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THE STRUCTURE OF nxn POSITIVE OPERATOR-MATRICES

by

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The structure of nxn positive operator-matrices

by T. Constantinescu

I Preliminaries

The purpose of this note is to describe the structure of nxn positive operator matrices. In the first part of the paper, we shall express the entries of a positive operator-matrix:

$$\begin{pmatrix} I, & S_{12}, & S_{13}, & \dots & S_{1n} \\ S_{12}^*, & I, & S_{23}, & \dots & S_{2n} \\ \vdots & & \ddots & & \\ S_{1n}^*, & \dots & & & I \end{pmatrix}$$

as functions depending on a family $\{\Gamma_{ij}\}_{\substack{i=1 \\ j>i}}^{\infty}$ of free parameters.((S_{ij} are operators acting on a Hilbert space $\mathcal{H}$.)

The result can be used to solve the similar problem for contractive operator-matrices.(A first attempt is given in [4] where the parametrization with arbitrary contractions is proved, but no explicit formulas appear.)

In the second part, we shall deduce an explicit construction of the Kolmogorov decomposition of a positive-definite kernel on $\mathbb{N}$. One more step is done in order to obtain a dilation result: there exists a family of unitary operators $W(n)$, $W(n)\in\mathcal{L}(\mathcal{K}_{n+1},\mathcal{K}_n)$ so that

$$S_{ij}=P^{\mathcal{K}_i}_{\mathcal{H}}\, W(i)\dots W(j-1)/\mathcal{H} \quad ,i> j.$$

We remark that all these facts are obtained by a slight modification of the proofs in [2] and [3]. The notations we will use, follow those of [1], [2], [3], but, for the sake of completeness, we indicate here the necessary changes.

First of all, let us mention that, in our situation, the parameters consist of the contractions Γ_{ij}, $i\geqslant 1$, $j\geqslant i+1$, so that for $j=i+1$, $i\geqslant 1$, Γ_{ij} acts on $\mathcal{H}$ and otherwise, Γ_{ij} acts from $\mathcal{D}_{\Gamma_{i+1,j}}$ into $\mathcal{D}_{\Gamma^*_{i,j-1}}$. Associated with a fixed set of parameters, we consider the operators: for $i \geq 1$ and $j \geq i+1$,

$$J^1_{ij} : \mathcal{H} \oplus \mathcal{D}_{\Gamma^*_{i,i+1}} \oplus \mathcal{D}_{\Gamma_{i+1,i+3}} \oplus \mathcal{D}_{\Gamma_{i+1,i+4}} \oplus \ldots \oplus \mathcal{D}_{\Gamma_{i+1,j}} \longrightarrow$$
$$\longrightarrow \mathcal{H} \oplus \mathcal{D}_{\Gamma_{i,i+1}} \oplus \mathcal{D}_{\Gamma_{i+1,i+3}} \oplus \ldots \oplus \mathcal{D}_{\Gamma_{i+1,j}}$$
$$J^1_{ij} = J(\Gamma_{i,i+1}) \oplus I \quad ;$$

for $j-1 \geq k > 1$,

$$J^k_{ij} : \mathcal{H} \oplus \mathcal{D}_{\Gamma_{i+1,i+2}} \oplus \ldots \oplus \mathcal{D}_{\Gamma_{i+1,i+k}} \oplus \mathcal{D}_{\Gamma^*_{i,i+k}} \oplus \ldots \oplus \mathcal{D}_{\Gamma_{i+1,j}} \longrightarrow$$
$$\longrightarrow \mathcal{H} \oplus \mathcal{D}_{\Gamma_{i+1,i+2}} \oplus \ldots \oplus \mathcal{D}_{\Gamma^*_{i,i+k-1}} \oplus \mathcal{D}_{\Gamma_{i,i+k}} \oplus \ldots \oplus \mathcal{D}_{\Gamma_{i+1,j}}$$
$$J^k_{ij} = I \oplus J(\Gamma_{i,i+k}) \oplus I \, ,$$

where, for an arbitrary contraction $T : \mathcal{H} \longrightarrow \mathcal{H}$,

$$J(T) : \mathcal{H} \oplus \mathcal{D}_{T^*} \longrightarrow \mathcal{H} \oplus \mathcal{D}_T$$

$$J(T) = \begin{pmatrix} T & , & D_{T^*} \\ D_T & , & -T^* \end{pmatrix} \, , \qquad D_T = (I - T^*T)^{\frac{1}{2}} \, , \; \mathcal{D}_T = \overline{D_T \mathcal{H}} \, ,$$

and I denotes the identity on the corresponding space. It is known that $J(T)$ is unitary, consequently J^k_{ij} are unitary too. These operators appear in the contractive intertwining dilations theory [1]. We also define the operators:

$$V_{ij} : \mathcal{H} \oplus \mathcal{D}_{\Gamma_{i+1,i+2}} \oplus \mathcal{D}_{\Gamma_{i+1,i+3}} \oplus \ldots \oplus \mathcal{D}_{\Gamma_{i+1,j}} \oplus \mathcal{D}_{\Gamma^*_{ij}} \longrightarrow$$
$$\longrightarrow \mathcal{H} \oplus \mathcal{D}_{\Gamma_{i,i+1}} \oplus \ldots \oplus \mathcal{D}_{\Gamma_{i,j-1}} \oplus \mathcal{D}_{\Gamma_{ij}}$$
$$V_{ij} = J^1_{ij} \, J^2_{ij} \ldots J^{j-1}_{ij} .$$

Finally, let us define some operators considered in [2] and [3] :

$$X_{ij} = X_{ij}(\Gamma_{i,i+1}, \Gamma_{i,i+2}, \ldots, \Gamma_{ij}) : \mathcal{H} \oplus \mathcal{D}_{\Gamma_{i+1,i+2}} \oplus \ldots \oplus \mathcal{D}_{\Gamma_{i+1,j}} \longrightarrow \mathcal{H}$$
$$X_{ij} = (\Gamma_{i,i+1}, D_{\Gamma^*_{i,i+1}} \Gamma_{i,i+2}, \ldots, D_{\Gamma^*_{i,i+1}} \ldots D_{\Gamma^*_{i,j-1}} \Gamma_{ij})$$
$$\widetilde{X}_{ij} = \widetilde{X}_{ij}(\Gamma_{j-1,j}, \Gamma_{j-2,j}, \ldots, \Gamma_{ij}) : \mathcal{H} \longrightarrow \mathcal{H} \oplus \mathcal{D}_{\Gamma^*_{j-2,j-1}} \oplus \ldots \oplus \mathcal{D}_{\Gamma^*_{i,j-1}}$$
$$\widetilde{X}_{ij} = (\Gamma_{j-1,j}, \Gamma_{j-2,j} D_{\Gamma_{j-1,j}}, \ldots, \Gamma_{ij} D_{\Gamma_{i+1,j}} \ldots D_{\Gamma_{j-1,j}})^t$$

("t" standing for matrix transpose). As proved in [3], X_{ij} and $\widetilde{X}_{ij}$ are contractions. Moreover, we mention the following identity from [2] :

$$V_{ij}=\begin{pmatrix} X_{ij} & , D_{\Gamma^*_{i,i+1}}\cdot D_{\Gamma^*_{i,i+2}}\cdots D_{\Gamma^*_{ij}} \\ D_{ij} & , \quad -Y_{ij}\end{pmatrix}$$

where

$$D_{ij}=\begin{pmatrix} D_{\Gamma_{i,i+1}} , & -\Gamma^*_{i,i+1}\Gamma_{i,i+2} , & \ldots , & -\Gamma^*_{i,i+1}D_{\Gamma^*_{i,i+2}}\cdots D_{\Gamma^*_{i,j-1}}\Gamma_{ij} \\ 0 , & D_{\Gamma_{i,i+2}} , & \ldots & \\ \vdots & & & \\ 0 , & 0 , & \ldots & D_{\Gamma_{ij}} \end{pmatrix}$$

and

$$Y_{ij}=(\Gamma^*_{i,i+1}D_{\Gamma^*_{i,i+2}}\cdots D_{\Gamma^*_{ij}}, \Gamma^*_{i,i+2}D_{\Gamma^*_{i,i+3}}\cdots D_{\Gamma^*_{ij}}, \ldots, \Gamma^*_{i,j-1}D_{\Gamma^*_{ij}}, \Gamma^*_{ij})^t .$$

II nxn positive operator-matrices

In this section we shall obtain a one-to-one correspondence between the set of sequences of positive operators $\{B_{1n}\}_{n=2}^{\infty}$,

$$B_{1n}=\begin{pmatrix} I , & S_{12}, & S_{13}, & \ldots S_{1n} \\ S^*_{12}, & I , & S_{23}, & \ldots S_{2n} \\ \vdots & & \ddots & \\ S^*_{1n}, & \ldots & & , I \end{pmatrix}$$

and the set of parameters Γ_{ij} introduced in I.

This result will be derived by the use of the well-known structure of a 2x2 positive operator, along the same line as in the proof of Theorem 1.2 in [2] . So , we shall consider the operators: $U_{ii} = I$, $i \geqslant 1$ and

$$U_{ij}: \mathcal{H}\oplus \mathcal{D}_{\Gamma^*_{i,i+1}}\oplus\cdots\oplus\mathcal{D}_{\Gamma^*_{i,j-1}}\oplus\mathcal{D}_{\Gamma^*_{ij}} \longrightarrow$$

$$\longrightarrow \mathcal{H}\oplus\mathcal{D}_{\Gamma_{i,i+1}}\oplus\cdots\oplus\mathcal{D}_{\Gamma_{ij}}$$

$$U_{ij}=V_{ij}(U_{i+1,j}\oplus I_{\mathcal{D}_{\Gamma^*_{ij}}})$$

and $F_{ii}=I$, $i \geqslant 1$,

$$F_{ij}=\begin{pmatrix} F_{i,j-1} & , U_{i,j-1}\widetilde{X}_{ij} \\ 0 & , D_{\Gamma_{ij}}\cdots D_{\Gamma_{j-1,j}}\end{pmatrix}$$

We need some preparations about the structure of the operators

F_{ij}.

2.1 LEMMA For $i \geqslant 1$, $j \geqslant i+1$,

$$U_{i,j-1}\widetilde{X}_{ij} = \begin{pmatrix} X_{i,j-1}U_{i+1,j-1}\widetilde{X}_{i+1,j} + D_{\Gamma^*_{i,i+1}}\cdots D_{\Gamma^*_{i,j-1}}\Gamma_{ij}D_{\Gamma_{i+1,j}}\cdots D_{\Gamma_{j-1,j}} \\ D_{i,j-1}U_{i+1,j-1}\widetilde{X}_{i+1,j} - Y_{i,j-1}\Gamma_{ij}D_{\Gamma_{i+1,j}}\cdots D_{\Gamma_{j-1,j}} \end{pmatrix}.$$

PROOF By a direct computation:

$$U_{i,j-1}\widetilde{X}_{ij} = V_{i,j-1}\begin{pmatrix} U_{i+1,j-1}, & 0 \\ 0 & , I \end{pmatrix}\begin{pmatrix} \widetilde{X}_{i+1,j} \\ \Gamma_{ij}D_{\Gamma_{i+1,j}}\cdots D_{\Gamma_{j-1,j}} \end{pmatrix} =$$

$$= \begin{pmatrix} X_{i,j-1} & , D_{\Gamma^*_{i,i+1}}\cdots D_{\Gamma^*_{i,j-1}} \\ D_{i,j-1} , & -Y_{i,j-1} \end{pmatrix}\begin{pmatrix} U_{i+1,j-1} & , 0 \\ 0 & , I \end{pmatrix}\begin{pmatrix} \widetilde{X}_{i+1,j} \\ \Gamma_{ij}D_{\Gamma_{i+1,j}}\cdots D_{\Gamma_{j-1,j}} \end{pmatrix} =$$

$$= \begin{pmatrix} X_{i,j-1}U_{i+1,j-1}\widetilde{X}_{i+1,j} + D_{\Gamma^*_{i,i+1}}\cdots D_{\Gamma^*_{i,j-1}}\Gamma_{ij}D_{\Gamma_{i+1,j}}\cdots D_{\Gamma_{j-1,j}} \\ D_{i,j-1}U_{i+1,j-1}\widetilde{X}_{i+1,j} - Y_{i,j-1}\Gamma_{ij}D_{\Gamma_{i+1,j}}\cdots D_{\Gamma_{j-1,j}} \end{pmatrix}. \blacksquare$$

2.2 LEMMA For $i \geqslant 1$, $j \geqslant i+1$,

$$F_{ij} = \begin{pmatrix} I & , X_{ij}F_{i+1,j} \\ 0 & , D_{ij}F_{i+1,j} \end{pmatrix}.$$

PROOF For every fixed i, we prove the equality by induction. The main step is as follows:

$$\begin{pmatrix} I, X_{ij}F_{i+1,j} \\ 0, D_{ij}F_{i+1,j} \end{pmatrix} = \begin{pmatrix} I, (X_{i,j-1}, D_{\Gamma^*_{i,i+1}}\cdots\Gamma_{ij})\begin{pmatrix} F_{i+1,j-1}, U_{i+1,j-1}\widetilde{X}_{i+1,j} \\ 0 \quad , D_{\Gamma_{i+1,j}}\cdots D_{\Gamma_{j-1,j}} \end{pmatrix} \\ 0, \begin{pmatrix} D_{i,j-1}, -Y_{i,j-1}\Gamma_{ij} \\ 0 \quad , \quad D_{\Gamma_{ij}} \end{pmatrix}\begin{pmatrix} F_{i+1,j-1}, U_{i+1,j-1}\widetilde{X}_{i+1,j} \\ 0 \quad , D_{\Gamma_{i+1,j}}\cdots D_{\Gamma_{j-1,j}} \end{pmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} F_{i,j-1} & , U_{i,j-1}\widetilde{X}_{ij} \\ 0 & , D_{\Gamma_{ij}}D_{\Gamma_{i+1,j}}\cdots D_{\Gamma_{j-1,j}} \end{pmatrix} = F_{ij}$$

where we used the inductive hypothesis and Lemma 2.1. ■

<u>2.3 LEMMA</u> For $i \geq 1$, $j \geq i+1$,

$$F_{ij} = V_{ij}\begin{pmatrix} X_{ij}^{*} , & F_{i+1,j} \\ D_{\Gamma_{ij}^*}\dots D_{\Gamma_{i,i+1}^*} , & 0 \end{pmatrix}.$$

<u>PROOF</u> Again, by a direct computation, using Lemma 2.2:

$$V_{ij}\begin{pmatrix} X_{ij}^{*} , & F_{i+1,j} \\ D_{\Gamma_{ij}^*}\dots D_{\Gamma_{i,i+1}^*} , & 0 \end{pmatrix} = \begin{pmatrix} X_{ij} , & D_{\Gamma_{i,i+1}^*}\dots D_{\Gamma_{ij}^*} \\ D_{ij} , & -Y_{ij} \end{pmatrix}\begin{pmatrix} X_{ij}^{*} , & F_{i+1,j} \\ D_{\Gamma_{ij}^*}\dots D_{\Gamma_{i,i+1}^*} , & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} X_{ij}X_{ij}^{*} + D_{\Gamma_{i,i+1}^*}\dots D_{\Gamma_{ij}^*}^{2}\dots D_{\Gamma_{i,i+1}^*} , & X_{ij}F_{i+1,j} \\ D_{ij}X_{ij}^{*} - Y_{ij}D_{\Gamma_{ij}^*}\dots D_{\Gamma_{i,i+1}^*} , & D_{ij}F_{i+1,j} \end{pmatrix} =$$

$$= \begin{pmatrix} I , & X_{ij}F_{i+1,j} \\ 0 , & D_{ij}F_{i+1,j} \end{pmatrix} = F_{ij} . \blacksquare$$

Given the above lemmas we shall prove the main result of this section.

<u>2.4.THEOREM</u> There exists a one-to-one correspondence between the set of the sequences of positive operators B_{1n} and the set of parameters Γ_{ij}, given by the formulas:

$$S_{i,i+1} = \Gamma_{i,i+1} \qquad , i \geq 1$$

$$S_{ij} = X_{i,j-1}U_{i+1,j-1}\tilde{X}_{i+1,j} + D_{\Gamma_{i,i+1}^*}\dots D_{\Gamma_{i,j-1}^*}\Gamma_{ij}D_{\Gamma_{i+1,j}}\dots D_{\Gamma_{j-1,j}}$$

$, i \geq 1,\ j \geq i+1.$

<u>PROOF</u> We only sketch the main parts of the proof. The first step is well-known:

$$B_{12} = \begin{pmatrix} I , & S_{12} \\ S_{12}^{*} , & I \end{pmatrix} \geq 0 \quad \text{if and only if } S_{12} = \Gamma_{12} \quad \text{is a}$$

contraction on $\mathcal{H}$. Further, we suppose for every nxn matrix to be true the following sentences(but we write them only for the matrix B_{1n}):

$(1)_n$ $(S_{12},\dots S_{1n})=X_{1n}F_{2n}$

$(2)_n$ $(S_{1n},\dots S_{n-1,n})^t=F^*_{1,n-1}U_{1,n-1}\widetilde{X}_{1n}$

$(3)_n$ $B_{1n}=F^*_{1n}F_{1n}$

$(4)_n$ there exists a unique contraction $\Gamma_{1,n+1}:\mathcal{D}_{\Gamma_{2,n+1}}\to\mathcal{D}_{\Gamma^*_{1,n}}$ so that

$$S_{1,n+1}=X_{1n}U_{2n}\widetilde{X}_{2,n+1}+D_{\Gamma^*_{12}}\dots D_{\Gamma^*_{1n}}\Gamma_{1,n+1}D_{\Gamma_{2,n+1}}\dots D_{\Gamma_{n,n+1}}$$

and we shall prove the same facts for $B_{1,n+1}$.

$(1)_{n+1}$ $(S_{12},\dots,S_{1,n+1})=(X_{1n}F_{2n},X_{1n}U_{2n}\widetilde{X}_{2,n+1}+D_{\Gamma^*_{12}}\dots\Gamma_{1,n+1}\dots D_{\Gamma_{n,n+1}})=$

$$=X_{1,n+1}F_{2,n+1}.$$

$(2)_{n+1}$

$$(S_{1,n+1},\dots S_{n,n+1})^t=\begin{pmatrix} X_{1n}U_{2n}\widetilde{X}_{2,n+1}+D_{\Gamma^*_{12}}\dots D_{\Gamma^*_{1n}}\Gamma_{1,n+1}D_{\Gamma_{2,n+1}}\dots D_{\Gamma_{n,n+1}} \\ F^*_{2n}U_{2n}\widetilde{X}_{2,n+1}\end{pmatrix}$$

$$=\begin{pmatrix} X_{1n}U_{2n}, & D_{\Gamma^*_{12}}\dots D_{\Gamma^*_{1n}} \\ F^*_{2n}U_{2n}, & 0\end{pmatrix}\cdot\widetilde{X}_{1,n+1}$$

But, using Lemma 2.3, it results:

$$F^*_{1n}U_{1n}=\begin{pmatrix} X_{1n}U_{2n}, & D_{\Gamma^*_{12}}\dots D_{\Gamma^*_{1n}} \\ F^*_{2n}U_{2n}, & 0\end{pmatrix}.$$

consequently,

$$(S_{1,n+1},\dots S_{n,n+1})^t=F^*_{1n}U_{1n}\widetilde{X}_{1,n+1}.$$

$(3)_{n+1}$ We can write:

$$B_{1,n+1}\equiv\begin{pmatrix} B_{1n}, & (S_{1,n+1},\dots,S_{n,n+1})^t \\ (S^*_{1,n+1},\dots,S^*_{n,n+1}), & I\end{pmatrix}=$$

$$=\begin{pmatrix} F^*_{1n}F_{1n}, & F^*_{1n}U_{1n}\widetilde{X}_{1,n+1} \\ \widetilde{X}^*_{1,n+1}U^*_{1n}F_{1n}, & I\end{pmatrix}=F^*_{1,n+1}F_{1,n+1}.$$

$(4)_{n+1}$ If $B_{1,n+2}\geq 0$ then

$$B_{2,n+2}\geq(S_{12},\dots,S_{1,n+2})^*(S_{12},\dots,S_{1,n+2})\quad\text{so}$$

there exists a contraction $K=(K_1,\ldots,K_{n+2})$ so that

$$(S_{12},\ldots,S_{1,n+2})=(K_1,\ldots K_{n+2})\begin{pmatrix} F_{2,n+1}, U_{2,n+1}\tilde{X}_{2,n+2} \\ 0 \quad , D_{\Gamma_{2,n+2}}\cdots D_{\Gamma_{n+1,n+2}} \end{pmatrix}.$$

We obtain $(K_1,\ldots,K_{n+1})=X_{1,n+1}$ and

$$S_{1,n+2}=X_{1,n+1}U_{2,n+1}\tilde{X}_{2,n+2} + D_{\Gamma_{12}^*}\cdots D_{\Gamma_{1,n+1}^*}\Gamma_{1,n+2}D_{\Gamma_{2,n+2}}\cdots D_{\Gamma_{n+1,n+2}}$$

where $\Gamma_{1,n+2}$ is a contraction uniquely determined by $S_{1,n+2}$.

The essential part of the Theorem is proved. ∎

2.5 REMARK

We can use Theorem 2.4 in order to find the structure of a mxn contractive operator-matrix

$$A=\begin{pmatrix} A_{11}, A_{12},\ldots, A_{1n} \\ A_{21},\ldots & \\ \vdots & \ddots \\ A_{m1},\ldots & A_{mn} \end{pmatrix}$$

We know that A is a contraction if and only if $\begin{pmatrix} I , A \\ A^*, I \end{pmatrix}$ is positive and now we use the algorithm obtained in Theorem 2.4.

In [4] , the same problem was solved, providing a factorization of the given contraction A into contractions of simple special form. But, from this factorization is not so easy to obtain explicite relations between A_{ij} and the parameters, as is done by the algorithm in Theorem 2.4. ∎

III Positive-definite kernels on $\mathbb{N}$

In this section we shall use Theorem 2.4 in order to obtain a concrete matricial construction for the minimal Kolmogorov decomposition of a positive-definite kernel on $\mathbb{N}$. At this end we first recall some of the standard definitions(see for example [5]).

We call a positive-definite kernel on $\mathbb{N}$, a map

$$K:\mathbb{N}\times\mathbb{N} \longrightarrow \mathcal{L}(\mathcal{H})$$

with the property that for each $n\in\mathbb{N}$ and each choice of vectors

$h_1,\dots,h_n$ in $\mathcal{H}$ and $p_1,\dots,p_n$ in $\mathbb{N}$, the following inequality holds:

$$\sum_{i,j=1}^{n}(K(p_i,p_j)h_j,h_i)\geq 0.$$

Without the loss of generality, we can suppose $K(p,p)=I$ for $p\in\mathbb{N}$, and in this case it is easy to see that K is positive-definite if and only if the matrices B_{1n} are positive for $n\in\mathbb{N}$, where

$$S^*_{ij}=K(i,j).$$

A Kolmogorov decomposition of K will be a map

$$V:\mathbb{N}\longrightarrow\mathcal{L}(\mathcal{H},\mathcal{H}_V)$$

where $\mathcal{H}_V$ is a Hilbert space, so that

$$K(i,j)=V(i)^*V(j),\quad i.j\in\mathbb{N},$$

and , if $\mathcal{H}_V=\bigvee_{n=1}^{\infty}V(n)\mathcal{H}$ then the decomposition is said to be minimal.

It is well-known that two minimal Kolmogorov decompositions are equivalent in a common sense.

Using Theorem 2.4, we associate parameters Γ_{ij} to every positive-definite kernel on $\mathbb{N}$. We define

$$\mathcal{K}_+=\mathcal{H}\oplus\bigoplus_{n=2}^{\infty}\mathcal{D}_{\Gamma_{1n}}$$

and we look at the spaces $\mathcal{H}\oplus\bigoplus_{k=2}^{n}\mathcal{D}_{\Gamma_{1k}}$ as being embeded in $\mathcal{K}_+$.

Regarding the minimal Kolmogorov decomposition we obtain:

<u>3.1 THEOREM</u> Let K be a positive-definite kernel on $\mathbb{N}$ and Γ_{ij} the assocíated parameters.Then

$$V:\mathbb{N}\longrightarrow\mathcal{L}(\mathcal{H},\mathcal{K}_+)$$

$$V(n)=\begin{pmatrix}U_{1,n-1}\widetilde{X}_{1n}\\ D_{\Gamma_{1n}}\dots D_{\Gamma_{n-1,n}}\end{pmatrix}$$

is the minimal Kolmogorov decomposition of K.

<u>PROOF</u> From the definitions:

$$F_{1n}=(V(1),V(2),\dots,V(n))$$

and using $(3)_n$ from the proof of Theorem 2.4, it results

$$B_{1n}=(V(1)^*,\dots,V(n)^*)^t(V(1),\dots V(n))$$

consequently,

$$S_{ij}=V(i)^*V(j)\ ,\ i,j\in\mathbb{N}$$

For the minimality we have to prove that

$$\mathcal{K}_+=\bigvee_{n=1}^{\infty}V(n)\mathcal{H}\ .$$

Let $f=(h_0,h_1,\dots)\in\mathcal{K}_+$, $f\perp V(n)\mathcal{H}$, $n\in\mathbb{N}$. But

$$V(n)h=(\ *\ ,*\ ,D_{\Gamma_{1n}}\dots D_{\Gamma_{n-1,n}}h\ ,0,0,\dots)\quad\text{for } h\in\mathcal{H}$$

so, succesively, $h_0=0$, $h_1=0$, ... and $\mathcal{K}_+=\bigvee_{n=1}^{\infty}V(n)\mathcal{H}$. ∎

We next show that the map V has a simple multiplicative structure. For this aim, we follow [3] in order to define the operators:

$$W_+(n)=W_+(\Gamma_{n,n+1},\Gamma_{n,n+2}\dots):\mathcal{H}\oplus\bigoplus_{k=2}^{\infty}\mathcal{D}_{\Gamma_{n+1,n+k}}\longrightarrow$$
$$\longrightarrow\mathcal{H}\oplus\bigoplus_{k=1}^{\infty}\mathcal{D}_{\Gamma_{n,n+k}}\ .$$

For this, let us denote for $k\geqslant 1$

$$W_k(n)=V_{n,n+k}(I\oplus\Gamma_{n,n+k})$$

and

$$W_+(n)=\underset{k\to\infty}{\text{s-lim}}\ W_k(n)\qquad\text{(Lemma 2.3 in [3]).}$$

<u>3.2 THEOREM</u> For $n>1$,

$$V(n)=W_+(1)W_+(2)\dots W_+(n-1)/\mathcal{H}$$

<u>PROOF</u> First, we argue as in Lemma 2.2 [3] that

$$W_+(1)\dots W_+(n)/\mathcal{H}=W_{n+1}(1)\dots W_{n+1}(n)/\mathcal{H}\ .$$

Further, we prove by induction that for $n>1$,

$$W_n(1)\dots W_n(n-1)(I,0_{n-1})^t=V(n).$$

The main step is as follows:

$$W_{n+1}(1)\dots W_{n+1}(n)(I,0_n)^t=\begin{pmatrix}W_n(1) & , * \\ 0,\dots, & D_{\Gamma_{1n}}, *\end{pmatrix}\dots\begin{pmatrix}W_n(n) & , * \\ 0,\dots, & D_{\Gamma_{n,2n}}, *\end{pmatrix}\begin{pmatrix}I\\0_n\end{pmatrix}=$$

$$=\begin{pmatrix}W_n(1) & ,* \\ 0,\dots,D_{\Gamma_{1n}} & ,*\end{pmatrix}\begin{pmatrix}W_n(2)\dots W_n(n)(I,0_{n-1})^t & ,\ 0 \\ 0 & ,\ 0\end{pmatrix}=$$

$$= \begin{pmatrix} W_n(1) \begin{pmatrix} U_{2n}\tilde{X}_{2,n+1} \\ D_{\Gamma_{2,n+1}} \cdots D_{\Gamma_{n,n+1}} \end{pmatrix} \\ D_{\Gamma_{1,n+1}} \cdots \quad D_{\Gamma_{n,n+1}} \end{pmatrix} = \begin{pmatrix} U_{1n}\tilde{X}_{1,n+1} \\ D_{\Gamma_{1,n+1}} \cdots D_{\Gamma_{n,n+1}} \end{pmatrix} = V(n+1). \blacksquare$$

Now, let us denote

$$W(n) : \mathcal{K}_{n+1} \longrightarrow \mathcal{K}_n$$

the unitary extensions of the isometries $W_+(n)$. We obtain the following dilation result:

<u>3.3 THEOREM</u> Let $\{B_{ij}\}_{\substack{i=1 \\ j>i}}^{\infty}$ be a family of contractions on a Hilbert space $\mathcal{H}$ so that $B_{1n} \geq 0$ for every $n \in \mathbb{N}$. Then, there exists a sequence of unitary operators $W(n) \in \mathcal{L}(\mathcal{K}_{n+1}, \mathcal{K}_n)$ with:

$$S_{ij} = P^{\mathcal{K}_i}_{\mathcal{H}} W(i) \ldots W(j-1) / \mathcal{H} \qquad , i \geq 1, j > i$$

and an appropriate minimality condition.

<u>PROOF</u> It is an immediate consequence of Theorem 3.1 and Theorem 3.2. $\blacksquare$

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