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by

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1. Introduction

We consider a standard process X with localy compact with denombrable base state space $(E, \mathcal{E})$, a regular point $a \in E$ and the local time L in a normalisated such that

$$(1.1) \qquad E^a\left(\int_0^\infty e^{-s}\, d L_s\right) = 1$$

(notations will be those in Blumenthal and Getoor /1/.

Our main result is Theorem 2.4. which gives necessary and sufficient conditions in order that a sequence A^n, $n \in N$ (or a family A^ε, $\varepsilon \to 0$) of increasing, adapted, right continuous with left hand limits (in short cadlag) processes converges to L. This convergence is an uniform L^2 convergence. To be more exact we introduce the following (possibly infinite) distance between two measurable processes M and N:

$$d_2(M, N) = \sup_x E^x(\sup_t (M_t - N_t)^2)$$

and for A^n we define

$$\bar{A}^n = \int_0^t e^{-s} \, d\, A^n_s$$

Theorem 2.4. gives necessary and sufficient condition in order that

$$d_2(\bar{A}^n, \bar{L}) \xrightarrow{n} 0$$

We mention that the above convergence implies $\lim_n d_2(A^n, L) = 0$ (see (2.14)).

Conditions on A^n will be expressed in terms of some parameters which controle the difference between an increasing process and the local time: the first one, Γ (see (2.6)) refer to the additivity property; the second one, $\bar{\Delta}$, (see (2.11)) refers to discontinuities and the two others, r and c (see (2.15) and (2.16)) are about the "support".

Before dealing with the problem of the convergence to the local time, we give a general theorem (Theorem 2.2, which is interesting by itself) of convergence (under d_2) of a sequence of increasing processes to an increasing process. This theorem is a consequence of Theorem 96. p.176 in Dellacherie and Meyer /2/ which ensures that if A^1 and A^2 are two optional increasing processes such that the left potential of $A^1 - A^2$ in dominated under module by an uniformly integrable martingale M, then

(1.2) $\quad E((A^1_\infty - A^2_\infty)^2) \leqslant 2\, E(M^2_\infty)(\, \| A^1_\infty \|^2_2 + \| A^2_\infty \|^2_2)$

By (1.2) and Doob's maximal inequality we get the inequality (2.12) for $d_2(A^1, A^2)$ (which yields Theorem 2.2). We mention that the same two inequalities are used by Getoor in /4/ in order to get a convergence theorem for downcrossing processes to the local time for processes with homogenous independent increments.

The theorem of convergence to the local (Theorem 2.4) will appear as a particular case of Theorem 2.2 (above mentioned). To this end we have to evaluate the difference between the potential of an increasing process and the potential of the local time. This is done by Lemma 2.3, and Theorem 2.4. follows immediately. As a corollary we give sufficient (but not necessary) conditions for almost sure convergence to the local time (this corollary follows from Theorem 2.4. by a Borel - Cantelli argument).

In sections 4, 5, 6 and 7 we are dealing with some particular approximation models. In section 4 we consider a sequence of continuous additive functionals which are normalised such that (1.1) holds for them and prove that if their supports are closed to a, this sequence of functionals converges to the local time L. The occupation time, the "area" and the "residual area" (see section 4.c) are exemples of this kind. In section 5 we give a general model of "downcrossings" and section 6 gives some informations about the Hausdorff - Besicovich measure of the set $\{t : X_t = a\}$ (it is a generalisation of 2.5. in Ito and Mc Keen /7/). Section 7 deals with the excursion model which was introduced by Fristed and Taylor /3/. We prove by our methods the main results in this work (in a slightly stronger form) and also some variants of them.

Except for section 6 (in which only the assumption that the point a is instantaneous is requiered) we have to assume that

$$\lim_{b \to a} m(1 - \varphi_a(b)\,\varphi_b(a)) = 0$$

where $\varphi_x(y) = E^y(e^{-T_x})$. We mention that this condition is true for processes with homogenous independent increments for which the local time exists and which have instantaneous states (see Getoor and Kerten /5/).

2. Main results

a. Notations and parameters

We consider a standard process $X = (\Omega, F, F_t, X_t, \theta_t, P^x)$ with state space $(E, \mathcal{E})$. For a function $f \in F$, $\|f\|_2^x = E^x(f^2)^{1/2}$ and $\|f\|_2 = \sup_x \|f\|_2^x$. For two measurable processes A and B we define:

$$d_2^x(A, B) = E^x(\sup_t (A_t - B_t)^2) \quad \text{and}$$

$$d_2(A, B) = \sup_x d_2^x(A, B)$$

We shall use the convention from Dellacherie and Meyer /2/: when speaking about an "increasing process" A we shall also assume without other mention that this process is cadlag, $A_t \in F_t$, $A_0 = 0$ and $A_\infty = \sup_t A_t$. For such a process we define

$$\overline{A}_t = \int_{[0,t]} e^{-s} \, dA_s$$

(with the convention that dA is concentrated on $\{t : A_t < \infty\}$

All the informations about A will be expressed in terms of A. We define u_A (we write u if no confusion is possible) to be the 1 - potential of A:

(2.1) $$u(x) = E^x(\overline{A}_\infty) = E^x\left(\int_0^\infty e^{-s} \, dA_s\right)$$

For the processes we shall deal with the following additional condition will also always be assumed:

(2.2) $u(x) < \infty$ for every $x \in E$

We start now to define the additivity parameter $\bar{\Gamma}$ (see (2.6)). We note first that by (2.2), $\bar{A}_\infty < \infty$ a.s. and so we may define:

(2.3) $$\Gamma_t = \bar{A}_\infty - \bar{A}_t - e^{-t}\,\bar{A}_\infty \circ \theta_t$$

We also define for every $x \in E$:

(2.4) $$Y_t^x = M_t^x - \bar{A}_t \quad \text{and}$$

$$Y_t = \Sigma\, 1_{(X_0 = x)}\, Y_t^x$$

where M^x is the cadlag version of $E^x(\bar{A}_\infty \mid F_t)$

$$Z_t = e^{-t}\,u(X_t)$$

$$\Gamma'_t = Y_t - Z_t$$

We note that for every $x \in E$ and every stopping time T

(2.5) $$E^x(\Gamma_T \mid F_T) = \Gamma'_T \quad P^x \text{ a.s.}$$

We are now able to define the additivity parameter $\bar{\Gamma}$:

(2.6) $$\bar{\Gamma} = d_2(\Gamma', 0)^{1/2} = \sup_x \,(E^x(\sup_t\,(\Gamma'_t)^2))^{1/2}$$

We list now some simple inequalities which give an idea about the way in which $\bar{\Gamma}$ works. The first one will be usefull through applications in order to evaluate $\bar{\Gamma}$:

(2.7) If $|\Gamma_T| \leq K$ a.s. for every stopping time T then $\bar{\Gamma} \leq K$

We return now to (2.3) and write it in the form

(2.8) $$\bar{A}_\infty = \bar{A}_T + e^{-T}\,\bar{A}_\infty \circ \theta_T + \Gamma_T$$

(this equality makes sense because $\bar{A}_\infty \circ \theta_T$ a.s. for every stopping time T (see (2.2))).

Using the strong Markov property we get an analogue of Dynkin's formula:

(2.9) $u(x) = E^x(\bar{A}_T) + E^x(e^{-T} u(X_T)) + E^x(\Gamma_T)$

and note that for the error which appears in the right side of (2.9) we have the inequality

(2.10) $|E^x(\Gamma_T)| \leq \bar{\Gamma}$ for every $x \in E$ and every stopping time T.

We go on and define the "discontinuity parameter" $\bar{\Delta}$:

(2.11) $$\Delta_t = \Delta\bar{A}_t = \bar{A}_t - \bar{A}_{t-}$$

$$\bar{\Delta} = d_2(\Delta, 0)^{1/2} = \sup_x(E^x(\sup_t \Delta_t^2))^{1/2}$$

We note that $E^x(\Delta\bar{A}_T) \leq \bar{\Delta}$ for every $x \in E$ and stopping time T.

b. A general result

Let A^1 and A^2 be two increasing processus and $c = (\|A^1_\infty\|_2 + \|\bar{A}^2_\infty\|_2) \vee 1$. To get a nice form for Lemma 2.1. we assume that

$$\|u_1 - u_2\| + d_2(\Gamma_1', \Gamma_2') + d_2(\Delta_1, \Delta_2) \leq 1/2$$

Lemma 2.1

(2.12) $d_2(\bar{A}^1, \bar{A}^2) \leq c C_1(\|u_1 - u_2\| + d_2(\Gamma_1', \Gamma_2') +$

$+ d_2(\Delta_1, \Delta_2)) \leq c\, C_2\, d_2(\bar{A}^1, \bar{A}^2)$

If $\bar{A}^1$ and $\bar{A}^2$ are previsible processus than

(2.13) $d_2(\bar{A}^1, \bar{A}^2) \leq c C_1(\|u_1 - u_2\| + d_2(\Gamma_1', \Gamma_2')) \leq$

$\leq c C_2\, d_2(\bar{A}^1, \bar{A}^2)$

with C_1 and C_2 universal constants.

Remark. If A^1 and A^2 are natural additive functionals of a Hunt process, they are previsible, $\Gamma_1' = \Gamma_2' = 0$ and so (2.13) becomes

$$d_2(\tilde{A}^1, \tilde{A}^2) \leqslant c\, C_1 \| u_1 - u_2 \| \leqslant c\, C_2\, d_2(\tilde{A}^1, \tilde{A}^2)$$

Remark. The following inequalities make clear the connection between $\tilde{A}$ and A under the metric d_2: for every $\varepsilon > 0$

$$(2.14) \qquad \sup_x E^x(\sup e^{-(2-\varepsilon)t}(A_t^1 - A_t^2)^2) \geqslant d_2(\tilde{A}^1, \tilde{A}^2) \geqslant$$

$$\geqslant \sup_x E^x(\sup_t e^{-2t}(\tilde{A}_t^1 - \tilde{A}_t^2)^2)$$

The above inequalities easily follow by using an integration by parts.

The following theorem is an immediate consequence of Lemma 2.1. and we shall omite the proof:

Theorem 2.2. Let A^n, $n \in N$ and A be increasing processes with $\sup_n \| A^n_\infty \|_2, \| A_\infty \|_2 < \infty$. In order that $\lim_n d_2(\tilde{A}^n, \tilde{A}) = 0$ it is necessary and sufficient that $\lim_n \| u_n - u \| = \lim_n d_2(\Gamma_n', \Gamma') = \lim_n d_2(\Delta_n, \Delta) = 0$. If A^n and A are previsible we may drop out the condition $\lim_n d_2(\Delta_n, \Delta) = 0$.

c. Convergence to the local time

Let $a \in E$ be the regular point fixed in the beginning of the paper and L the local time in a satisfying (1.1).

For an increasing process A we define two parameters r and c which give an idea about the "support" of A: for a nearly Borel set $B \subseteq E$

$$(2.15) \qquad r(B) = \sup_x E^x(\tilde{A}_{T_B}) = \sup_x E^x\Big(\int_{[0,T_B)} e^{-s}\, d A_s\Big)$$

with $T_B = \inf\{t > 0 : X_t \in B\}$.

We note that if A is a C.A.F. and B is closed then $r(B) = 0$ iff $\sup A \subseteq B$.

We also define

(2.16) $c(B) = \sup\{1 - \varphi_a(x)\varphi_x(a) : x \in \overline{B}\}$

with $\varphi_x(y) = E^y(e^{-T_x})$, $T_x = T_{\{x\}}$ for $x, y \in E$.

Lemma 2.3. If A is an increasing process such that u and $u(a) = 1$ and B is a nearly Borel set containing the point a, then

(2.17) $$|u(x) - \varphi_a(x)| \leqslant 2\overline{\Gamma} + \overline{\Delta} + r(B) + (2\overline{\Gamma} + c(B))/(1 - c(B))$$

for every $x \in E$.

Theorem 2.4. Let A^n $n \in N$ be increasing processes with $u_n < \infty$ and $u_n(a) = 1$.

(i) If $\lim_n \overline{\Gamma}_n = \lim_n \overline{\Delta}_n = 0$ and for every $\varepsilon > 0$ we may find a nearly Borel set B_ε containing the point a such that $c(B_\varepsilon) \leqslant \varepsilon$ and $\lim_n r_n(B_\varepsilon) = 0$ then $\lim_n d_2(\overline{A}^n, \overline{L}) = 0$.

(ii) If $\lim_n d_2(\overline{A}^n, \overline{L}) = 0$ then $\lim_n \overline{\Gamma}_n = \lim_n \overline{\Delta}_n = \lim_n r_n(\{a\}) = 0$.

Corollary 2.5. Let A^n $n \in N$ be increasing processes such that $u_n < \infty$ and $u_n(a) = 1$. If $\sum_n \overline{\Gamma}_n < \infty$, $\sum_n \overline{\Delta}_n < \infty$ and for every $n \in N$ we may find a nearly Borel set B_n containing the point a such that $\sum_n c(B_n) < \infty$ and $\sum_n r_n(B_n) < \infty$ then

$$\lim_n \sup_t |\overline{A}^n_t - \overline{L}_t| = 0 \quad \text{a.s.}$$

3. Proofs

Proof of Lemma 2.1.

Let us prove at first the inequalities in the right side of (2.12) and (2.13)]. It is easy to see that $\|u_1 - u_2\| \leqslant d_2(\bar{A}^1, \bar{A}^2)$ and $d_2(\Delta_1, \Delta_2) \leqslant 2\, d_2(\bar{A}^1, \bar{A}^2)$. We note that

$$d_2(\Gamma_1', \Gamma_2') \leqslant 2(\sup_x E^x(\sup_t(Y_1^x(t) - Y_2^x(t))^2) +$$

$$+ \sup_x E^x(\sup_t(Z_1(t) - Z_2(t))^2)$$

The second term in the right side of the above inequalities is bounded by $\|u_1 - u_2\| \leqslant d_2(\bar{A}^1, A^2)$. For the first one we write

$$\sup_t(Y_1^x(t) - Y_2^x(t))^2 \leqslant 2(\sup_t E^x(\bar{A}^1_\infty - \bar{A}^2_\infty | F_t)^2 +$$

$$+ \sup_t (\bar{A}^1_t - \bar{A}^2_t)^2)$$

We apply Doob's maximal inequalities for the submartingale $E^x(\bar{A}^1 - \bar{A}^2 | F_t)^2$ and get

$$E^x(\sup_t E^x(\bar{A}^1 - \bar{A}^2 | F_t)^2) \leqslant 4\, E^x((A^1 - A^2)^2) \leqslant$$

$$\leqslant 4\, d_2(A^1, A^2)$$

which ends the proof of the inequalities in the right side of (2.12) and (2.13).

Let us now prove the inequality in the left side of (2.12). We write

$$\bar{A}^1_t - \bar{A}^2_t = (\bar{A}^2_\infty - \bar{A}^2_t) - (\bar{A}^1_\infty - \bar{A}^1_t) + (\bar{A}^1_\infty - \bar{A}^2_\infty)$$

which yields

$$\bar{A}^1_t - \bar{A}^2_t = Y_2^x(t) - Y_1^x(t) + E^x(\bar{A}^1_\infty - \bar{A}^2_\infty | F_t) \quad P^x \text{ a.s.}$$

As above, we use Dood's maximal inequality for $E^x(\bar{A}^1_\infty - \bar{A}^2_\infty | F_t)^2$ and get

(3.1) $\quad d_2^x(\bar{A}^1, \bar{A}^2) \leqslant 2\, d_2^x(Y_1^x, Y_2^x) + 8\, E^x((\bar{A}^1_\infty - \bar{A}^2_\infty)^2)$

By $\Upsilon_i^x(t) = \Gamma_i'(t) + Z_i(t)$ $i = 1,2$ P^x a.s.

and $d_2^x(Z_1, Z_2) \leqslant \| u_1 - u_2 \|$ we get

$$(3.2) \qquad d_2^x(\Upsilon_1^x, \Upsilon_2^x) \leqslant 2(\|u_1 - u_2\| + d_2^x(\Gamma_1', \Gamma_2'))$$

To bound the second term in the right side of (3.1) we shall use theorem 96. p.176 in Dellacherie et Meyer /2/ for the process $\bar{A}^1 - \bar{A}^2$ ($c < \infty$ thus this process has integrable variation). Its left potential is

$$(3.3) \qquad E^x(\bar{A}^1 - \bar{A}^2 \mid F_t) - (\bar{A}^1_{t-} - \bar{A}^2_{t-})$$

which under module is dominated by the martingale $H_t = E^x(H \mid F_t)$ with

$$H = \sup_t |\Delta_t^1 - \Delta_t^2| + \sup_t |\Upsilon_1^x(t) - \Upsilon_2^x(t)|$$

By the above mentioned theorem we get

$$(3.4) \qquad E^x((\bar{A}^1 - \bar{A}^2)^2) \leqslant 2\, E^x(H \int_{[0,\infty)} d|\bar{A}^1_s - \bar{A}^2_s|) \leqslant$$

$$\leqslant 2 \|H\|_2^x (\|A^1_\infty\|_2^x + \|A^2_\infty\|_2^x) \leqslant 2c\|H\|_2^x$$

But $E^x(H^2) \leqslant 2(d_2^x(\Upsilon_1^x, \Upsilon_2^x) + d_2^x(\Delta^1, \Delta^2))$, thus (3.1), (3.2) and (3.4) imply the first inequality in (2.12).

If A^1 and A^2 are previsible we may apply theorem 96 (above mentioned) for previsible processes and so we have to consider the potential of $\bar{A}^1 - \bar{A}^2$ which is $E^x(\bar{A}^1 - \bar{A}^2 \mid F_t) - (\bar{A}^1_t - \bar{A}^2_t)$. In this case $H = \sup_t |\Upsilon_1^x(t) - \Upsilon_2^x(t)|$ and the parameter Δ disappears.

Proof of Lemma 3.3.

Since $X(T_a) = a$ and $X(T_x) = x$ a.s., by applying twice (2.9) we get

$$u(a) = E^a(\bar{A}_{T_x}) + E^a(\Gamma_{T_x}) + \varphi_x(a)(E^x(\bar{A}_{T_a}) + E^x(\Gamma_{T_a}) +$$

$$+ \varphi_a(x)\, u(a)$$

and since $u(a) = 1$ we get

$$1 - \varphi_a(x)\, \varphi_x(a) = E^a(\bar{A}_{T_x}) + E^a(\Gamma_{T_x}) + \varphi_x(a)\, E^x(\bar{A}_{T_a}) +$$

$$+ \varphi_x(a)\, E^x(\Gamma_{T_a})$$

and since $\bar{A}_{T_x} \geqslant 0$, by (2.10)

$$E^x(\bar{A}_{T_a}) \leqslant (2\bar{\Gamma} + (1 - \varphi_a(x)\, \varphi_x(a)))/\, \varphi_x(a)$$

which by the definition of $c(B)$ yields

$$\text{(3.5)} \qquad \sup_{x \in \bar{B}} E^x(\bar{A}_{T_a}) \leqslant (2\bar{\Gamma} + c(B))/(1 - c(B))$$

We put $T = T_B$. Because $T \leqslant T_a$, T_a is strong termined and $T_a \circ \Theta_{T_a} = 0$ a.s.

$$\text{(3.6)} \qquad T_a = T + T_a \circ \Theta_T$$

We also note that

$$\text{(3.7)} \qquad E^x(\bar{A}_T) = E^x(\bar{A}_{T-}) + E^x(\Delta_T) \leqslant r(B) + \bar{\Delta}$$

We are now able to prove (2.17). By (2.9)

$$\text{(3.8)} \qquad u(x) = E^x(\bar{A}_T) + E^x(\Gamma_T) + E^x(e^{-T} u(X_T))$$

By applying (2.9) to the last term in the right of (3.8) we may write it in the form

$$\text{(3.9)} \qquad E^x(e^{-T} e^{X_T}(\bar{A}_{T_a})) + E^x(e^{-T} E^{X_T}(\Gamma_{T_a})) +$$

$$+ E^x(e^{-T} E^{X_T}(e^{-T_a} u(a)))$$

Since $u(a) = 1$, (3.6) implies that the last term in (3.9) is $\varphi_a(x)$ and so we may write (3.8) in the form

$$(3.10)\quad u(x) - \varphi_a(x) = E^x(\bar{A}_T) + E^x(\Gamma_T) + E^x(e^{-T} E^{X_T}(\Gamma_{T_a})) + $$
$$+ E^x(e^{-T} e^{X_T}(\bar{A}_{T_a}))$$

We consider (3.10) under module and dominate the terms in its right side in the following way: the first one by $r(B) + \bar{\Delta}$ (see 3.7), the two following ones by $2\bar{\Gamma}$ (see 2.10) and because $X_T \in \bar{B}$ a.s. the last one is dominated by $(2\bar{\Gamma} + c(B))/(1 - c(B))$ (see (3.5)).

Proof of theorem 2.4 (i)

We shall first prove that for sufficiently large n

$$(3.11)\qquad \| \bar{A}^n_\infty \|_2 \leqslant 11$$

To this end we consider B such that $c(B) \leqslant 1/2$ and by lemma 2.3. we get $\| u_n \| \leqslant 1 + 6\bar{\Gamma}_n + \bar{\Delta}_n + r_n(B) + 2$ which for sufficiently large n is less than 4. We write now

$$(3.12)\qquad E^x((\bar{A}^n_\infty)^2) = 2\, E^x\left(\int_0^\infty e^{-s}(\bar{A}^n_\infty - \bar{A}^n_s)\, d\, A^n_s\right) +$$
$$+ E^x\left(\int_0^\infty e^{-s}\, \Delta \bar{A}^n_s\, d\, A^n_s\right)$$

The first term in the right side of (3.12) is equal to

$$2\, E^x\left(\int_0^\infty e^{-s}\, Y^n_s\, d\, A^n_s\right) \leqslant 2\, E^x(\sup_s Y^n_s \cdot A^n_\infty) \leqslant$$
$$\leqslant 2\, d_2^x(Y^n, 0)^{1/2}\, \| \bar{A}^n_\infty \|_2$$

which by (3.2) is dominated by

$$2(\| u_n \| + \bar{\Gamma}_n)\, \| \bar{A}^n_\infty \| \leqslant 10\, \| \bar{A}^n_\infty \|_2$$

(for $\bar{\Gamma}_n \leqslant 1$).

We dominate the second term in the right side of (3.12) by $\Delta_n \cdot \| A^n_\infty \|_2 \leqslant \| \bar{A}^n_\infty \|_2$ $(\Delta_n \leqslant 1)$ and so we get $\| \bar{A}^n_\infty \|_2^2 \leqslant 11 \cdot \| \bar{A}^n_\infty \|_2$ which yields (3.11).

Now we may apply Lemma 2.3. to get $\lim_n \| u_n - u \| = 0$ (with u_n associated to A^n and u associated to L by (2.1)), and then Theorem 2.2. to get $\lim_n d_2(\bar{A}^n, \bar{L}) = 0$.

The point (ii) of Theorem 2.4. is immediate.

Proof of Corollary 2.5.

By (2.12) and (2.17)

$$d_2(\bar{A}^n, \bar{L}) \leqslant K(\bar{\Gamma}_n + \bar{\Delta}_n + c(B_n) + r(B_n))$$

where K is an universal constant. Thus, by our assumptions

$$\sum_n d_2(\bar{A}^n, \bar{L}) < \infty$$

Chebyshev's inequality and a Borel Cantelli argument yields $\lim_n \sup_t |\bar{A}^n_t - \bar{L}_t| = 0$ a.s.

4. Approximation by continuous additive functionals

a. A general result

We are in the context defined in section 2.a. and assume that

(4.1) $$\lim_{b \to a} (1 - \varphi_a(b)\, \varphi_b(a)) = 0$$

Let us consider a family $(A^\varepsilon, \varepsilon \to 0)$ of C.A.F's $a_\varepsilon = E^a(\int_0^\infty e^{-s}\, d A^\varepsilon_s)$ and $u_\varepsilon(x) = a_\varepsilon^{-1} E^x(\bar{A}^\varepsilon_\infty)$.

The following theorem is an immediate consequence of Theorem 2.4. and Corollary 2.5.

Theorem 4.1. Assume that (4.1) holds and $u_\varepsilon < \infty$ for every $\varepsilon > 0$.

(i) If $\lim_{\varepsilon \to 0} d(\text{supp } A^\varepsilon, a) = 0$ then

$$\lim_{\varepsilon \to 0} d_2(a_\varepsilon^{-1} \bar{A}^\varepsilon, \bar{L}) = 0$$

(ii) If we choose $\varepsilon_n > 0$ such that $\sum_n c(\operatorname{supp} A^{\varepsilon_n}) < \infty$ then

$$\lim_{\varepsilon \to 0} \sup_t |a_\varepsilon^{-1} \bar{A}_t^\varepsilon - \bar{L}_t| = 0 \quad \text{a.s.}$$

We consider now a C.A.F. A, such that $a \in \operatorname{supp} A$ and put $S_\varepsilon = \{x : d(a,x) \leqslant \varepsilon\}$, $c_\varepsilon = \sup\{1 - \varphi_a(x)\, \varphi_x(a) : x \in S_\varepsilon\}$, $a_\varepsilon = E^a(\int_0^\infty e^{-s}\, 1_{S_\varepsilon}(X_s)\, dA_s)$ and $A_t^\varepsilon = a_\varepsilon^{-1} \int_0^\infty 1_{S_\varepsilon}(X_s) dA_s$.

The following corollary is an immediate consequence of Theorem 4.1.:

<u>Corollary 4.2.</u> We assume that (4.1) holds and $u_A < \infty$

(i) If $\lim_{\varepsilon \to 0} c_\varepsilon = 0$ then $\lim_{\varepsilon \to 0} d_2(\bar{A}^\varepsilon, \bar{L}) = 0$

(ii) If $\sum_n c_{\varepsilon_n} < \infty$ then $\lim_{\varepsilon \to 0} \sup_t |\bar{A}_t^{\varepsilon_n} - \bar{L}_t| = 0$ a.s.

b. <u>Approximation by local times</u>

We assume that every x in a neighbourhood of the point a is regular and denote by L^x the local time in x with the usual normalisation

$$E^x(\int_0^\infty e^{-s}\, dL_s^x) = 1$$

<u>Proposition 4.3.</u> Assume that (4.1) holds.

Then $\lim_{x \to a} d_2(\bar{L}^x, \bar{L}^a) = 0$

Proof. Put $a_x = E^a(\int_0^\infty e^{-s}\, dL_s^x) = \varphi_x(a) \longrightarrow 1$ as $x \to a$. By Theorem 4.1. $\lim_{x \to a} d_2(a_x^{-1} \bar{L}^x, \bar{L}^a) = 0$ thus our Lemma is proved.

c. <u>Occupation time, area and residual area</u>

Let us put

$$A_t^{\varepsilon} = \int_0^t 1_{S_\varepsilon}(X_s)ds \qquad a_\varepsilon = E^a\left(\int_0^\infty e^{-s}\, dA_s^{\varepsilon}\right)$$

$$M_t^{\varepsilon} = \varepsilon t - \int_0^t \varepsilon \wedge d(X_s,a)ds \qquad m_\varepsilon = E^a\left(\int_0^\infty e^{-s}\, dM_s^{\varepsilon}\right)$$

$$N_t^{\varepsilon} = \int_0^t d(X_s,a)\; 1_{S_\varepsilon}(X_s)ds \qquad n_\varepsilon = E^a\left(\int_0^\infty e^{-s}\, dN_s^{\varepsilon}\right)$$

(clearly $A_t^\varepsilon, M_t^\varepsilon, N_t^\varepsilon < \infty$ for $t < \infty$ and $a_\varepsilon, m_\varepsilon, n_\varepsilon < \infty$)
A is the wellknown occupation time, M is the "residual area" (which was introduced in section 9 in Fristed and Taylor /3/) and N is the "area" (the intuitive interpretation of M and N are immediate from a picture).

A^ε, M^ε and N^ε are C.A.F's with the support included in S_ε and so the following proposition is an immediate consequence of Theorem 4.1.

Proposition 4.4. Assume that (4.1) holds.

(i) $\lim_{\varepsilon\to 0} d_2(a_\varepsilon^{-1}\overline{A}^\varepsilon, \overline{L}) = 0$ (the same for M and N)

(ii) If $\sum_n c_{\varepsilon_n} < \infty$ then $\lim_n \sup_t |a_{\varepsilon}^{-1}\overline{A}_t^{\varepsilon_n} - \overline{L}_t| = 0$ a.s. (the same for M and N)

5. Downcrossings

We are in the general context defined in section 2.a. and assume that (4.1) holds. By the obvious inequality $\varphi_x(y) \geq \varphi_x(a)\, \varphi_a(y)$ we get

(5.1) $$\lim_{x,y\to a} (1 - \varphi_x(y)\, \varphi_y(x)) = 1$$

We give now a general model of "downcrossings".

Let us consider a ~~system~~ $M_1, \ldots, M_p$ of disjoint closed subsets of E and denote $N_i = \bigcup_{j\neq i} M_j$ and $N = \bigcup_j M_j$. For this system of sets we define the following sequence of stopping times:

$$T = \sum_{i=1}^{p} 1_{M_i}(X_o) T_{N_i} + \infty \cdot 1_{C_N}(X_o)$$

$$T_o = T_N \quad \text{and} \quad T_{k+1} = T_k + T \circ \theta_{T_k}$$

Since X is quasi-left-continuous and $X(T_{M_i}) \in M_i$ a.s. we may conclude that $\sup_k T_k = \infty$. We put

(5.2) $$b = \sum_k E^a(e^{-T_k})$$

and we shall need that

(5.3) $$0 < b < \infty$$

Since $b \geq E^a(e^{-T_x})$ for $x \in N$, by (5.1) $b > 0$ if $d(N, a)$ is small enough.

To get the other inequality we write

$$E^a(e^{-T_{k+1}}) = E^a(e^{-T_k} E^{X_{T_k}}(e^{-T_1})) \leq$$

$$\leq E^a(e^{-T_k}) \sup_{x \in N} E^x(e^{-T_1})$$

thus a sufficient condition in order that $b < \infty$ is

(5.4) $$\sup_{x \in N} E^x(e^{-T_1}) < 1$$

To get (5.4) one has to assume that one of the following two conditions holds for N:

(5.5) $\text{card } N < \infty$

(5.6) $\lim_{y \to x} (1 - \varphi_x(y)\, \varphi_y(x)) = 0$ for $y, x \in N$

Since $E^x(e^{-T_1}) < 1$ for every $x \in N$, under (5.5) clearly (5.4) holds.

To prove that under (5.6), (5.4) holds it will suffice to prove that

(5.7) $$\sup_{x \in M_i} E^x(e^{-T_{N_i}}) < 1 \quad \text{for } i \leq p$$

If (5.7) is false we may find $\bar{x} \to x$, x_n, $x \in M_i$ such that $E^{x_n}(e^{-T_{N_i}}) \to 1$ as $n \to \infty$. We write

$$(5.8)\quad E^x(e^{-T_{N_i}}) = E^x(e^{-T_{x_n}}, T_{N_i} > T_{x_n}) E^{x_n}(e^{-T_{N_i}}) +$$

$$+ E^x(e^{-T_{N_i}}, T_{N_i} < T_{x_n})$$

But

$$(5.9)\quad E^x(e^{-T_{x_n}}\; T_{N_i} > T_{x_n}) =$$

$$= \varphi_{x_n}(x) - E^x(e^{-T_{N_i}}\, \varphi_{x_n}(X_{T_{N_i}}), T_{N_i} < T_{x_n}) \geq$$

$$\geq \varphi_{x_n}(x) - E^x(e^{-T_{N_i}}, T_{N_i} < T_{x_n}).$$

Using (5.8), (5.9) and $\lim_n E^{x_n}(e^{-T_{N_i}}) = 1$ we get $E^x(e^{-T_{N_i}}) = 1$ which is contradictory.

We go on and present our construction:

$B_t = k$ on $T_k \leq t < T_{k+1}$ and

$A_t = b^{-1} B_t$

We note that $A_t = b^{-1} \sum_{i=0}^{k} e^{-T_i}$ on $T_k \leq t < T_{k+1}$

We shall now study the parameters of A. It is clear that

$$\bar{\Delta} \leq 1/b$$

To get the same inequality for   , it will suffice that $|\Gamma_S| \leq 1/b$ for every stopping time S (see (2.7)). To prove this we consider the following two possibilities

(a) $T_k \leq S < T_{CM_i}$ and $X(T_k) \in M_i$

(b) $T_{CM_i} \leq S < T_{k+1}$ and $X(T_k) \in M_i$

In the first case $S + T_0 \circ \Theta_S = S$ and $S + T_p \circ \Theta_S = T_{p+k}$ for $p \geqslant 1$, and so

$$\Gamma_S = b^{-1}\left(\sum_{p>k} e^{-T_p} - \sum_{p=0} e^{-(S+T_p \circ \Theta_S)} \right) = -1/b\ e^{-S}$$

In the second case $S + T_p \circ \Theta_S = T_{p+k+1}$ for $p \geqslant 0$ and so $\Gamma_S = 0$. So we may conclude that

$$\overline{\Gamma} \leqslant 1/b$$

Since $X(T_k) \in N$ a.s., $r(N) = 0$. In order to evaluate $\overline{\Delta}$ and $\overline{\Gamma}$ we shall need an evaluation of $1/b$. For this purpose let us consider $x \in M_1$, $y \in M_2$ and define

$$S = T_x + T_y \circ \Theta_{T_x}$$

$$S_0 = T_x \quad \text{and} \quad S_{k+1} = S_k + S \circ \Theta_{S_k} \quad \text{for } k \geqslant 1$$

Obviously $S_k \geqslant T_k$ and so

(5.10) $b \geqslant \sum_k E^a(e^{-S_k}) = \varphi_x(a)/(1 - \varphi_x(y)\varphi_y(x))$

Let us consider now for every $n \in N$ a finite sistem $M_1^n, \ldots, M_{p_n}^n$ for which (5.5) or (5.6) holds and A^n the asociated increasing processes.

<u>Theorem 5.1.</u> Assume that (4.1) holds

(i) If $\lim_n d(N_n, a) = 0$ then $\lim_n d_2(\overline{A}^n, \overline{L}) = 0$

(ii) If $\sum_n c(N_n) < \infty$ then $\lim_n \sup_t |\overline{A}_t^n - \overline{L}_t| = 0$ a.s.

<u>Proof.</u> By (5.10) and (5.1), $\lim_n d(N_n,a) = 0$ implies that $\lim_n \overline{\Delta}_n = \lim_n \overline{\Gamma}_n = \lim_n c(N_n) = 0$ thus (i) is a consequence of Theorem 2.4. (i). For sufficiently large n, $1/b_n \leqslant 2c(N_n)$ thus (ii) is a consequence of Corollary 2.5.

Remark. For the system of sets $\{x_n\},\{y_n\}$ with $x_n, y_n \to a$ we get a generalisation of (1.9) in Getoor /4/, and for the system $(\varepsilon, \lambda\varepsilon)$, $(-\lambda\varepsilon, -\varepsilon)$ we get the limit theorem stated in the same work in the low part of p.3 (only the L^2 convergence). For the system $\{x_n\}, \{a\}$ we get Corollary 6.5. in Fristed and Taylor /3/.

Remark. Another model of downcrossings may be the following: consider the system of sets $M_1, \ldots, M_p$ and assume that B increases with one after X has visited $M_1, M_2, \ldots, M_p$ in this precise order. The evaluation of b changes in

$$b \geqslant \varphi_x(a)/(1 - \varphi_{x_1}(x_2)\,\varphi_{x_3}(x_3) \ldots \varphi_{x_{p-1}}(x_p))$$

where $x_i \in M_i$ are fixed points. Proofs go like above.

6. The Hausdorff Besicovitch dimension

We are in the general context defined in section 2.a. and we assume that the point a is instantaneous. We shall give a theorem of convergence to the local time which yields the same inequality for the Hausdorff Besicovitch dimension of $\Lambda_t(\omega) = \{t : X_t(\omega) = a\}$ as 2.5. in Ito and Mc Keen /7/. We do not consider the same model as there because an evaluation of Γ would be very difficult.

For a fixed $\varepsilon > 0$ we define

$$T_\varepsilon = \varepsilon + T_a \circ \theta_\varepsilon$$

$$T_0^\varepsilon = 0 \quad \text{and} \quad T_{k+1}^\varepsilon = T_k^\varepsilon + T_\varepsilon \circ \theta_{T_k^\varepsilon} \quad \text{for } k \geqslant 0$$

Since $X(T_k^\varepsilon) = a$ a.s. a simple calculation shows that

$$\sum_k E^a(e^{-T_k^\varepsilon}) = (1 - E^a(e^{-T_\varepsilon}))^{-1}$$

We define now:

$$B_t^\varepsilon = k \quad \text{on} \quad T_k^\varepsilon \leqslant t < T_{k+1}^\varepsilon$$

$$b_\varepsilon = 1 - E^a(e^{-T_\varepsilon}) = 1 - e^{-\varepsilon} E^a(\varphi_a(X_\varepsilon))$$

$$A_t^\varepsilon = b_\varepsilon \cdot B_t^\varepsilon \quad \text{and} \quad u_\varepsilon(x) = E^x(\tilde{A}_\infty^\varepsilon)$$

It is clear that $u_\varepsilon(a) = 1$ and $u_\varepsilon < \infty$. Since $X_{T_k^\varepsilon} = a$ a.s., $\gamma_\varepsilon(\{a\}) = 0$. It is also clear that $\overline{\Delta}_\varepsilon \leqslant b_\varepsilon$ and for $\overline{\Gamma}_\varepsilon \leqslant 3\, b_\varepsilon$ see the deterministic calculation in the appendix at the end of this section.

<u>Theorem 6.1.</u> If a is an instaneous point, then

(i) $\lim\limits_{\varepsilon \to 0} d_2(\overline{A}^\varepsilon, \overline{L}) = 0$

(ii) $\lim\limits_{\varepsilon \to 0} \sup\limits_t |\tilde{A}_t^\varepsilon - \overline{L}_t| = 0$ a.s.

<u>Proof</u>. Since φ_a is 1 - excessive, $\varepsilon \to \varphi_a(X_\varepsilon)$ is almost sure right continuous on the trajectories thus

$$\lim_{\varepsilon \to 0} b_\varepsilon = 0$$

and so Theorem 2.4. implies (i).

To prove (ii) we shall first prove that $\varepsilon \to b_\varepsilon$ is continuous. The right continuity follows from the above mentioned argument and to prove the left continuity it will suffice to show that $T = \sup\limits_{\varepsilon < \varepsilon_0} T_\varepsilon = T_{\varepsilon_0}$. The inequality $T \leqslant T_{\varepsilon_0}$ is immediate and it is also clear that $T \geqslant \varepsilon_0$. If $T > \varepsilon_0$, by the quasi-left-continuity $X_T = \lim\limits_{\varepsilon \uparrow \varepsilon_0} X_{T_\varepsilon} = a$ and so $T \geqslant T_{\varepsilon_0}$. If $T = \varepsilon_0$ the same argument shows that $X_{\varepsilon_0} = a$ thus, since a is regular, $T_{\varepsilon_0} = \varepsilon_0 + T_a \circ \theta_{\varepsilon_0} = \varepsilon_0$ and the proof is complete.

Now we may choose $\varepsilon_n > 0$, $n \in N$ such that $b_{\varepsilon_n} = 1/n^2$. By Corollary 4.5.

(6.1) $$\lim_n \sup_t |\overline{A_t}^{\varepsilon_n} - \overline{L}_t| = 0 \text{ a.s.}$$

For $\varepsilon_{n+1} \leqslant \varepsilon < \varepsilon_n$

$$b_{\varepsilon_{n+1}} B_t^{\varepsilon_n} \leqslant b_\varepsilon B_t^\varepsilon \leqslant b_{\varepsilon_n} \overline{B}_t^{\varepsilon_{n+1}}$$

thus, since $\lim_n b_{\varepsilon_{n+1}}/b_{\varepsilon_n} = 1$, (6.1) implies (ii).

Corollary 6.2. If from some $\delta > 0$

$$\limsup_{\varepsilon \to 0} \varepsilon^{1/\delta} / (1 - e^{-\varepsilon} E^a(\varphi_a(X_\varepsilon))) = c < \infty$$

then the Hausdorff Besicovitch dimension of

$\Lambda_t(\omega) = \{t : X_t(\omega) = a\}$ is almost sure greater then δ.

Proof. Consider k_t defined by $T_{k_t}^\varepsilon \leqslant t < T_{k_t+1}^\varepsilon$

$\bigcup_{k \leqslant k_t} [T_k^\varepsilon, T_k^\varepsilon + \varepsilon]$ is a closed covering of Λ_t with intervals of length ε.

$$\limsup_{\varepsilon \to 0} \sum_{k \leqslant k_t} ((T_k^\varepsilon + \varepsilon) - T_k^\varepsilon)^{1/\delta} =$$

$$= \limsup_{\varepsilon \to 0} \varepsilon^{1/\delta} B_t^\varepsilon \leqslant c \limsup_{\varepsilon \to 0} b_\varepsilon B_t^\varepsilon = L_t < \infty \text{ a.s.}$$

Appendix

In order to prove that $\overline{\Gamma}_\varepsilon \leqslant 3 b_\varepsilon$ (used in the proof of Theorem 6.1) we'll give a deterministic calculation which clearly implies the above inequality (see Lemma 6.3).

Consider $A \subseteq R_+$ which is assumed to be left closed ($x_n \downarrow x$, $x_n \in A \Rightarrow x \in A$). For a fixed $\varepsilon > 0$ we define

$$D_t = \inf\{u > t : A \cap (t, u) \neq \emptyset\}$$

The following simple proprierties of D will be usefull:

$$\text{(i) } s \leqslant t \Longrightarrow D_s \leqslant D_t, \quad \text{(ii) } t \leqslant D_t,$$

$$\text{(iii) } t < a \text{ and } a \in A \text{ implies } D_t \leqslant a,$$

$$\text{(iv) } D_t \in A.$$

We also define

$$T_o = 0 \quad \text{and} \quad T_{k+1} = D_{T_k+\varepsilon} \quad \text{for } k \geqslant 0$$

(we mention that T_k^{ε} in the definition of B^{ε} are equal to T_k defined with respect to $\{t : X_t = a\}$).

We consider now a fixed $t \in [T_p, T_{p+1})$ and define

$$S_o = t \quad \text{and} \quad S_{k+1} = D_{S_k+\varepsilon} \quad \text{for } k \geqslant 0$$

(we mention that $t + T_k^{\varepsilon} \circ \Theta_t = S_k$).

We put

$$k_o = \inf\{k : T_{k+p+1} > T_{k+p} + 2\varepsilon\}$$

$$= \infty \text{ if the above set is void}$$

It is easy to check that

(a) $T_{k_o+p+1} > T_{k_o+p} + 2\varepsilon$ for $k_o < \infty$

(b) $S_k \in [T_{p+k}, T_{p+k+1}]$ for every $k \in N$

(c) If $S_{k_o} = T_{p+k_o+1}$, than $S_k = T_{p+k+1}$ for $k \geqslant k_o$

(d) If $S_{k_o} \neq T_{p+k_o+1}$ then $S_k = T_{p+k}$ for $k > k_o$ and

$$S_{k_o} \in [T_{p+k_o}, T_{p+k_o} + \varepsilon]$$

We are now able to state our lemma:

<u>Lemma 6.3.</u> If $\mu = \sum e^{-T_k}$ and $\nu = \sum e^{-S_k}$ then

(6.2) $$\left| \mu - \sum_{k \leqslant p} e^{-T_k} - \nu \right| \leqslant 3$$

(6.3) $$\Big|\mu - \sum_{k\leqslant p-1} e^{-T_k} - \nu\Big| \leqslant 2$$

Proof. Clearly (6.2) follows from (6.3). To prove (6.3) we note first that T_k, $S_k \geqslant k\varepsilon$ and so we may change the order in the above sums. Consider first that $k_o < \infty$ and $T_{p+k_o+1} \neq S_{k_o}$. Then, by (d) the term in the left side of (6.3) is

$$\Big|\sum_{k\leqslant k_o} (e^{-T_{p+k}} - e^{-S_k})\Big| \leqslant$$

$$\leqslant \sum_{k<k_o} \Big|e^{-T_{p+k}} - e^{-T_{p+k+1}}\Big| + \Big|e^{-T_{p+k_o}} - e^{-S_{k_o}}\Big| \leqslant$$

$$\leqslant \sum_{k<k_o} e^{-T_{p+k}} \Big|T_{p+k+1} - T_{p+k}\Big| + \varepsilon e^{-T_{p+k_o}} \leqslant$$

$$\leqslant \varepsilon \sum_{k=0}^{\infty} e^{-k\varepsilon} = \varepsilon/(1-e^{-\varepsilon}) \leqslant 2$$

(for small ε).

If $k_o < \infty$ and $T_{p+k_o+1} = S_{k_o}$ an analogue argument works, and if $k_o = \infty$ the argument holds with the sums over k till ∞.

7. Excursions

Through this section we shall study a model of increasing processes constructed in terms of excursions from the point a. This model was conceived in Fristed and Taylor /3/. Although the language is those of excursions we are not essentialy related with results in the theory of Poisson point processes. So in sections a and b we do not use them, but to get a nicer form for our theorems, in sections c we use such results (presented in Ito /7/ and Fristed and Taylor /3/). In section d we deal with counting constructions which may be regarded as a particular form of the excursion model (other possible applications of this model are presented in Fristed and Taylor /3/ but we have already presented them in the other sections of our paper).

a. Construction

We are in the general context defined in section 2.a. and assure that the point a is instantaneous. Since the set $\Lambda(\omega) = \{t > 0 : X_t(\omega) = a\}$ is closed, CZ is a countable union of open intervals $(\alpha(\omega), \beta(\omega))$. For such an interval we define the excursion W by

$$W(t) = X(t + W^-) \quad \text{for} \quad t < W^+ - W^-$$
$$= a \qquad \text{for} \quad t \geq W^+ - W^-$$

with $W^- = \alpha$ and $W^+ = \beta$

We shall ignore the first interval (α, β), (if such an interval exists) and so we may assume that $W^- \geq T_a$. For a positive and measurable function $f : D[0,\infty) \to R$ we define:

$$\varphi^t(\omega) = \sum_{W^- > t} e^{-\alpha W^-} f(W)$$

$$\psi^t(\omega) = \sum_{W^+ > t} e^{-\alpha W^+} f(W)$$

$$\varphi_t(\omega) = \sum_{W^- \leq t} e^{-\alpha W^-} f(W)$$

$$\psi_t(\omega) = \sum_{W^+ \leq t} e^{-\alpha W^+} f(W)$$

and note that

$$(7.1) \quad e^{-\alpha t} \varphi^0(\theta_t(\omega)) = \varphi^t(\omega) \quad \text{and}$$

$$e^{-\alpha t} \psi^0(\theta_t(\omega)) = \psi^t(\omega) - e^{-\alpha \tilde{W}^+} f(\tilde{W})$$

with $\tilde{W}$ the excursion defined by $\tilde{W}^- \leq t < \tilde{W}^+$

We denote

$$b = E^a(\varphi^o(\omega)) \quad \text{and} \quad c = E^a(\psi^o(\omega))$$

We assume that $0 < b < \infty$ and $0 < c < \infty$ and define

$$B_t = b^{-1} \sum_{W^- \leq t} f(W) \quad \text{and} \quad C_t = c^{-1} \sum_{W^+ \leq t} f(W)$$

We note that

$$\bar{B}_t = \int_o^t e^{-s}\, dB_s = b^{-1} \sum_{W^- \leq t} e^{-W^-} f(W)$$

$$\bar{C}_t = \int_o^t e^{-s}\, dC_s = c^{-1} \sum_{W^+ \leq t} e^{-W^+} f(W)$$

We shall now study the parameters of B. Clearly, $u_B(a) = 1$ and if $\tilde{b} = E^a(\sup_t e^{-2W^-} f^2(W))$ then

(7.2) $$\bar{\Delta}_B \leq \tilde{b}/b^2$$

To get

(7.3) $$\bar{\Gamma}_B \leq 10\, \tilde{b}/b^2$$

we write

$$\Gamma_t' = E^x(b^{-1} \varphi^o | F_t) - b^{-1} \varphi_t - e^{-t} u_B(X_t) =$$

$$= b^{-1} E^x(e^{-\tilde{W}} f(\tilde{W}) \,|\, F_t) - b^{-1} e^{-\tilde{W}} f(\tilde{W}) \quad P^*\text{a.s.}$$

with $\tilde{W}$ such that $\tilde{W}^- \leq t < \tilde{W}^+$ (to get the lost equality we use $\varphi^o = \varphi^t + \varphi_t$ and (7.1). The inequality (7.3) follows by Doob's maximal inequality.

Since $W^- \geq T_a$ we get

(7.4) $$\gamma_B(\{a\}) = E^a(\bar{B}_{T_a^-}) = 0$$

We are now dealing with the parameters of C. As for B, $u_C(a) = 1$ and

(7.5) $$\bar{\Delta}_C = \tilde{c}/c$$

with $\tilde{c} = E^a(\sup e^{-2W^+} f^2(W))$.

Since $\psi_t \in F_t$, $E^x(c^{-1}\psi_t | F_t) = \psi_t$ and by

(7.1)

$$\Gamma'_t = E^x(c^{-1}\psi^t | F_t) - e^{-t} u_C(X_t) =$$

$$= c^{-1} e^{-\tilde{W}^+} f(\tilde{W}) P^x \quad \text{a.s.}$$

We conclude that

(7.6) $$\bar{\Gamma}_C \leqslant \tilde{c}/c^2$$

Finally, since $W^+ \geqslant T_a$

(7.7) $$\gamma_C(\{a\}) = E^a(\bar{C}_{T_a}) = 0$$

b. General results

Let us consider for every $\varepsilon > 0$ a positive and measurable function $f_\varepsilon : D[0,\infty) \to R$ and B^ε, C^ε the increasing processes associated to f_ε.

Theorem 7.1.

(i) We assume that $0 < b_\varepsilon < \infty$ for every $\varepsilon > 0$. Then $\lim_{\varepsilon \to 0} d_2(\bar{B}^\varepsilon, \bar{L}) = 0$ iff $\lim_{\varepsilon \to 0} \tilde{b}_\varepsilon / b_\varepsilon^2 = 0$.

(ii) We assume that $0 < c_\varepsilon < \infty$ for every $\varepsilon > 0$ Then $\lim_{\varepsilon \to 0} d_2(\bar{C}^\varepsilon, \bar{L}) = 0$ iff $\lim_{\varepsilon \to 0} \tilde{c}_\varepsilon / c_\varepsilon^2 = 0$.

Proof. It is an immediate consequence of Theorem 2.4. and (7.2) - (7.7).

Theorem 7.2.

(i) If $0 < b_\varepsilon < \infty$ for every $\varepsilon > 0$, then

$$\sum_n \tilde{b}_{\varepsilon_n} / b^2_{\varepsilon_n} \quad \text{implies}$$

(7.8) $$\lim_n \sup_t |\bar{B}_t^{\varepsilon_n} - \bar{L}_t| = 0 \quad \text{a.s.}$$

(ii) If $0 < c_\varepsilon < \infty$ for every $\varepsilon > 0$, then $\sum_n \tilde{c}_{\varepsilon_n} / c^2_{\varepsilon_n}$ implies that (7.8) holds for C^{ε_n}, $n \to N$.

Proof. It is an immediate consequence of Corollary 2.5. and (7.2) - (7.7).

Corollary 7.3. If for some $K > 0$, $f_\varepsilon \leq K$ for every $\varepsilon > 0$ then

(i) $\lim_{\varepsilon \to 0} b_\varepsilon = \infty$ implies $\lim_{\varepsilon \to 0} d_2(\tilde{B}^\varepsilon, \bar{L}) = 0$

(ii) $\lim_{\varepsilon \to 0} c_\varepsilon = \infty$ implies $\lim_{\varepsilon \to 0} d_2(\tilde{C}^\varepsilon, \bar{L}) = 0$

Proof. It is an immediate consequence of Theorem 7.1.

Corollary 7.4. If $\varepsilon \to f_\varepsilon$ increases when ε decreases and $f_\varepsilon \leq K$ for every $\varepsilon > 0$ $(K \in R_+)$ then

(i) $\lim_{\varepsilon \to 0} b_\varepsilon = \infty$ implies

$$\lim_{\varepsilon \to 0} \sup_t |\tilde{B}_t^\varepsilon - \bar{L}_t| = 0 \text{ a.s.} \tag{7.9}$$

(ii) $\lim_{\varepsilon \to 0} c_\varepsilon = \infty$ implies that (7.9) holds for $(\tilde{C}^\varepsilon, \varepsilon \to 0)$.

Proof. Let us consider at first that $\varepsilon \longrightarrow b_\varepsilon$ is continuous. Then we may choose $\varepsilon_n > 0$ such that $b_{\varepsilon_n} = 1/n$. Since $\sum_n \tilde{b}_{\varepsilon_n}/b^2_{\varepsilon_n} \leq \sum_n K/n^2 < \infty$ (7.8) holds for $(B^{\varepsilon_n}, n \in N)$.

If $\varepsilon_{n+1} \leq \varepsilon < \varepsilon_n$ then

$$b^{-1}_{\varepsilon_n} \sum_{W^- \leq t} f_{\varepsilon_{n+1}}(W) \geq b^{-1}_\varepsilon \sum_{W^- \leq t} f_\varepsilon(W) \geq$$

$$\geq b^{-1}_{\varepsilon_{n+1}} \sum_{W^- \leq t} f_{\varepsilon_{n+1}}(W)$$

thus, since $\lim_n b_{\varepsilon_n}/b_{\varepsilon_{n+1}} = 1$, (7.9) holds for $(B^\varepsilon, \varepsilon \to 0)$.

If $\varepsilon \to b_\varepsilon$ is not continuous, the analysis argument of p.88 in Fristed and Taylor /3/ works and we get (7.9).

The same proof works for (ii).

Corollary 7.5. Assume that $\varepsilon \to f_\varepsilon$ decreases when increases and $f_\varepsilon \leqslant K$ for every $\varepsilon > 0$ ($K \in R_+$). Then

(i) If $\varepsilon \to b_\varepsilon$ is continuous and $\lim_{\varepsilon \to 0} b_\varepsilon = \infty$ then (7.9) holds for (B^ε, $\varepsilon \to 0$).

(ii) If $\varepsilon \to c_\varepsilon$ is continuous and $\lim_{\varepsilon \to 0} c_\varepsilon = \infty$ then (7.9) holds for (C^ε, $\varepsilon \to \infty$).

Proof. The arguments are simillar of those in Corollary 7.4. but the analysis argument in Fristed and Taylor /3/ no longer works in this case, thus we have to preserve the condition on the continuity of $\varepsilon \to b_\varepsilon$ ($\varepsilon \to c_\varepsilon$).

c. The connection with the characteristic measure

Let us denote by ν the characteristic measure of the excursion Poisson point process (see Ito /6/) and assume for a positive and measurable function $f : D[0,\infty) \to R$ that

(7.10) $$\int f \, d\nu < \infty$$

By Lemma 3.4. pg.84 in Fristed and Taylor /3/

(7.11/ $$E^a\left(\int_0^\infty e^{-\theta s} \, d \sum_{W^- \leqslant t} f(W)\right) = h(\theta) \int f(W) \, d\nu(W)$$

Using (7.11) for the function $g = e^{-\sigma} f, (\sigma(W) = W^+ - W^-)$ we get

(7.12) $$E^a\left(\int_0^\infty e^{-\theta s} \sum_{W^+ \leqslant t} f(W)\right) = h(\theta) \int e^{-\sigma} f \, d\nu$$

with $h(\theta)$ a constant independent of f and

Theorem 7.6.

$$\lim_{\varepsilon \to 0} d_2(\overline{B}^\varepsilon, \overline{L}) = 0 \quad \text{iff}$$

$$\lim_{\varepsilon \to 0} \int f_\varepsilon^2 \, d\nu / \left(\int f_\varepsilon \, d\nu\right)^2 = 0$$

Proof. The implication from left to right is a consequence of Theorem 3.5. p.85 in Fristed and Taylor /3/ (see also Lemma 2.2. pg.79 in the same work and the remark after Lemma 2.1 in our work).

To get the other implication we use (7.11) and write

$$b_\varepsilon = h(1) \int f_\varepsilon \, d\nu \quad \text{and}$$

$$\tilde{b}_\varepsilon = E^a(\sup e^{-2W^-} f_\varepsilon^2(W)) \leqslant E^a(\sum e^{-2W^-} f_\varepsilon^2(W)) =$$

$$= h(2) \int f_\varepsilon^2 \, d\nu$$

Thus the implication from right to left follows from Theorem 7.1. (i).

Theorem 7.7.

$$\lim_{\varepsilon \to 0} \int e^{-2\sigma} f_\varepsilon^2 \, d\nu / \left(\int e^{-\sigma} f_\varepsilon \, d\nu \right)^2 = 0 \quad \text{implies}$$

$$\lim_{\varepsilon \to 0} d_2(\overline{C}^\varepsilon, \overline{L}) = 0$$

Proof. By (7.12)

$$c_\varepsilon = h(1) \quad e^{-\sigma} f_\varepsilon \, d\nu \quad \text{and}$$

$$\tilde{c}_\varepsilon \leqslant E^a(\sum e^{-2W^+} f_\varepsilon^2(W)) = h(2) \int e^{-2\sigma} f_\varepsilon \, d\nu$$

thus our assertion is an immediate consequence of Theorem 7.1. (ii).

d. Counting constructions

Let us consider a measurable set $\mathcal{E}_\varepsilon \subseteq D[0,\infty)$ such that $\nu(\mathcal{E}_\varepsilon) < \infty$ and define

$$N_\varepsilon(t) = \text{card} \{ W : W \in \mathcal{E}_\varepsilon, \; W^- \leqslant t \}$$

Corollary 7.8. If $\lim_{\varepsilon \to 0} \nu(\mathcal{E}_\varepsilon) = 0$ then

$$\lim_{\varepsilon \to 0} d_2(\nu(\mathcal{E}_\varepsilon)^{-1} \overline{N}_\varepsilon, \overline{L}) = 0$$

If $\mathcal{E}_\varepsilon$ increases as ε decreases

$$\lim_{\varepsilon \to 0} \sup_t \left| \nu(\mathcal{E}_\varepsilon)^{-1} \overline{N}_\varepsilon(t) - \overline{L}(t) \right| = 0 \text{ a.s.}$$

<u>Proof.</u> It is particular case of Corollary 7.3. and 7.4.

Let us now take $\mathcal{E}_\varepsilon = \{ w : w^+ - w^- > \varepsilon \}$

We denote by n the Levy measure of the subordinator $S = L^{-1}$ and $N_{[0,1]\times\mathcal{E}_\varepsilon} = \operatorname{card}\{ s \leqslant 1 : \Delta S_s > \varepsilon \}$

We have:

$$\nu(\mathcal{E}_\varepsilon) = E^a(N_{[0,1]\times\mathcal{E}_\varepsilon}) = n((\varepsilon, \infty))$$

(see Ito /6/).

We define

$$M_\varepsilon(t) = \operatorname{card}\{ w : w^+ - w^- > \varepsilon, \quad w^- \leqslant t \}$$

and get a generalisation of /5/ pg.43 in Ito and Mc Keen /7/:

<u>Corollary 7.9.</u> If the point a is instantaneous and $m_\varepsilon = n(\varepsilon, \infty)$ then

$$\lim_{\varepsilon \to 0} d_2(m_\varepsilon^{-1} \overline{M}_\varepsilon, \overline{L}) = 0$$

$$\lim_{\varepsilon \to 0} \sup_t \left| m_\varepsilon^{-1} \overline{M}_\varepsilon(t) - \overline{L}(t) \right| = 0 \text{ a.s.}$$

<u>Proof.</u> Since the point a is instantaneous, $\lim_{\varepsilon \to 0} m_\varepsilon = \lim_{\varepsilon \to 0} n((\varepsilon, \infty)) = \infty$ and Corollary 7.9. is a particular case of Corollary 7.8.

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