{\partial^2 s}{\partial x_n^2} \in L^{3/2}(Y)$ and hence $s \in W^{(2)}_{3/2}(Y)$. Consequently $\nabla s \in \underset{\sim}{W}^{(1)}_{3/2}(Y) \subseteq \underset{\sim}{L}^3(Y)$ and as from Lemma 3.1 $\underset{\sim}{u} \in \underset{\sim}{H}^1(Y) \subseteq \underset{\sim}{L}^6(Y)$, it follows $(\underset{\sim}{u} \nabla s) \in L^2(Y)$ and recalling (1.15) we finally get $\frac{\partial^2 s}{\partial x_n^2} \in L^2(Y)$.

The estimation (3.17) reduces inside the domain to (3.13). On the boundary the estimation of $\frac{\partial^2 s}{\partial x_i \partial x_j}$ with $(i,j) \neq (n,n)$ can be made analogously; then, the estimation of $\frac{\partial^2 s}{\partial x_n^2}$ follows staightly from (1.15).

The second part of the theorem can be proved with the technique of

Lemma 3.1; the pressure p, solution of the Neumann problem (3.2)-(3.3), has the property $p \in H^3(\Omega)$, because, this time, in concordance with (3.18), we have $(s+w_h)g \in H^2(\Omega)$. □

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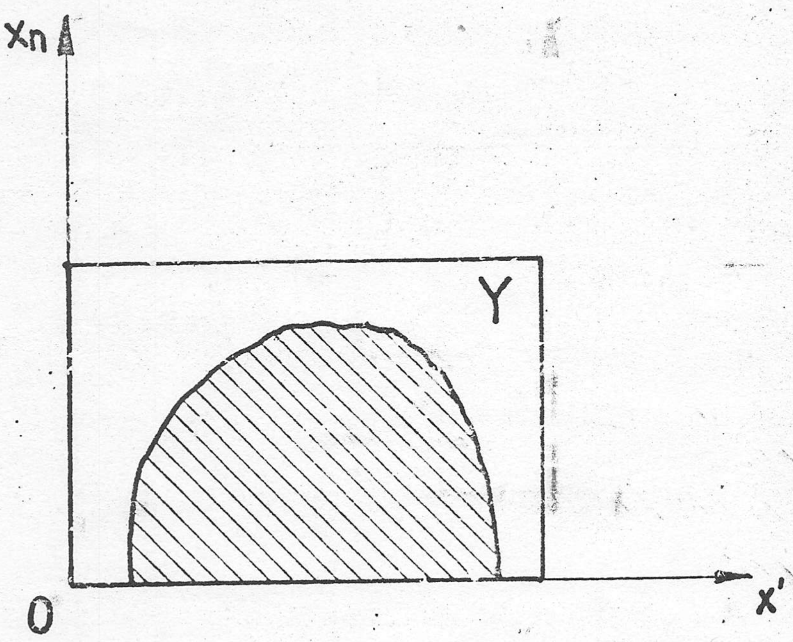

Fig. 3.1 - The support of S (hachured area)

STEADY CONVECTION IN POROUS MEDIA-II. "LOW RAYLEIGH NUMBER"-CASE AND ASYMPTOTIC EXPANSIONS

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Abstract. In this part of the paper we prove that for sufficiently low Rayleigh numbers uniqueness holds; moreover, this only solution is analytic and regular with respect to the Rayleigh number. In connection with these general results, we study by the perturbation method the case of the domain confined between two concentric spheres which are maintained at different uniform temperatures. Using the first three terms of the asymptotic expansions, two special values of the Rayleigh number are pointed out; they separate different types of flows - with one, two or three cells.

4. "LOW RAYLEIGH NUMBER" - CASE

Recalling the Friedrichs' inequality (2.7) we can specify what we meant previously by sufficiently low Rayleigh numbers:

$$a < a_o \tag{4.1}$$

The relation (4.1) will be the framework of this section, first of all because of the following result.

Theorem 4.1. If (4.1) holds then the solution of the problem (2.2) is unique.

Proof. Let $(\underset{\sim}{u}_1, s_1)$ and $(\underset{\sim}{u}_2, s_2)$ be solutions of (2.2); denoting with $\underset{\sim}{u}=\underset{\sim}{u}_1-\underset{\sim}{u}_2$, $s=s_1-s_2$ and subtracting the corresponding relations of (2.3)-(2.4), we get

$$B(\underset{\sim}{u},\underset{\sim}{v})=a(s,\underset{\sim}{g}.\underset{\sim}{v}) \qquad (\forall) \quad \underset{\sim}{v}\in \underset{\sim}{H} \tag{4.2}$$

$$A(s,t)+b(\underset{\sim}{u},s_2+w_h,t)+b(\underset{\sim}{u}_1,s,t)=0 \qquad (\forall) \quad t\in H_o^1(\Omega) \tag{4.3}$$

Taking in (4.2) $\underset{\sim}{v}=\underset{\sim}{u}$ and in (4.3) t=s one can easily obtain the following estimations:

$$|\underset{\sim}{u}|\leqslant a|s| \tag{4.4}$$

$$\|s\|\leqslant\frac{1}{2}|\underset{\sim}{u}| \tag{4.5}$$

From (4.4)-(4.5), with (2.7), it follows

$$(1-\frac{a}{a_o})\|s\|\leqslant 0 \tag{4.6}$$

that is, because of (4.1), $\|s\|=0$; finally (4.4) with (2.7) implies $|\underset{\sim}{u}|=0$. □

Another important thing is that in the presence of (4.1) we can fix $w_h\in H^2(\Omega)$ (which satisfy the properties (1.6)-(1.7)) because in this case $h_o=2a_o^{-2}(a_o-1)$ verify the condition (2.9) for any $a\in(0,a_o)$. Denoting with $w=w_{h_o}$ and $(\underset{\sim}{u}(a),s(a))\in E_o$,

$$E_o=(\underset{\sim}{H}xH_o^1(\Omega))\cap(\underset{\sim}{H}^2(\Omega)xH^2(\Omega)) \ ,$$

the unique solution of (2.2), then the system (2.3)-(2.4) take the form:

$$B(\underset{\sim}{u}(a),\underset{\sim}{v})=a(s(a)+w,\underset{\sim}{g}.\underset{\sim}{v}) \qquad (\forall) \quad \underset{\sim}{v}\in\underset{\sim}{H} \tag{4.7}$$

$$A(s(a),t)+b(\underset{\sim}{u}(a),s(a)+w,t)=(\mathrm{div}(\underset{\sim}{A}\nabla w),t) \qquad (\forall) \quad t\in H_o^1(\Omega) \tag{4.8}$$

<u>Proposition 4.1</u>. $(\underset{\sim}{u}(a+h),s(a+h))\longrightarrow(\underset{\sim}{u}(a),s(a))$ strongly in $\underset{\sim}{H}xH_o^1(\Omega)$, as $h\rightarrow 0$.

Proof. From (4.7)-(4.8) with $\underset{\sim}{v}=\underset{\sim}{u}(a+h)-\underset{\sim}{u}(a)$ and $t=s(a+h)-s(a)$, we have

$$\left|\underset{\sim}{u}(a+h)-\underset{\sim}{u}(a)\right| \leqslant \frac{2(a+h)}{a_o}\left\| s(a+h)-s(a)\right\| + \frac{h}{2}$$

$$\left\| s(a+h)-s(a)\right\| \leqslant \frac{1}{2}\left|\underset{\sim}{u}(a+h)-\underset{\sim}{u}(a)\right|$$

and the proposition follow immediately. □

For any $a\in(0,a_o)$ and any $h\neq 0$ with the property $(a+h)\in(0,a_o)$ we define $D_h\underset{\sim}{u}(a)=h^{-1}(\underset{\sim}{u}(a+h)-\underset{\sim}{u}(a))$ and $D_h s(a)=h^{-1}(s(a+h)-s(a))$.

In order to examine the analytic nature of $(\underset{\sim}{u}(a),s(a))$ we differentiate formally (4.7)-(4.8) with respect to a, defining $(\underset{\sim}{u}'(a),s'(a))\in E_o$ by:

$$B(\underset{\sim}{u}'(a),\underset{\sim}{v})=a(s'(a),\underset{\sim}{g}.\underset{\sim}{v})+(s(a)+w,\underset{\sim}{g}.\underset{\sim}{v}) \quad (\forall) \quad \underset{\sim}{v}\in \underset{\sim}{H} \tag{4.9}$$

$$A(s'(a),t)+b(\underset{\sim}{u}'(a),s(a)+w,t)+b(\underset{\sim}{u}(a),s'(a),t)=0 \quad (\forall) \quad t\in H_o^1(\Omega) \tag{4.10}$$

Remark 4.1. The existence and uniqueness of $(\underset{\sim}{u}'(a),s'(a))$ in $\underset{\sim}{H}\times H_o^1(\Omega)$ can be proved like as the Theorems 2.2 and 4.1, while the regularity property follows with the methods of § 3.

Proposition 4.2. $(D_h\underset{\sim}{u}(a),D_h s(a)) \longrightarrow (\underset{\sim}{u}'(a),s'(a))$ strongly in $\underset{\sim}{H}\times H_o^1(\Omega)$, as $h\longrightarrow 0$.

Proof. From (4.7)-(4.8) for any $(\underset{\sim}{v},t)\in \underset{\sim}{H}\times H_o^1(\Omega)$ it follows

$$B(D_h\underset{\sim}{u}(a),\underset{\sim}{v})=a(D_h s(a),\underset{\sim}{g}.\underset{\sim}{v})+(s(a+h)+w,\underset{\sim}{g}.\underset{\sim}{v}) \tag{4.11}$$

$$A(D_h s(a),t)+b(D_h\underset{\sim}{u}(a),s(a+h)+w,t)+b(\underset{\sim}{u}(a),D_h s(a),t)=0 \tag{4.12}$$

Subtracting (4.11), (4.12) respectively from (4.9), (4.10), results

$$B(D_h\underset{\sim}{u}(a)-\underset{\sim}{u}'(a),v)=a(D_h s(a)-s'(a),\underset{\sim}{g}.\underset{\sim}{v})+(s(a+h)-s(a),\underset{\sim}{g}.\underset{\sim}{v}) \qquad (4.13)$$

$$A(D_h s(a)-s'(a),t)+b(\underset{\sim}{u}(a),D_h s(a)-s'(a),t)=$$

$$=b(D_h\underset{\sim}{u}(a)-\underset{\sim}{u}'(a),t.s(a+h)+w)-b(\underset{\sim}{u}'(a),s(a+h)-s(a),t) \qquad (4.14)$$

Taking in (4.13) $\underset{\sim}{v}=D_h\underset{\sim}{u}(a)-\underset{\sim}{u}'(a)$ and in (4.14) $t=D_h s(a)-s'(a)$, we get the estimations:

$$\left|D_h\underset{\sim}{u}(a)-\underset{\sim}{u}'(a)\right| \leqslant \frac{2a}{a_o}\left\|D_h s(a)-s'(a)\right\| + \left|s(a+h)-s(a)\right| \qquad (4.15)$$

$$\left\|D_h s(a)-s'(a)\right\| \leqslant \frac{1}{2}\left|D_h\underset{\sim}{u}(a)-\underset{\sim}{u}'(a)\right| + \left|\underset{\sim}{u}'(a)\right|_4 \left|s(a+h)-s(a)\right|_4 \qquad (4.16)$$

If we use besides (4.15)-(4.16) the Proposition 4.1, the proof is completed. □

Denoting with $(\underset{\sim}{u}^{(o)}(a),s^{(o)}(a))=(\underset{\sim}{u}(a),s(a))$, $(\underset{\sim}{u}^{(1)}(a),s^{(1)}(a))=(\underset{\sim}{u}'(a),s'(a))$ we define recursively for any $m \geqslant 2$ $(\underset{\sim}{u}^{(m)}(a),s^{(m)}(a)) \in E_o$ by:

$$B(\underset{\sim}{u}^{(m)}(a),v)=(as^{(m)}(a)+ms^{(m-1)}(a),\underset{\sim}{g}.\underset{\sim}{v}) \qquad (\forall)\ \underset{\sim}{v}\in \underset{\sim}{H} \qquad (4.17)$$

$$A(s^{(m)}(a),t)+b(\underset{\sim}{u}^{(m)}(a),w,t)+\sum_{k=0}^{m}\binom{m}{k}b(\underset{\sim}{u}^{(m-k)}(a),s^{(k)}(a),t)=0,$$

$$(\forall)\quad t\in H_o^1(\Omega) \qquad (4.18)$$

Following the way of the Proposition 4.1 and 4.2, one can prove by complete induction the propositions:

$$P_1(n):(\underset{\sim}{u}^{(m-1)}(a+h),s^{(m-1)}(a+h)) \xrightarrow[(h\to 0)]{} (\underset{\sim}{u}^{(m-1)}(a),s^{(m-1)}(a))$$

strongly in $HxH_o^1(\Omega)$.

$$P_2(n):(D_h\underset{\sim}{u}^{(m-1)}(a),D_h s^{(m-1)}(a)) \xrightarrow[(h\to 0)]{} (\underset{\sim}{u}^{(m)}(a),s^{(m)}(a))$$

strongly in $HxH_o^1(\Omega)$.

Thus we have

<u>Proposition 4.3</u>. The solution of the problem (2.2) is of class C^∞ with respect to any $a\in(0,a_o)$.

We reconsider now (4.17)-(4.18) by puting $\underset{\sim}{v}=\underset{\sim}{u}^{(n)}(a)$ and $t=s^{(n)}(a)$; it follows

$$\left|\underset{\sim}{u}^{(m)}(a)\right|\leqslant\frac{2a}{a_o}\left\|s^{(m)}(a)\right\|+\frac{2m}{a_o}\left\|s^{(m-1)}(a)\right\| \tag{4.19}$$

$$\left\|s^{(m)}(a)\right\|\leqslant\frac{1}{2}\left|\underset{\sim}{u}^{(m)}(a)\right|+\sum_{k=1}^{m-1}\binom{m}{k}\left|\underset{\sim}{u}^{(m-k)}(a)\right|_4\left|s^{(k)}(a)\right|_4 \tag{4.20}$$

Passing through the Neumann problem of the pressure, equivalent to (4.17), like in Lemma 3.1 we get

$$\left\|\underset{\sim}{u}^{(m)}(a)\right\|_1\leqslant c_o\left\|as^{(m)}(a)+ms^{(m-1)}(a)\right\| \quad (\forall)\quad m\geqslant 1 \tag{4.21}$$

Thus from (4.19)-(4.20) we have

$$\left\|s^{(m)}(a)\right\|\leqslant\frac{a}{a_o}\left\|s^{(m)}(a)\right\|+\frac{m}{a_o}\left\|s^{(m-1)}(a)\right\|+$$

$$+c_1\sum_{k=1}^{m-1}\binom{m}{k}\left\|s^{(k)}(a)\right\|\left(\left\|s^{(m-k)}(a)\right\|+(m-k)\left\|s^{(m-k-1)}(a)\right\|\right) \tag{4.22}$$

After some easy reductions of (4.22) we are conducted to the following estimation

$$\frac{1}{m!}\left\|s^{(m)}(a)\right\|\leqslant A_m \quad (\forall)\quad m\geqslant 1 \tag{4.23}$$

where A_m is the sequence defined by

$$A_m=c_2\sum_{k=1}^{m-1}A_kA_{m-k}\quad (\forall)\quad m\geqslant 2,\quad A_1=\left\|s'(a)\right\| \tag{4.24}$$

(everywhere $c_i > 0$ are certain constants). As A_m can be determined from (4.24):

$$A_m = \frac{1}{m}\binom{2m-2}{m-1} c_2^{m-1} \| s'(a) \|^m \quad (\forall) \quad m \geqslant 1 \tag{4.25}$$

and from (4.9)-(4.10) with the standard estimation method one can obtain

$$\| s'(a) \| \leqslant \frac{a_o}{4(a_o - a)} \tag{4.26}$$

it follows

$$\frac{1}{m!} \| s^{(m)}(a) \| \leqslant (c_3(a_o - a))^{-m} \tag{4.28}$$

With the comparison test we have finally

Proposition 4.4. The Taylor's series

$$\mathcal{T}(a,h) = \sum_{m \geqslant 0} \frac{h^m}{m!} (\underset{\sim}{u}^{(m)}(a), s^{(m)}(a)) \stackrel{\text{not}}{=\!=} (\underset{\sim}{U}(a,h), S(a,h)) \tag{4.29}$$

is well defined in $\underset{\sim}{H} \cap \underset{\sim}{H}^1(\Omega) \times H_o^1(\Omega)$; moreover its radius of convergence is at least equal to $c_3(a_o - a)$.

Our next objective is to show that $(\underset{\sim}{u}(a), s(a))$ is regular. This will be completed by the following proposition:

Proposition 4.5. $\mathcal{T}(a,h) = (\underset{\sim}{u}(a+h), s(a+h))$ for any $h \in \mathbb{R}$ with $|h| < c_3(a_o - a)$.

Proof. Defining the partial sums

$$\underset{\sim}{U}_m(a,h) = \sum_{n=0}^{m} \frac{h^n}{n!} \underset{\sim}{u}^{(n)}(a) \quad \text{and} \quad S_m(a,h) = \sum_{n=0}^{m} s^{(n)}(a) \tag{4.30}$$

and using (4.7), (4.9) and (4.17) we get

$$B(\underset{\sim}{U}_m(a,h),\underset{\sim}{v})=(a+h)(S_m(a+h)+w,\underset{\sim}{g}\cdot\underset{\sim}{v}) - \frac{h^{m+1}}{m!}(s^{(m)}(a),\underset{\sim}{g}\cdot\underset{\sim}{v}) \quad (\forall)\ \underset{\sim}{v}\in\underset{\sim}{H} \qquad (4.31)$$

Passing (4.31) to the limit, using Proposition 4.4 and the relation (4.28), it follows

$$B(\underset{\sim}{U}(a,h),v)=(a+h)(S(a+h)+w,\underset{\sim}{g}\cdot\underset{\sim}{v}) \qquad (\forall)\ \underset{\sim}{v}\in\underset{\sim}{H} \qquad (4.32)$$

We do the same thing with the second equation of the system; with (4.8), (4.10) and (4.18) we get

$$A(S_m(a,h),t)+b(\underset{\sim}{U}_m(a,h),S_m(a,h)+w,t)=(\mathrm{div}(\underset{\sim}{A}\nabla w),t)+$$

$$+\sum_{n=1}^{m}\sum_{k=0}^{m-n} b\left(\frac{h^{n+k}}{(n+k)!}\underset{\sim}{u}^{(n+k)}(a),\frac{h^{m-k}}{(m-k)!}s^{(m-k)}(a),t\right) \quad (\forall)\quad t\in H_o^1(\Omega) \qquad (4.33)$$

and as the last term of (4.33) is bounded by

$$\left(\sum_{n=1}^{m}\sum_{k=0}^{m-n}\rho^{m+n}\right)\|t\| \qquad \text{with some } \rho\in(0,1) \qquad (4.34)$$

passing (4.33) to the limit, we obtain

$$A(S(a,h),t)+b(\underset{\sim}{U}(a,h),S(a,h)+w,t)=(\mathrm{div}(\underset{\sim}{A}\nabla w),t), \quad (\forall)\ t\in H_o^1(\Omega) \quad (4.35)$$

Since the system (4.32), (4.35) is identical with the system (4.7)-(4.8) which has a unique solution (Theorem 4.1) the proof is completed. □

Now it is abvious that with the Proposition 4.3, 4.4 and 4.5 we have proved in fact

<u>Theorem 4.2</u>. The solution of the problem (2.2) is analytic and regular with respect to any $a\in(0,a_o)$.

5. ASYMPTOTIC EXPANSIONS

This section have to be considered as a completion of the works [4] and [5], in which the problem of the natural convection in a fluid-saturated porous media, placed between two water-proof concentric spheres, was studied by strightforward expansions in terms of the Rayleigh number. For all that we start with recalling this problem.

Let be the radii R_1, R_2 $(R_1 < R_2)$ and let be the center of the spheres 0, the origin of the co-ordinate system $0x_1x_2x_3$, so that $0x_3$ is antiparallel to the gravity $(\underset{\sim}{g}=-g\underset{\sim}{e}_3$, $g>0)$. We adopt here a particular form of the system (1.1)-(1.3) (in fact the classical form [1]):

$$\operatorname{div} \underset{\sim}{u}=0 \tag{5.1}$$

$$\frac{\mu}{\varkappa}\underset{\sim}{u}+\nabla p= (1-\alpha'(T-T_0))\underset{\sim}{g} \tag{5.2}$$

$$\rho c_v \underset{\sim}{u}\nabla T=k\Delta T \tag{5.3}$$

for any $r=\sqrt{x_1^2+x_2^2+x_3^2}\in(R_1,R_2)$, where α, c_v, μ and ρ stand respectively for the coefficient of thermal expansion, the specific heat (at constant volume), the viscosity and the density of the saturation fluid, while the permeability $\varkappa$ and the thermal conductivity k are specific features of the porous medium.

Assuming that heating or cooling is uniformly applied along the boundaries, the system (5.1)-(5.3) have to be solved subject the following conditions:

$$\underset{\sim}{u}.\underset{\sim}{\nu}=0 \qquad \text{for} \quad r=R_1 \quad \text{and} \quad r=R_2 \tag{5.4}$$

$$T=T_1 \qquad \text{for} \quad r=R_1 \tag{5.5}$$

$$T=T_2 \qquad \text{for} \quad r=R_2 \tag{5.6}$$

We suppose $T_1 \neq T_2$, because the case $T_1=T_2$ is trivial. Now we pass to a non--dimensional form by defining

$$\underset{\sim}{x}^* = R_1^{-1}\underset{\sim}{x} \quad ; \quad q^* = R_1^{-1}R_2 > 1$$

$$T^* = (T_1-T_2)^{-1}(T-T_2) \quad ; \quad \underset{\sim}{u}^* = c_v R_1 k^{-1}\underset{\sim}{u}$$

$$h^* = \mu^{-1}k^{-1}\rho c_v \mathcal{H}(p+\rho g(1+\tfrac{1}{2}(T_o-T_2))x_3)$$

$$a^* = \mu^{-1}k^{-1}\rho^2 c_v R_1 \mathcal{H} g\alpha(T_1-T_2)$$

Remark 5.1. The Rayleigh number a^* covers the case $T_1 > T_2$ (with $a^* > 0$) as well as the case $T_1 < T_2$ (with $a^* < 0$).

Omiting conventionally the index $*$, we obtain

$$\operatorname{div} \underset{\sim}{u}=0 \qquad \text{for} \quad r\in(1,q) \tag{5.7}$$

$$\underset{\sim}{u}+\nabla h=aT\underset{\sim}{e}_3 \qquad \text{for} \quad r\in(1,q) \tag{5.8}$$

$$\underset{\sim}{u}\nabla T=\Delta T \qquad \text{for} \quad r\in(1,q) \tag{5.9}$$

$$\underset{\sim}{u}.\underset{\sim}{\nu}=0 \qquad \text{for} \quad r=1 \quad \text{and} \quad r=q \tag{5.10}$$

$$T=1 \qquad \text{for} \quad r=1 \tag{5.11}$$

$$T=0 \qquad \text{for} \quad r=q \tag{5.12}$$

Refering now to the spherical polar co-ordinate system (r,θ,φ) the problem (5.7)-(5.12) becomes:

$$\frac{\partial}{\partial r}(r^2\sin\theta\, u_r)+\frac{\partial}{\partial\theta}(r\sin\theta\; u_\theta)+\frac{\partial}{\partial\varphi}(ru_\varphi)=0 \quad \text{for } r\in(1,q) \tag{5.13}$$

$$u_r+\frac{\partial h}{\partial r}=aT\cos\theta \qquad \text{for} \quad r\in(1,q) \tag{5.14}$$

$$u_\theta+\frac{1}{r}\frac{\partial h}{\partial\theta}=-aT\sin\theta \qquad \text{for} \quad r\in(1,q) \tag{5.15}$$

$$u_\varphi + \frac{1}{r\sin\theta}\frac{\partial h}{\partial\varphi}=0 \quad \text{for} \quad r\in(1,q) \tag{5.16}$$

$$\frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\theta}\frac{\partial T}{\partial\varphi}\right)\right)=$$

$$=u_r\frac{\partial T}{\partial r}+\frac{u_\theta}{r}\frac{\partial T}{\partial\theta}+\frac{u_\varphi}{r\sin\theta}\frac{\partial T}{\partial\varphi} \quad \text{for} \quad r\in(1,q) \tag{5.17}$$

$$u_r=0 \quad \text{for} \quad r=1 \tag{5.18}$$

$$u_r=0 \quad \text{for} \quad r=q \tag{5.19}$$

$$T=1 \quad \text{for} \quad r=1 \tag{5.20}$$

$$T=0 \quad \text{for} \quad r=q \tag{5.21}$$

Proposition 1. For any $a\neq 0$ and $q>1$ the total heat transfer is different from the heat transfer which occurs by pure conduction alone.

Proof. Suppose that there exists $a\neq 0$ and $q>1$ for which $\underset{\sim}{u}\nabla T\equiv 0$; then from equation (5.17) together with the boundary conditions (5.20)-(5.21) we find

$$T=\frac{1}{q-1}\left(\frac{q}{r}-1\right) \tag{5.22}$$

Coming back to $\underset{\sim}{u}\nabla T\equiv 0$, we see that it reduces to $u_r=0$, and thus from (5.13)-(5.21) remain to verify

$$\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta)+\frac{\partial u_\varphi}{\partial\varphi}=0 \tag{5.23}$$

$$\frac{\partial h}{\partial r}=\frac{a}{q-1}\left(\frac{q}{r}-1\right)\cos\theta \tag{5.24}$$

$$u_\theta+\frac{1}{r}\frac{\partial h}{\partial\theta}=\frac{-a}{q-1}\left(\frac{q}{r}-1\right)\sin\theta \tag{5.25}$$

$$u_\varphi + \frac{1}{r\sin\theta}\frac{\partial h}{\partial\varphi} = 0 \tag{5.26}$$

for any $r\in(1,q)$. From (5.24) results

$$h = \frac{a}{q-1}((q\ \ln r - r)\cos\theta + qf(\theta,\varphi)) \tag{5.27}$$

where $f=f(\theta,\varphi)$, via the elimination of u_θ and u_φ between (5.23), (6.25) and (6.26), have to verify

$$\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial f}{\partial\theta}) + \frac{\partial^2 f}{\partial\varphi^2} = (-1+\ln r)\sin 2\theta \tag{5.28}$$

which is obviously impossible. □

Now if we look for asymptotic solutions of the form

$$(\underset{\sim}{u},h,T)=\sum_{m\geqslant 0}(\underset{\sim}{u}_m,h_m,T_m)\frac{a^m}{m!} \tag{5.29}$$

where the a-independent functions $(\underset{\sim}{u}_m,h_m,T_m)$ are supposed to satisfy the set of equations obtained by equating the coefficients of the different powers of "a" which occur in the formal expansion of the system (5.13)-(5.21), then it can be easily verified that (6.29) are the Taylor series in the origin, and we can refer to the results of § 4. Thus we know that the successive problems of the expansions have unique solutions; that's why in the present case they are independent of φ. This fact, together with (5.13), allow us to introduce a stream function Ψ given by

$$r^2\sin\theta\ u_r = \frac{\partial\Psi}{\partial\theta} \qquad r\sin\theta\ u_\theta = -\frac{\partial\Psi}{\partial r} \tag{5.30}$$

Thus, also eliminating h, the system (5.13)-(5.21) becomes:

$$\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)=a\left(\cos\theta\frac{\partial T}{\partial\theta}+r\sin\theta\frac{\partial T}{\partial r}\right) \tag{5.31}$$

for $r\in(1,q)$

$$\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)=\frac{\partial T}{\partial r}\frac{\partial\psi}{\partial\theta}-\frac{\partial T}{\partial\theta}\frac{\partial\psi}{\partial r} \tag{5.32}$$

for $r\in(1,q)$

$$T=1 \quad \text{and} \quad \psi=0 \quad \text{for } r=1 \tag{5.33}$$

$$T=0 \quad \text{and} \quad \psi=0 \quad \text{for } r=q \tag{5.34}$$

We underline the fact that the introduction of the stream function was done under the hypothesis that the unknows have the form (6.29). We don't know if $(\underset{\sim}{u},h,T)$ are φ-independent in general.

Taking (ψ,T) under the form

$$(\psi,T)=\sum_{m\geq 0}(\psi_m,T_m)a^m \tag{5.35}$$

then (ψ_m,T_m) can be defined recursively, for any $m\geq 1$, by

$$r^2\frac{\partial^2\psi_m}{\partial r^2}+\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi_m}{\partial\theta}\right)=r^2\sin\theta\left(\cos\theta\frac{\partial T_{m-1}}{\partial\theta}-r\sin\theta\frac{\partial T_{m-1}}{\partial r}\right) \tag{5.36}$$

for $r\in(1,q)$

$$\psi_m=0 \quad \text{for} \quad r=1 \quad \text{and} \quad r=q \tag{5.37}$$

$$\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial T_m}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T_m}{\partial\theta}\right)=\sum_{k=1}^{m}\left(\frac{\partial T_{m-k}}{\partial r}\frac{\partial\psi_k}{\partial\theta}-\frac{\partial T_{m-k}}{\partial\theta}\frac{\partial\psi_k}{\partial r}\right) \tag{5.38}$$

for $r\in(1,q)$

$$T_m=0 \quad \text{for} \quad r=1 \quad \text{and} \quad r=q \tag{5.39}$$

where the first terms are

$$\Psi_o=0 \quad \text{and} \quad T_o = \frac{1}{q-1}\left(\frac{q}{r} - 1\right) \tag{5.40}$$

The expressions of (Ψ_1, T_1) and (Ψ_2, T_2), as we already mentioned, can be found in [4] and [5] with various discussions (configurations of the streamlines and isothermal lines, the Nusslet number and so on). Now we want to emphasize two facts. First, that if we make $q \to \infty$ then we find [2] as a limit case of the present problem, that is the natural convection about a heated sphere with the boundary conditions $\Psi=0$, $T=1$ as $r=1$ and $|\underset{\sim}{u}| \to 0$, $T \to 0$ as $r \to \infty$. Second, that a multicellular flow can be obtain even with the following approximation

$$\Psi \simeq a\Psi_1 + a^2\Psi_2 \tag{5.41}$$

Trying with (5.41) to get the streamlines on which $\Psi=0$, we get besides

$$r=q \quad , \quad r=1 \quad , \quad \theta=0 \quad \text{and} \quad \theta=\pi \tag{5.42}$$

the (r,θ)-points which verify

$$\left(\frac{q+1}{q} + \frac{1}{r}\right) + \frac{a\cos\theta}{12(q^5-1)}H(r,q)=0 \tag{5.43}$$

where

$$H(r,q)=r^{-2}q^2(q+1)(q^2+q+1)-r^{-1}q(5q^4+3q^3+2q^2+3q+5)+$$

$$+(q^2+q+1)^{-1}((q+1)(4q^6+6q^4+7q^3+6q^2+4)+r(q^6-6q^5-3q^4-2q^3-3q^2-6q+1))$$

Let's denote

$$a_1 = \frac{12(q^5-1)(q+2)}{qH(q,q)} > 0 \quad \text{and} \quad a_2 = \frac{q^2(2q+1)}{(q+2)} a_1 > a_1$$

(for instance when $q=2,5$, $a_1 \simeq 17$ and $a_2 \simeq 142$); thus in the case $T_1 > T_2$ we have the following discussion:

1) $a \in (0,a_1)$ - (5.43) has no solution and we find the usual unicellular flow.

2) $a \in (a_1,a_2)$ - in the relatively stagnant and stable cold region from the bottom of the enclosure appears a second cell (Fig. 5.1).

3) $a \in (a_2,\infty)$ - a third small cell appears at the top of the inner sphere (Fig. 5.2).

If $T_1 < T_2$ the discussion is somewhat similar, the configurations being upside down in comparison with the previous cases.

Experimental works [3] have mentioned the small cell which appears at the top of the inner sphere, but no distinct motion have been observed on the bottom of the enclosure, yet.

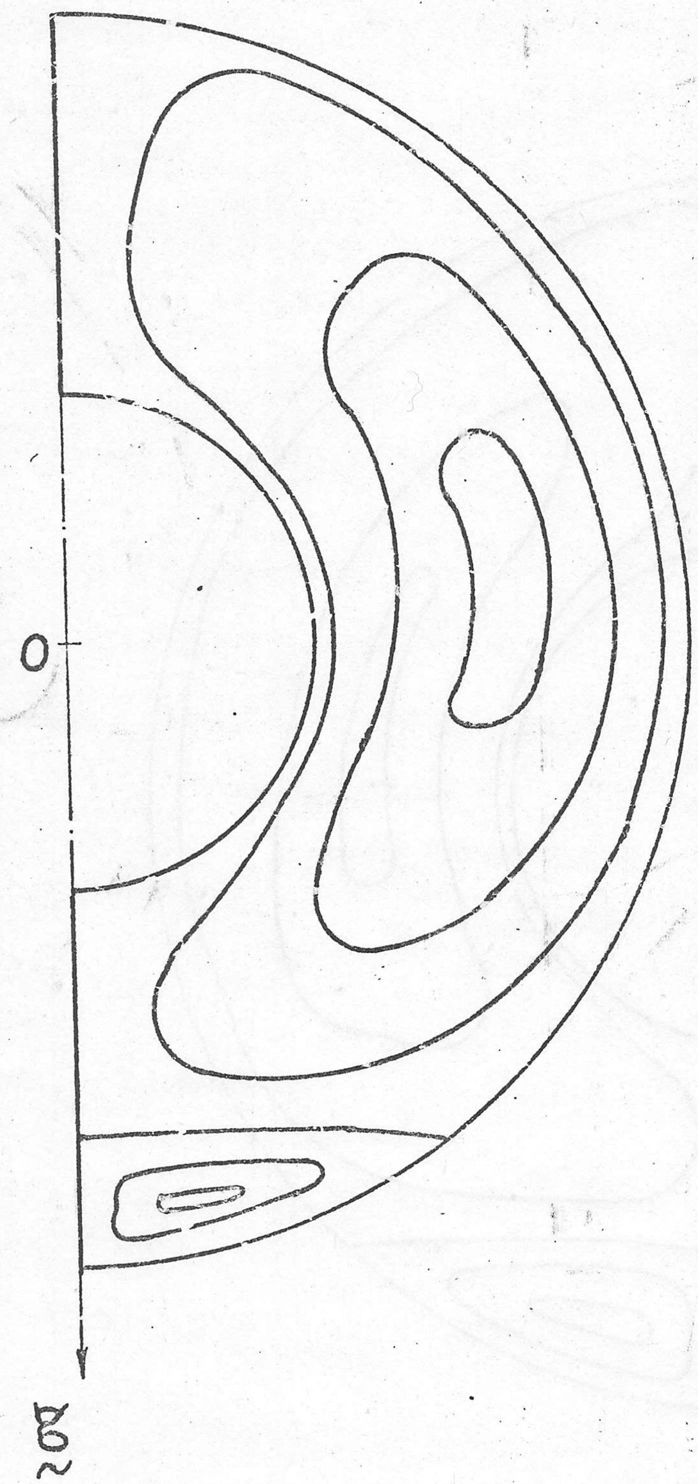

Fig. 5.1 - Streamlines pattern for $a \in (a_1, a_2)$

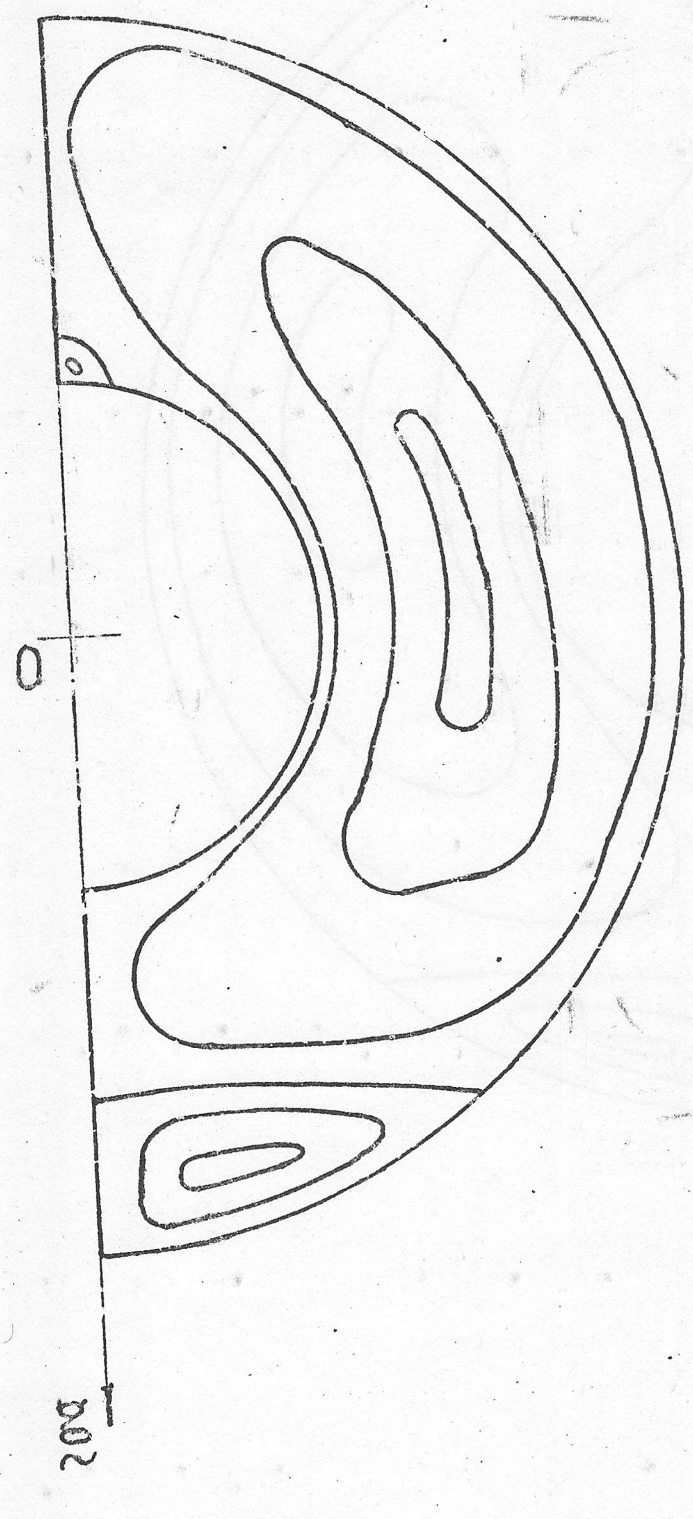

Fig. 5.2 - Streamlines pattern for $a > a_2$

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STEADY CONVECTION IN POROUS MEDIA - III
THE STRUCTURE OF THE SOLUTIONS

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Abstract. The present final part of this work deals with some general and generic properties of the set of solutions, analogous to that obtained for the Navier-Stokes equations (see [1]-[3]). Thus, in the first section we succeed in proving that the set of solutions is homeomorphic to a compact set of $\mathbb{R}^m$ (m sufficiently large), emphasizing the leading part of the temperature for this problem. In the second section we show that for "almost all" data the set of solutions is finite, while the continuum of the soluitons (with respect to the Rayleigh number) is an one-dimensional manifold of class C^1.

6. GENERAL PROPERTIES

We introduce $A_o : \mathcal{D}(\Omega) \longrightarrow L^2(\Omega)$ given by

$$A_o(T) = -\mathrm{div}(\underset{\sim}{A}\nabla T) = -A_{ij}\frac{\partial^2 T}{\partial x_i \partial x_j} \quad ;$$

this operator is positive defined and self-adjoint in the sense of Lagrange. Consequently, it admits a self-adjoint extension $\tilde{A}_o$ with the properties:

a) the domain of $\tilde{A}_o$, $\qquad D(\tilde{A}_o) = D(\tilde{A}_o^*) \cap H_o^1(\Omega)$

b) the range of $\tilde{A}_o$, $\qquad R(\tilde{A}_o) = L^2(\Omega)$

c) $(\tilde{A}_o(S), S) \geqslant \|S\|^2 \qquad (\forall) \quad S \in D(\tilde{A}_o)$

As $\tilde{A}_o^{-1} : L^2(\Omega) \longrightarrow L^2(\Omega)$ is a self-adjoint operator, from property a), via Rellich's lemma it follows that $\tilde{A}_o^{-1}$ is compact too. Hence, there exists a sequence $\{r_k\}_k$ of orthonormal eigenvectors of $\tilde{A}_o^{-1}$ which form an orthonormal basis in $L^2(\Omega)$; also, if we denote $\lambda_k^{-1} > 0$ the eigenvalues of $\tilde{A}_o^{-1}$, they have the property

$$1 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \ldots \longrightarrow \infty$$

Since $\tilde{A}_o^{-1} r_k = \lambda_k^{-1} r_k$ it is equivalent with $\tilde{A}_o r_k = \lambda_k r_k$, $\{\lambda_k\}_k$ are the eigenvalues of $\tilde{A}_o$, corresponding to the same eigenvectors.

Lemma 6.1. $A(r_k, T) = \lambda_k (r_k, T)$, $(\forall)\ T \in H_o^1(\Omega)$

Proof. For any $T \in \mathcal{D}(\Omega)$ we have

$$A(r_k,T) = \int_\Omega A_{ij} \frac{\partial T}{\partial x_i} \frac{\partial r_k}{\partial x_j} = -\int_\Omega r_k \frac{\partial}{\partial x_j}(A_{ij} \frac{\partial T}{\partial x_i}) = -\int_\Omega r_k \frac{\partial}{\partial x_i}(A_{ij} \frac{\partial T}{\partial x_j}) =$$

$$=(r_k, A_o(T)) = (A_o^*(r_k), T) \tag{6.1}$$

Because $\tilde{A}_o = A_o^* \big|_{D(A_o^*) \cap H_o^1(\Omega)}$, (6.1) becomes:

$$A(r_k,T) = \lambda_k (r_k,T) \qquad (\forall)\ T \in \mathcal{D}(\Omega)$$

and as $\mathcal{D}(\Omega)$ is dense in $H_o^1(\Omega)$, the relation follows by continuity. □

Lemma 6.2. $|Q_m S| \leqslant c \lambda_{m+1}^{-1/2} \|Q_m S\|$, $(\forall)\ S \in H_o^1(\Omega)$ where $Q_m = I - P_m$ with P_m the orthogonal projection onto the space spanned by $r_1, \ldots, r_m$.

Proof. Let $Q_m S = \sum_{i \geqslant m+1} \sigma_i r_i$; it follows

$$A(Q_m S, Q_m S) = \sum_{i \geq m+1} \sigma_i A(r_i, Q_m S) = \sum_{i \geq m+1} \sigma_i \lambda_i (r_i, Q_m S) =$$

$$= \sum_{i \geq m+1} \lambda_i \sigma_i^2 \geq \lambda_{m+1} \sum_{i \geq m+1} \sigma_i^2 = \lambda_{m+1} |Q_m S|^2 ;$$

on the other hand, evidently $A(Q_m S, Q_m S) \leq c \, \|Q_m S\|^2$. □

<u>Lemma 6.3</u>. $\lambda_m \geq c(\Omega) m^{2/3}$

<u>Proof</u>. This can be easily obtained using an estimation which can be found in [4]:

$$|S(x)| \leq c_1 \|S\|_2^{3/4} |S|^{1/4} \quad \text{a.e. on } \Omega, \quad (\forall) \quad S \in H^2(\Omega) \tag{6.2}$$

Let $\sigma_i \in \mathbb{R}$, $i \in \{1,2,\dots,m\}$ and let $S(x) = \sum_{i=1}^{m} \sigma_i r_i(x)$ in (6.2); it follows

$$\left| \sum_{i=1}^{m} \sigma_i r_i(x) \right| \leq c_1 \left\| \sum_{i=1}^{m} \sigma_i r_i \right\|_2^{3/4} \left| \sum_{i=1}^{m} \sigma_i r_i \right|^{1/4} \tag{6.3}$$

From the classical regularity result for the Dirichlet problem [5]

$$\|S\|_2 \leq c_2 |\tilde{A}_o(S)| \qquad (\forall) \quad S \in D(\tilde{A}_o) \tag{6.4}$$

and with the same S as in (6.3) we get:

$$\left\| \sum_{i=1}^{m} \sigma_i r_i \right\|_2 \leq c_2 \left| \tilde{A}_o\left(\sum_{i=1}^{m} \sigma_i r_i\right) \right| = c_2 \left| \sum_{i=1}^{m} \sigma_i \lambda_i r_i \right| \leq$$
$$\leq c_2 \lambda_m \left| \sum_{i=1}^{m} \sigma_i r_i \right| \tag{6.5}$$

Introducing (6.5) in (6.3) it follows:

$$\left|\sum_{i=1}^{m}\sigma_i r_i(x)\right| \leq c_3\lambda_m^{3/4}\left|\sum_{i=1}^{m}\sigma_i r_i\right| = c_3\lambda_m^{3/4}\left(\sum_{i=1}^{m}\sigma_i^2\right)^{1/2}$$

Thus we obtain the relation

$$\sum_{i=1}^{m}|r_i(x)|^2 \leq c_4\lambda_m^{3/2}$$

which by integration, via the fact that $|r_i|=1$ $(\forall)$ i, completes the proof. □

In this section we denote with $\mathcal{S}$ the set of the solutions of the problem (2.2) and with $F_m : \mathcal{S} \longrightarrow L^2(\Omega)$ the operator $F_m(\underset{\sim}{u},s)=P_m s$, which is obviously continuous if $\mathcal{S}$ is endowed with the topology of $\underset{\sim}{H} \times H_o^1(\Omega)$.

Lemma 6.4. F_m is injective if m is sufficiently large.

Proof. Let $(\underset{\sim}{u}_1,s_1)$, $(\underset{\sim}{u}_2,s_2)$ and let $(\underset{\sim}{u},s)=(\underset{\sim}{u}_1-\underset{\sim}{u}_2,s_1-s_2)$; from (3.6) results

$$B(\underset{\sim}{u},\underset{\sim}{v})+(as,\underset{\sim}{g}.\underset{\sim}{v})=0 \qquad (\forall)\ \underset{\sim}{v}\in\underset{\sim}{H} \tag{6.6}$$

$$A(s,t)+b(\underset{\sim}{u},s_2+w_n,t)+b(\underset{\sim}{u}_1,s,t)=0 \qquad (\forall)\ t\in H_o^1(\Omega) \tag{6.7}$$

Choosing $t=Q_m s$ in (6.7) and using Theorem 3.2, we obtain

$$\|Q_m s\| \leq \frac{1}{2}|\underset{\sim}{u}|+|\underset{\sim}{u}_1|_\infty \cdot |P_m s| \leq \frac{1}{2}|\underset{\sim}{u}|+c_4|P_m s| \tag{6.8}$$

As from (6.6) with $\underset{\sim}{v}=\underset{\sim}{u}$ we have

$$|\underset{\sim}{u}| \leq a|s| \leq a|P_m s|+a|Q_m s| \tag{6.9}$$

the relation (6.8) becomes

$$\|Q_m s\| \leq \frac{a}{2}|Q_m s| + (\frac{a}{2} + C_4)|P_m s| \tag{6.10}$$

Using Lemma 6.2 from (6.10) it follows

$$(1 - \frac{ac}{2}\lambda_{m+1}^{-1/2})\|Q_m s\| \leq (\frac{a}{2} + C_4)|P_m s| \tag{6.11}$$

From Lemma 6.3 we find that for sufficiently large m the relation

$$(1 - \frac{ac}{2}\lambda_{m+1}^{-1/2}) > 0 \tag{6.12}$$

holds. Hence, if $|P_m s|=0$ from (6.11) results $|Q_m s|=0$; using once more Lemma 6.2 it follows $|Q_m s|=0$, and finally, from (6.9) $|\underset{\sim}{u}|=0$. □

<u>Theorem 6.1</u>. $\mathcal{S} \subseteq H \times H_0^1(\Omega)$ is homeomorphic to a compact set of $\mathbb{R}^m$, m sufficiently large so that (6.12) is satisfied.

<u>Proof</u>. From Theorem 3.2 and from Sobolev's embedding theorems it results that $\mathcal{S}$ is compact in $\underset{\sim}{H} \times H_0^1(\Omega)$. If m satisfy (6.12), then $F_m(\mathcal{S})$ is compact in $\mathbb{R}^m$ and F_m^{-1} is continuous on this set; this follows from (6.9), (6.11) and

$$\|P_m s\|^2 \leq A(P_m s, P_m s) \leq \lambda_m |P_m s|^2 .$$ □

<u>Theorem 6.2</u>. $\mathcal{S}$ is compact in E_0.

<u>Proof</u>. Let $(\underset{\sim}{u}_i, s_i)$ be a sequence of elements of $\mathcal{S} \subseteq E_0$. As $\{(\underset{\sim}{u}_i, s_i)\}_i$ is bounded in E_0 (see Theorem 3.2) it contains a subsequence (still denoted i) which is weakly convergent in E_0 and strongly convergent in $\underset{\sim}{H} \cap \underset{\sim}{H}^1(\Omega) \times H_0^1(\Omega)$ to some element $(\underset{\sim}{u}_0, s_0) \in E_0$. Puting $(\underset{\sim}{v}_i, t_i) = (\underset{\sim}{u}_i - u_0, s_i - s_0)$, the relation (6.7) becomes :

$$\tilde{A}_o(t_i)+\underset{\sim}{v}\nabla(s_i+w_h)+\underset{\sim o}{u}\nabla t_i=0 \quad \text{in} \quad L^2(\Omega) \tag{6.13}$$

From (6.13) we obtain

$$\left|\tilde{A}_o(t_i)\right| \leqslant c_1 \left\|\underset{\sim i}{v}\right\|_1 \left\|s_i+w_h\right\|_2 + \left|\underset{\sim o}{u}\right|_\infty \cdot \left\|t_i\right\| \tag{6.14}$$

We can see now that from (6.14), via (6.4), it results $\|t_i\|_2 \longrightarrow 0$. Taking in account the corresponding pressures, the equation that $q_i=p_i-p_o$ have to verify is

$$-\operatorname{div}(\underset{\sim}{K}\nabla q_i)=a\operatorname{div}(\underset{\sim\sim}{Kg}t_i) \quad \text{in} \quad \Omega \tag{6.15}$$

$$(\underset{\sim}{K}\nabla q_i)\cdot\underset{\sim}{\nu}=0 \quad \text{on} \quad \partial\Omega \tag{6.16}$$

From the already used regularity properties of the Neumann problem (see [5]) we get

$$\|q_i\|_3 \leqslant c_2 \left\| a\operatorname{div}(\underset{\sim\sim}{Kg}t_i)\right\|_1 \leqslant c_3 \|t_i\|_2 \longrightarrow 0 \tag{6.17}$$

Now, the fact that $\left\|\underset{\sim i}{v}\right\|_2 \longrightarrow 0$ follows straightly from (1.14). □

<u>Remark 6.1.</u> (An implicit representation of the solutions). If m satisfy (6.12), then Lemma 6.4 allow us to define for any $\xi \in P_m(H^1_o(\Omega)\cap H^2(\Omega))$ the only element $(\underset{\sim}{u}(\xi),\sigma(\xi))\in \underset{\sim}{H}\times Q_m(H^1_o(\Omega)\cap H^2(\Omega))$ which verify

$$P_H(\underset{\sim\sim}{Bu}(\xi)+a(\xi+\sigma(\xi)+w_h)\underset{\sim}{g})=0 \tag{6.18}$$

$$-\operatorname{div}(\underset{\sim}{A}\nabla\sigma(\xi))+Q_m(\underset{\sim}{u}(\xi)\nabla(\xi+\sigma(\xi)+w_h))=\operatorname{div}(\underset{\sim}{A}\nabla(\xi+Q_m w_h)) \tag{6.19}$$

where P_H is the projection of $\underset{\sim}{L}^2(\Omega)$ on $\underset{\sim}{H}$.

Obviously, $(\underset{\sim}{u}(\xi), \xi+\sigma(\xi))$ is a solution of our problem, if, and only if, ξ verify

$$-\mathrm{div}(\underset{\sim}{A}\nabla\xi)+P_m(\underset{\sim}{u}(\xi)\nabla(\xi+\sigma(\xi)+w_h))=\mathrm{div}(\underset{\sim}{A}\nabla P_m w_h) \tag{6.20}$$

7. GENERIC PROPERTIES

The results of this section are based on Smale's density theorem [6] in the improved form of [7]. That's why we start by recalling this theorem.

Let E, F be Banach spaces and $L:E \longrightarrow F$ a linear continuous operator; then L is a Fredholm operator if:

1^o. The kernel of L is finite dimensional.

2^o. The image of L is closed and has finite codimension.

The index of a Fredholm operator is defined by

$$\mathrm{ind}\, L=\dim(\ker L)-\dim(F/R(L))$$

If $f:E\longrightarrow F$ is a C^1 map, then f is a Fredholm map if for every $x\in E$, $Df(x)$ is a Fredholm operator. For such an operator an index can be defined by $\mathrm{ind}\, f=\mathrm{ind}\, Df(x)$, because $\mathrm{ind}\, Df(x)$ is independent of x (see [8]).

A point $x\in E$ is a regular point of f if $Df(x)$ is surjective; otherwise, x is a critical point of f. The image of the critical points under f is the set of critical values of f and its complement is the set of regular values of f.

Now we can state the Smale theorem.

Theorem 7.1. If E, F are Banach spaces and $f:E \to F$ a Fredholm map of class C^q, with $q > \max\{0, \operatorname{ind} f\}$, then the set of regular values of f is residual (the countable intersection of open dense sets) in F. Moreover, if y is a regular value of f, then $f^{-1}(y)$ is either empty or a manifold of dimension ind f (if ind f=0 then $f^{-1}(y)$ is discrete).

Remark 7.1. By Baire's Category theorem it results that the set of regular values of f is dense in F.

Now we reconsider the system (1.13)-(1.17), defining $\underset{\sim}{v} = \frac{1}{a}\underset{\sim}{u}$, $T=S+w_h+T_o$ and introducing an arbitrary amount of heat $Q \in L^2(\Omega)$ in (1.14); the corresponding system that we are interested in, is the following:

$$\operatorname{div} \underset{\sim}{v}=0 \quad \text{in } \Omega \tag{7.1}$$

$$P_H(\underset{\sim}{B}\underset{\sim}{v}-T\underset{\sim}{g})=\underset{\sim}{0} \quad \text{in } \underset{\sim}{L}^2(\Omega) \tag{7.2}$$

$$-\operatorname{div}(\underset{\sim}{A}\nabla T)+a\underset{\sim}{v}\nabla T=Q \quad \text{in } \Omega \tag{7.3}$$

$$\underset{\sim}{v}.\underset{\sim}{\nu}=0 \quad \text{on } \partial\Omega \tag{7.4}$$

$$T=\tau \quad \text{on } \partial\Omega \tag{7.5}$$

Let's denote with $\mathcal{S}(a,Q,\tau)$ the set of the weak solutions $(\underset{\sim}{v},t)$ of the problem (7.1)-(7.5); it can be proved that $\mathcal{S}(a,Q,\tau)$ is a non--empty compact subset of $\underset{\sim}{H} \cap \underset{\sim}{H}^2(\Omega) \times H^2(\Omega)$; one can use the same techniques as in the previous sections and everything holds identically except the weak maximum principle (Theorem 3.1) which take the following form:

Theorem 7.2. If $(\underset{\sim}{v},T) \in \underset{\sim}{H} \times H^1(\Omega)$ is a solution of (7.1)-(7.5), then $T \in L^\infty(\Omega)$ and

$$|T|_\infty \leq \frac{1}{2} + c(\Omega)|Q| \tag{7.6}$$

Proof. Passing through the Neumann problem of the pressure like in Lemma 3.1, we get $\underset{\sim}{u}\in \underset{\sim}{H}^1(\Omega)$; thus we can make the duality product of (7.3) with any $S\in H^1_o(\Omega)$:

$$A(T,S)+ab(\underset{\sim}{v},T,S)=(Q,S) \tag{7.7}$$

For any $k>0$ we define $S_k=\mathrm{sgn}(T)\max\left\{|T|-\frac{1}{2}-k,0\right\}$; according to Proposition 3.2, $S_k\in H^1_o(\Omega)$ and denoting with

$$A_k=\left\{x\in\Omega \,\middle|\, |T|>\frac{1}{2}+k\right\}$$

we see that $\nabla S_k=\nabla T$ on A_k and $\nabla S_k=0$ elsewhere. Choosing $S=S_k$ in (7.7) we have

$$c_1|S_k|^2_{L^6(A_k)}\leqslant \|S_k\|^2\leqslant A(S_k,S_k)=A(S_k,S_k)+b(\underset{\sim}{u},S_k,S_k)=$$

$$=A(T,S_k)+b(\underset{\sim}{u},T,S_k)=(Q,S_k)\leqslant|Q|(\text{meas }A_k)^{1/3}|S_k|_{L^6(A_k)}$$

that is

$$|S_k|_{L^6(A_k)}\leqslant c_2(\text{meas }A_k)^{1/3}|Q| \tag{7.8}$$

As for any $h>k$

$$(h-k)(\text{meas }A_h)^{1/6}\leqslant|S_k|_{L^6(A_h)}\leqslant|S_k|_{L^6(A_k)}$$

with (7.8) it follows

$$\text{meas }A_h\leqslant c_3\,\frac{|Q|^6}{(h-k)^6}(\text{meas }A_k)^2 \tag{7.9}$$

Because meas A_k is a nonincreasing function (after k), it results from a classical Lemma (see [9]) that meas $A_d=0$ for $d=C_4(\text{meas } A_o)^{1/6}.|\Omega|$. □

Defining now the Banach spaces

$$E_1=\left\{(\underset{\sim}{v},T)\in \underset{\sim}{H}\cap \underset{\sim}{H}^2(\Omega)\times H^2(\Omega) \;\middle|\; P_H(\underset{\sim}{B}\underset{\sim}{v}-T\underset{\sim}{g})=\underset{\sim}{0}\right\} \tag{7.10}$$

$$F=L^2(\Omega)\times H^{3/2}(\partial\Omega) \tag{7.11}$$

and the map $f_1:E_1\to F$ by

$$f_1(\underset{\sim}{v},T)=(-\text{div}(\underset{\sim}{A}\nabla T)+a\underset{\sim}{v}\nabla T\ ,\ T/\partial\Omega\ ) \tag{7.12}$$

the system (7.1)-(7.5) is equivalent, in the sense of the Propositions 2.1 and 2.2, with

$$f_1(\underset{\sim}{v},T)=(Q,\tau) \tag{7.13}$$

Proposition 7.1. $f_1:E_1\to F$ defined by (7.10)-(7.12) is a Fredholm map with ind $f_1=0$.

Proof. The Fréchet differential of f_1, given by

$$\left\langle Df_1(\underset{\sim}{v},T),(\underset{\sim}{u},S)\right\rangle=(-\text{div}(\underset{\sim}{A}\nabla S)+a\underset{\sim}{u}\nabla T+a\underset{\sim}{v}\nabla S,S/\partial\Omega\ ) \tag{7.14}$$

has the form (L+K), with

$$L(\underset{\sim}{u},s)=(-\text{div}(\underset{\sim}{A}\nabla S),S/\partial\Omega\ )$$

an isomorphism from E_1 onto F, and

$$K(\underset{\sim}{u},S)=a(\underset{\sim}{u}\nabla T+\underset{\sim}{v}\nabla S,\ 0/\partial\Omega\ )$$

a compact operator from E_1 with values in F, because for any $(\underset{\sim}{v},T)\in$
$\in \underset{\sim}{H}\cap\underset{\sim}{H}^2(\Omega)\times H^2(\Omega)$, the operator

$$(\underset{\sim}{u},S)\longmapsto(\underset{\sim}{u}\nabla T+\underset{\sim}{v}\nabla S)$$

is continuous from $\underset{\sim}{H}\cap\underset{\sim}{H}^1(\Omega)\times H^1(\Omega)$ with values in $L^2(\Omega)$. Hence $\mathcal{D}f_1(\underset{\sim}{v},T)$ is a Fredholm operator and

$$\text{ind } \mathcal{D}f_1(v,T)=\text{ind}(L+K)=\text{ind } L=0 \;. \qquad \square$$

Theorem 7.3. For every $a\geqslant 0$ there exists a dense open set $\mathcal{O}_1$ in F, such that for any $(Q,\tau)\in\mathcal{O}_1$ the set $\mathcal{S}(a,Q,\tau)$ is finite. Moreover, if $\mathcal{O}_\alpha$ is an connected component of $\mathcal{O}_1$, then for any $(Q,\tau)\in\mathcal{O}_\alpha$ the number of elements of $\mathcal{S}(a,Q,\tau)$ is constant and every element of $\mathcal{S}(a,Q,\tau)$ is a C^∞ function of the pair (Q,τ).

Proof. Obviously f_1 is of class C^∞ and from Proposition 7.1 we find that we are in the hypothesis of Theorem 7.1. Denoting with $\mathcal{O}_1$ the set of regular values of f_1, and having in mind that $\mathcal{S}(a,Q,\tau)$ is compact in $\underset{\sim}{H}\cap\underset{\sim}{H}^2(\Omega)\times H^2(\Omega)$ and hence in E_1, everything can be proved exactly like in [2]. $\square$

We pass now to the second application of the Smale's theorem. This time we put

$$E_2=E_1\times\mathbb{R}_+ \tag{7.14}$$

and $f_2:E_2\longrightarrow F$ defined by

$$f_2(\underset{\sim}{v},T,a)=(-\text{div}(\underset{\sim}{A}\nabla T)+a\underset{\sim}{v}\nabla T,\; T/\partial\Omega) \tag{7.15}$$

Proposition 7.2. $f_2:E_2\longrightarrow F$ defined by (7.14)-(7.15) is a Fredholm map with ind $f_2=1$.

Proof. The Fréchet differential of f_2, given by

$$\left\langle Df_2(\underset{\sim}{v},T,a),(\underset{\sim}{u},S,b)\right\rangle=(-\mathrm{div}(\underset{\sim}{A}\nabla S)+a\underset{\sim}{v}\nabla S+a\underset{\sim}{u}\nabla T+b\underset{\sim}{v}\nabla T,\ S/\partial\Omega) \tag{7.16}$$

can be put under the form (L+K), where

$$L(\underset{\sim}{u},S,b)=(-\mathrm{div}(\underset{\sim}{A}\nabla S)+b\underset{\sim}{v}\nabla T,\ S/\partial\Omega)$$

for any $b\in\mathbb{R}_+$ is an isomorphism from E_1 onto F, and

$$K(\underset{\sim}{u},S,b)=a(\underset{\sim}{u}\nabla T+\underset{\sim}{v}\nabla S,\ 0/\partial\Omega)$$

which is, like in Proposition 7.1, a compact operator. It follows that $Df_2(\underset{\sim}{v},T,a)$ is a Fredholm operator and

$$\mathrm{ind}\ Df_2(\underset{\sim}{v},T,a)=\mathrm{ind}(L+K)=\mathrm{ind}(L)=1 \qquad \square$$

Theorem 7.4. There exists a dense open set $\mathcal{O}_2$ in F, such that for any $(Q,\tau)\in\mathcal{O}_2$ the set

$$\mathcal{S}(Q,\tau)=\left\{(\underset{\sim}{v},T,a)\in E_2\ \middle|\ (\underset{\sim}{v},T)\in\mathcal{S}(a,Q,\tau)\right\}$$

is an one-dimensional manifold.

Proof. The map f_2 is of class C^∞; as an exercise one can check

$$\left\langle (D^2f(\underset{\sim}{u},R,a))(\underset{\sim}{w},T,c),(\underset{\sim}{v},S,b)\right\rangle=(c\underset{\sim}{u}\nabla S+a\underset{\sim}{w}\nabla S+b\underset{\sim}{u}\nabla T+b\underset{\sim}{w}\nabla R+a\underset{\sim}{v}\nabla T+$$
$$+c\underset{\sim}{v}\nabla R,\ 0/\partial\Omega)\ .$$

As usual, $\mathcal{O}_2$ is the set of regular values of f_2 and as $\mathcal{S}(Q,\tau)=$ $=f_2^{-1}(Q,\tau)$, $(\forall)$ $(Q,\tau)\in F$, almost everything follows from Theorem 7.1; we have only to prove that $\mathcal{O}_2$ is open.

Let $(Q_n, \tau_n) \in F \setminus \mathcal{O}_2$ be a sequence converging in F to (Q, τ). Let $(\underset{\sim}{u}_n, T_n, 0) \in \mathcal{S}(Q_n, \tau_n)$; as $\{(Q_n, \tau_n)\}_n$ is bounded in F, from the corresponding estimations of Section 3, it results that $\{(\underset{\sim}{u}_n, T_n)\}_n$ is bounded in $\underset{\sim}{H} \cap \underset{\sim}{H}^2(\Omega) \times H^2(\Omega)$. Hence, there exists $(\underset{\sim}{u}, T) \in \underset{\sim}{H} \cap \underset{\sim}{H}^2(\Omega) \times H^2(\Omega)$ for which, passing, just in case, to a sequence, we have

$$(\underset{\sim}{u}_n, T_n) \rightharpoonup (\underset{\sim}{u}, T) \text{ weakly in } \underset{\sim}{H} \cap \underset{\sim}{H}^2(\Omega) \times H^2(\Omega) \tag{7.17}$$

Let's notice that it follows at once $(\underset{\sim}{u}, T, 0) \in \mathcal{S}(Q, \tau)$. Moreover, as ind $f_2 = 1$, we have

$$\dim \operatorname{Ker} Df_2(\underset{\sim}{u}_n, T_n, 0) = \dim \operatorname{Coker} Df_2(\underset{\sim}{u}_n, T_n, 0) + 1 \geqslant 1$$

and thus, there exists $(\underset{\sim}{v}_n, S_n, a) \in E_2$ with $(\underset{\sim}{v}_n, S_n, a) \neq \underset{\sim}{0}$ such that

$$\langle Df_2(\underset{\sim}{u}_n, T_n, 0), (\underset{\sim}{v}_n, S_n, a) \rangle = 0$$

that is

$$P_H(\underset{\sim}{B}\underset{\sim}{v}_n - S_n \underset{\sim}{g}) = 0 \quad \text{in} \quad \underset{\sim}{L}^2(\Omega) \tag{7.18}$$

$$-\operatorname{div}(\underset{\sim}{A} \nabla S_n) + a_n \underset{\sim}{u}_n \nabla T_n = 0 \quad \text{in} \quad \Omega \tag{7.19}$$

$$S_n = 0 \quad \text{on} \quad \partial\Omega \tag{7.20}$$

Let's notice that $a_n = 0$ implies $(\underset{\sim}{v}_n, S_n) = \underset{\sim}{0}$ and we can suppose that $a_n = 1$, $(\underset{\sim}{v}_n, S_n, a_n)$ being defined up to the multiplication by a constant. From the system (7.18)-(7.20) with $a_n = 1$ it follows that $\{(\underset{\sim}{v}_n, S_n)\}_n$ is bounded in $\underset{\sim}{H} \cap \underset{\sim}{H}^2(\Omega) \times H^2(\Omega)$, as the "given" term $\underset{\sim}{u}_n \nabla T_n$ is bounded in $L^2(\Omega)$. Hence, we can extract a new subsequence (still denoted n) such that

$$(\underset{\sim}{v}_n, S_n) \longrightarrow (\underset{\sim}{v}, S) \quad \text{weakly in } \underset{\sim}{H} \cap \underset{\sim}{H}^2(\Omega) \times H^2(\Omega) \tag{7.21}$$

Now passing to the limit the system (7.18)-(7.20), with (7.17) and (7.21) in mind, we get

$$\langle Df_2(\underset{\sim}{u}, T, 0), (\underset{\sim}{v}, S, 1) \rangle = 0$$

with $(\underset{\sim}{v}, S, 1) \in E_2$; as $(\underset{\sim}{v}, S, 1) \neq \underset{\sim}{0}$ the element $(\underset{\sim}{u}, T, 0)$ is not a regular point, and hence $(Q, \zeta) \in F \setminus \mathcal{O}_2$. □

Remark 7.2. From the theorems 7.3 and 7.4 it follows that (generical) there are no bifurcation points and that fluctuations can appear only where the projection of $\mathcal{S}(Q, \zeta)$ on the Rayleigh number-axis overlaps; also, as long as the Rayleigh number remains bounded, the number of overlaps is finite.

These results are in a close connection with the problem of natural convection because, as we have seen in the theorems 7.3 and 7.4, Q can approximate as well as we want the null value.

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