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ADRIAN CARABINEANU

In this paper we solve some problems in the filtration theory using the theory of the spherical functions.

These problems are in fact boundary-value problems of the potential theory, considered for diverse canonical domains.

Depending on the domain in which we consider the filtration motion, we shall choose a convenient system of orthogonal curvilinear coordinates. This choice will permit us to solve the problems with the help of the method of separation of variables.

1. The Filtration Motion determined by a Three Dimensional Source in the Presence of a Spherical Inhomogeneity.

Let us consider a sphere S whose radius is equal to R. We denote by O the center of the sphere S and let Oxyz be a cartesian reference system. We shall consider that the three-dimensional source is situated within the sphere S, on the Oz axis and we denote by ρ the distance between the source and the centre of S(fig.1).

We suppose that the porous medium is characterized by the filtration coefficient k_1 in the interior of the sphere S and by the filtration coefficient k_2 in the exterior of this sphere.

We denote by φ_1 the velocity potential in the interior of the sphere, by φ_2 the velocity potential in the exterior of the sphere and by φ_0 the velocity potential in the absence of the inhomogeneity. Let us introduce the notations:

(1) $\varphi_1^* = \varphi_1 - \varphi_0$

(2) $\varphi_2^* = \varphi_2 - \varphi_0$

The conditions that we impose on the spherical surface S are

([1],pag.45) :

(3) $k_2 \varphi_1^* - k_1 \varphi_2^* = (k_1 - k_2)\varphi_o$

(4) $\frac{\partial \varphi_1^*}{\partial n} = \frac{\partial \varphi_2^*}{\partial n}$ ($\frac{\partial}{\partial n}$ is the normal derivative)

Because of the spherical inhomogeneity, it is convenient to introduce the system of spherical coordinates (r, θ, φ) given by the relations :

(5) $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, $z = r \cos\theta$

As shown in [1], pag.376, when the medium is homogeneous the potential is :

(6) $\varphi_o(r,\theta,\varphi) = -\frac{Q}{4\pi\sqrt{r^2 + \rho^2 - 2r\rho\cos\theta}}$ (Q is the rate of flow of the source).

φ_1^* and φ_2^* are harmonic functions; therefore in the system of spherical coordinates they will have the following expressions ([2],pag.294) :

(7) $\varphi_1^* = \sum_{n=o}^{\infty} A_n \left(\frac{r}{R}\right)^n P_n(\cos\theta)$

(8) $\varphi_2^* = \sum_{n=o} B_n \left(\frac{R}{r}\right)^{n+1} P_n(\cos\theta)$

(P_n, $n \in N$ are the polynomials of Legendre)

Using the expression of the generating function(2 ,pag.75) of the polynomials of Legendre , we shall expand the function φ_o into a series of polynomials of Legendre on the surface S :

(9) $\varphi_o = -\frac{Q}{4\pi R} \sum_{n=o}^{\infty} \left(\frac{\rho}{R}\right)^n P_n(\cos\theta)$

Replacing in the relations (3) and (4) the expressions (7),(8) and (9) of φ_1^*, φ_2^* and φ_o and identifying the coefficients of $P_n(\cos\theta)$, $n \in N$, we obtain :

(1o) $n A_n = -(n+1) B_n$

$$(11)\quad k_2A_n - k_1B_n = (k_2 - k_1)\,\frac{Q}{4\pi R}\left(\frac{\rho}{R}\right)^n \quad ; n\in N$$

(1o) and (11) may be considered as a system of linear algebraic equations. Solving this system we get the unknowns A_n and B_n:

$$(12)\quad A_n = \frac{(k_2 - k_1)(n+1)}{k_1 n + k_2(n+1)}\;\frac{Q}{4\pi R}\left(\frac{\rho}{R}\right)^n$$

$$(13)\quad B_n = \frac{(k_1 - k_2)n}{k_1 n + k_2(n+1)}\;\frac{Q}{4\pi R}\left(\frac{\rho}{R}\right)^n$$

From (12),(13),(7),(8),(6),(2) and (1) we establish the expressions of the potentials φ_1 and φ_2.

The Influence of the Inhomogeneity on the Rate of Flow.

We shall denote by Q^* the possible variation of the rate of flow of the three-dimensional source due to the spherical inhomogeneity.

In order to calculate Q^*, we shall integrate the normal derivative of φ_1^* on the spherical surface S.

Taking into account that $\dfrac{\partial\varphi_1^*(r,\theta,\varphi)}{\partial n} = \dfrac{\partial\varphi_1^*(r,\theta,\varphi)}{\partial r}$

for $r = R$ and $d\sigma = R\sin\theta\, d\varphi\, d\theta$ we get from (7) and (12) :

$$(14)\quad Q^* = \iint_S \frac{\partial\varphi_1^*}{\partial n}\,d\sigma = \int_0^{2\pi}\!\!\int_0^{\pi} \frac{\partial\varphi_1^*(R,\theta,\varphi)}{\partial r}\,\sin\theta\, d\varphi\, d\theta =$$

$$= \frac{Q}{2}\int_0^{\pi}\sum_{n=0}^{\infty}\frac{(k_2 - k_1)n(n+1)}{k_1 n + k_2(n+1)}\left(\frac{\rho}{R}\right)^n P_n(\cos\theta)\,\sin\theta\, d\theta$$

The series which appears in (14) is uniformly convergent and therefore we can commute the operations of summation and integration. Taking into acount the relation ([2],pag.82) :

$$(15)\quad \int_0^{\pi} P_n(\cos\theta)\,\sin\theta\, d\theta = \int_{-1}^{1} P_n(x)\,dx = 0$$

we obtain from (14) :

(16) $Q^{*} = o$

Consequently the existence of the spherical inhomogeneity does not influence the rate of flow.

2. The Semi-Ellipsoidal Well.

We consider a well in which the water raises up to the level H. (Fig.2). Let :

(17) $S_1 : \dfrac{x^2 + y^2}{a^2} + \dfrac{z^2}{b^2} = 1 ;\ x < o$

be the equation of the surface of supply.

We suppose that the surface :

(18) $S_2 : x = o,\ \dfrac{y^2}{a^2} + \dfrac{z^2}{b^2} - 1 > o$

is tight.

The water from the well percolates a porous domain D_1:

(19) $D_1 : x < o,\ \dfrac{x^2 + y^2}{a^2} + \dfrac{z^2}{b^2} - 1 > o$

We denote by φ the potential of the filtration velocity. On the surface of supply S_1, φ is constant ([1],pag.44) :

(2o) $\varphi = -k\left(H + \dfrac{p_a}{\gamma'}\right)$ on S_1.

In (2o) k represents the filtration coefficient, p_a the atmospheric pressure and γ' the specific weight of the water. We denote by φ_o the value of φ on S_1 and by φ_∞ the value of φ at infinity, i.e. :

(21) $\varphi_\infty = \lim\limits_{x^2 + y^2 + z^2 \to \infty} \varphi(x,y,z)$

In order to study the motion within the domain D_1 we shall find at first the expression of φ and then we shall calculate the rate of flow.

The Expression of the Velocity Potential.

The surface S_2 is tight. Therefore the normal component of the velocity on this surface vanishes:

(22) $\frac{\partial\varphi}{\partial n} = \frac{\partial\varphi}{\partial x} = 0$ on S_2

Consequently we can extend the domain of definition of φ to :

(23) $D_1' : \frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} > 1$

through the agency of the relation :

(24) $\varphi(x,y,z) = \varphi(-x,y,z)$

If we consider that $a < b$, we deduce that S_1', the boundary of D_1' is an elongated rotation ellipsoid. For this reason we shall introduce the system of the degenerated ellipsoidal coordinates (α,β,γ) by the relations ([2],pag.299) :

(25) $x = c \sinh\alpha \sin\beta \cos\gamma$

(25') $y = c \sinh\alpha \sin\beta \sin\gamma$

(25") $z = c \cosh\alpha \cos\beta$

(25''') $0 \leqslant \alpha < \infty, \; 0 \leqslant \beta \leqslant \pi, \; -\pi \leqslant \gamma \leqslant \pi$

Let us consider the notations :

(26) $c = \sqrt{b^2 - a^2}$

(26') $\sinh\alpha_0 = \frac{a}{c}$

From (23), (25),(25'),(25"),(25'''),(26) and (26') we obtain the equation of S_1' in degenerated ellipsoidal coordinates :

(27) $\alpha = \alpha_0$

From the above considerations we deduce that we have to solve the following Dirichlet problem : to find a harmonic function $\varphi(\alpha,\beta,\gamma)$ in the domain :

(28) $D_1' : \alpha_0 \leqslant \alpha < \infty \; ; \; 0 \leqslant \beta \leqslant \pi \; ; \; -\pi \leqslant \gamma \leqslant \pi$

taking into consideration the following conditions imposed on the boundary and at infinity :

(29) $\varphi(\alpha_0,\beta,\gamma) = \varphi_0$

(3o) $\lim_{\alpha\to\infty} \varphi(\alpha,\beta,\gamma) = \varphi_\infty$

In degenerated ellipsoidal coordinates, the expression of the harmonic functions which vanish at infinity and do not depend on the variable γ is ([2],pag. 3o4) :

(31) $\varphi(\alpha,\beta,\gamma) = \sum_{n=0}^{\infty} N_n\, Q_n(\cosh\alpha)\, P_n(\cos\beta) + \varphi_\infty$

P_n and Q_n are the spherical functions of first respective second kind and N_n, n = o,1,2,... are coefficients which must be calculated

From (29),(3o) and (31) we get :

(32) $\varphi_0 - \varphi_\infty = \sum_{n=0}^{\infty} N_n\, Q_n(\cosh\alpha_0)\, P_n(\cos\beta)$

It is known ([2],pag.263) that :

(41) $Q_0(z) = \frac{1}{2}\ln\frac{z+1}{z-1}$

From (32) and (33) we obtain by identification of the coefficients of $P_n(\cos\beta)$, n = o,1,2 ... :

(34) $N_0 = \dfrac{\varphi_0-\varphi_\infty}{\ln\coth\frac{\alpha_0}{2}}$

(34') $N_n = 0$ for n = 1,2,3,...

From (31),(34) and (34') we find the expression of φ :

(35) $\varphi(\alpha,\beta,\gamma) = (\varphi_0-\varphi_\infty)\dfrac{\ln\coth\frac{\alpha}{2}}{\ln\coth\frac{\alpha_0}{2}} + \varphi_\infty$

The Calculation of the Rate of Flow.

In ellipsoidal coordinates, the equation of the surface of supply S_1, is :

(36) $\alpha = \alpha_0\,;\ 0 \leq \gamma \leq \pi$

The parameters of Lamé are ([2],pag.3oo) :

(37) $H_\alpha = H_\beta = c(\sinh^2\alpha + \sin^2\beta)^{1/2}$

(37') $H_\gamma = c\sinh\alpha\sin\beta$

The normal component of the filtration velocity, on the surface of supply $\alpha = \alpha_o$ is :

$$(38)\quad v_n = \frac{1}{H_{\alpha_o}} \frac{\partial \varphi}{\partial \alpha} = \frac{1}{c(\sinh^2 \alpha_o + \sin^2 \beta)^{1/2}} \frac{\partial \varphi}{\partial \alpha}$$

On S_1 the surface element is :

$$(39)\quad d\sigma = c^2 \sinh \alpha_o \sin\beta \sqrt{\sinh^2 \alpha_o + \sin^2 \beta}\, d\beta\, d\gamma$$

Therefore the rate of flow is :

$$(4o)\quad Q = \iint_{S_1} \frac{\partial \varphi}{\partial n}\, d \quad = c \int_0^{\pi}\int_0^{\pi} \frac{\partial \varphi}{\partial \alpha} \sinh \alpha_o \sin\beta\, d\beta\, d\gamma$$

From (35) it follows :

$$(41)\quad \frac{\partial \varphi}{\partial \alpha} = \frac{\varphi_\infty - \varphi_o}{\sinh \alpha_o \ln(\coth \frac{\alpha_o}{2})};\quad \alpha = \alpha_o$$

From (41),(4o),(34) and (34') we get the final expression of the rate of flow :

$$(42)\quad Q = \frac{2\pi(\varphi_\infty - \varphi_o)\sqrt{b^2 - a^2}}{\ln \frac{b + (b^2 - a^2)^{1/2}}{a}}$$

3.The Problem of the Well in the Shape of a Spherical Calotte.

Let us consider a well in which the water raises up to the level H.We supposr that the bottom of the well (which is a surface of supply) is a spherical calotte:

$$(43)\quad S_1 : x^2 + y^2 + (z - R\cos\beta_o) = R^2 \; ; \; z < o.$$

The water from the well percolates the porous domain(fig.3) :

$$(44)\quad D_1 : x^2 y^2 + (z - R\cos\beta_o)^2 > R^2 \; ; \; z < o.$$

The boundary of D_1 is composed from the surfaces S_1 and :

$$(45)\quad S_2 : x^2 + y^2 - R^2 \sin^2\beta_o > o; \; z = o.$$

We consider that S_2 is tight and we search on for the expression of the velocity potential and the rate of flow .

The Expression of the Velocity Potential in the General Case.

On S_2 the normal component of the velocity vanishes. Hence :

(46) $\frac{\partial \varphi}{\partial n} = \frac{\partial \varphi}{\partial z}$ on S_2 (φ is the velocity potential).

From (44) and (46) we deduce that we can extend the domain of definition of φ to the domain :

(47) $D : x^2 + y^2 + (z - R\cos\beta_0)^2 > R^2$

by means of the relation :

(48) $\varphi(x,y,z) = \varphi(x,y,-z)$

Further, we shall introduce the system of toroidal coordinates (α,β,γ) by means of the relations ([2], pag.311) :

(49) $x = \dfrac{c \sinh\alpha \cos\gamma}{\cosh\alpha - \cos\beta}$

(49') $y = \dfrac{c \sinh\alpha \sin\gamma}{\cosh\alpha - \cos\beta}$

(49") $z = \dfrac{c \sin\beta}{\cosh\alpha - \cos\beta}$

(49''') $0 \leqslant \alpha < \infty$, $-\pi \leqslant \beta \leqslant \pi$, $-\pi \leqslant \gamma \leqslant \pi$

If we assign to c the value :

(50) $c = R\sin\beta_0$

the domain D is defined in toroidal coordinates by the relations :

(51) $D : -\beta_0 \leqslant \beta \leqslant \beta_0$

and the equation of its boundary is :

(52) $|\beta| = \beta_0$

The equation of S_2 is :

(53) $S_2 : \beta = 0$

S_1 is a surface of supply. Hence :

(54) $\varphi(\alpha,\beta,\gamma)\big|_{S_1} = \varphi(\alpha,\beta_0,\gamma) = \varphi_0 = -k\left(H + \dfrac{p_a}{\gamma'}\right)$

From (49),(49') and (49") we deduce that :

(55) $\lim\limits_{\substack{\alpha \to 0 \\ \beta \to 0}} (x^2 + y^2 + z^2) = \infty$

We denote by φ_∞ the value of the velocity potential at infinity. Therefore :

(56) $\varphi(o,o,\gamma) = \varphi_\infty$ = const.

In toroidal coordinares, the solutions of Laplace's equation, independent of γ, can be obtained by summation of functions of the following shape : ([2],pag.319).

(57) $u = (2\cosh\alpha - 2\cos\beta)^{1/2}(M_\tau \cosh\tau\beta + N_\tau \sinh\tau\beta)\, P_{-\frac{1}{2}+i\tau}(\cosh\alpha)$

In the above relation M_τ and N_τ are real coefficients and $P_{-\frac{1}{2}+i\tau}$ are spherical functions of the first kind.

The following formula, known from the theory of integral representation of the spherical functions ([2],pag. 320) :

(58) $$(2\cosh\alpha - 2\cos\beta_o)^{-1/2} = \int_0^\infty \frac{\cosh(\pi-\beta_o)\tau}{\cosh\pi\tau} P_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, d\tau$$

as well the formulas (54),(56) and (57) suggests us the expression of the velocity potential :

(59) $$\varphi = (\varphi_o - \varphi_\infty)(2\cosh\alpha - 2\cos\beta)^{1/2} \int_0^\infty \frac{\cosh\tau(\pi-\beta_o)\cosh\tau\beta}{\cosh\tau\pi\,\cosh\tau\beta_o} P_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, d\tau + \varphi_\infty$$

<u>The Expression of the Velocity Potential and of the Rate of Flow in a Particular Case.</u>

Let us consider that in the common points of S_1 and S_2, the a angle between the two surfaces is $\frac{\pi}{4}$ (i.e. $\beta_o = \frac{\pi}{4}$)

From (59) it follows that :

(60) $$\varphi = (\varphi_o - \varphi_\infty)(2\cosh\alpha - 2\cos\beta)^{1/2}. \int_0^\infty \frac{\cosh\frac{3\pi\tau}{4}\cosh\tau\beta}{\cosh\pi\tau\cosh\frac{\pi\tau}{4}} P_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, d\tau + \varphi_\infty$$

Successive computations lead us to the following expressions of the velocity potential :

(61) $$\varphi = (\varphi_0 - \varphi_\infty)\,(2\cosh\alpha - 2\cos\beta)^{1/2}\cdot \int_0^\infty \frac{(2\cosh\frac{\pi\tau}{2} - 1)\cosh\tau\beta}{\cosh\pi\tau}\, P_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, d\tau + \varphi_\infty$$

(62) $$\varphi = (\varphi_0 - \varphi_\infty)\,(2\cosh\alpha - 2\cos\beta)^{1/2}\cdot \int_0^\infty \frac{\cosh\tau(\frac{\pi}{2}-\beta) + \cosh\tau(\frac{\pi}{2}+\beta) - \cosh\tau\beta}{\cosh\pi\tau}\, P_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, d\tau + \varphi_\infty$$

(63) $$\varphi = (\varphi_0 - \varphi_\infty)\,(2\cosh\alpha - 2\cos\beta)^{1/2}\cdot \int_0^\infty \frac{\cosh\tau(\pi - (\frac{\pi}{2}+\beta)) + \cosh\tau(\pi - (\frac{\pi}{2}-\beta)) - \cosh\tau(\pi-(\pi-\beta))}{\cosh\pi\tau}\, P_{-\frac{1}{2}+i\tau}(\cosh\alpha) + \varphi_\infty$$

From (58) and (63) we obtain :

(64) $$\varphi = (\varphi_0 - \varphi_\infty)\left(\sqrt{\frac{\cosh\alpha - \cos\beta}{\cosh\alpha + \sin\beta}} + \sqrt{\frac{\cosh\alpha - \cos\beta}{\cosh\alpha - \sin\beta}} - \sqrt{\frac{\cosh\alpha - \cos\beta}{\cosh\alpha + \cos\beta}}\right) + \varphi_\infty$$

In order to compute the rate of flow, we have to integrate the normal component of the velocity on the surface of supply. Taking into acount the expression of the parameters of Lamé :

(65) $$H_\alpha = H_\beta = \frac{c}{\cosh\alpha - \cos\beta}$$

(65') $$H_\gamma = \frac{c\sinh\alpha}{\cosh\alpha - \cos\beta}$$

we get :

(66) $$v_n = -\frac{\cosh\alpha - \cos\beta}{c}\frac{\partial\varphi}{\partial\beta} \text{ on } S_1 \ (\text{i.e. } \beta = \frac{\pi}{4})$$

From (64) and (66) it follows :

$$(67)\quad v_n = \frac{\varphi_0-\varphi_\infty}{\sqrt{2}\,c}\left(1 - \frac{\left(\cosh\alpha - \frac{\sqrt{2}}{2}\right)^{3/2}}{\left(\cosh\alpha + \frac{\sqrt{2}}{2}\right)^{3/2}}\right)$$

On the surfaces β = const. the element of area is :

$$(68)\quad d\sigma = \frac{c^2 \sinh\alpha}{(\cosh\alpha - \cos\beta)^2}\, d\alpha\, d\gamma$$

From (67) and (68) we obtain :

$$(69)\quad Q = \frac{c(\varphi_\infty - \varphi_0)}{2}\int_0^\infty\int_0^\pi\left(1 - \frac{\left(\cosh\alpha - \frac{\sqrt{2}}{2}\right)^{3/2}}{\left(\cosh\alpha + \frac{\sqrt{2}}{2}\right)^{3/2}}\right)\frac{\sinh\alpha}{\left(\cosh\alpha - \frac{\sqrt{2}}{2}\right)^2}\, d\alpha\, d\gamma$$

Computing the integral from the right part of (69) and taking into acount the relation (5o) we find the rate Q of flow :

$$(7o)\quad Q = \frac{R(\varphi_\infty - \varphi_0)\pi(12 - \sqrt{2})}{4}$$

Figures

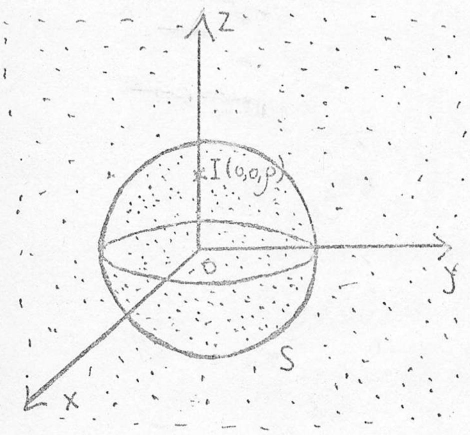

Fig.1

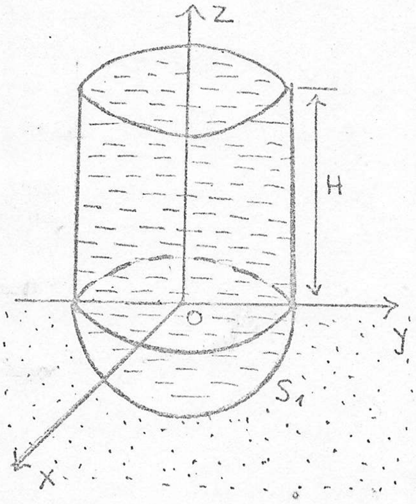

Fig.2

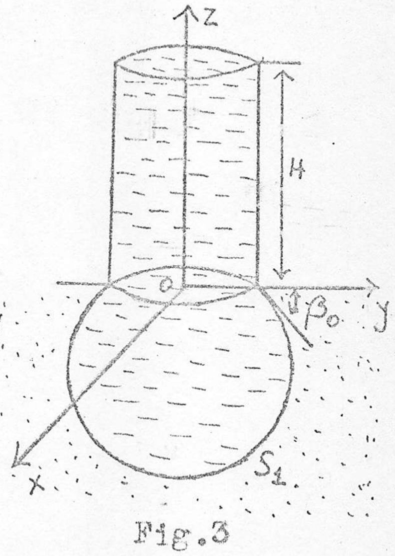

Fig.3

References

1) St.I.Gheorghiţă - Metode matematice în hidrogazodinamica subterană. Ed.Acad. R.S.R.,Bucureşti 1966

2) N. N. Lebedev - Funcţii speciale şi aplicaţiile lor. Ed. Tehnică,Bucureşti,1957