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by

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ANALYTIC OPERATORS AND SPECTRAL DECOMPOSITIONS

F.-H.Vasilescu

1. INTRODUCTION

Let K be a compact subset of $\mathbb{C}^n$ and let us consider the Fréchet space $A'(K)$ of all analytic functionals carried by K (see, for instance, [9], Chapt. IX). It is not necessarily true that $u \in A'(K_1) \cap A'(K_2)$ implies $u \in A'(K_1 \cap K_2)$. However, if one takes only compact subsets of $\mathbb{R}^n$ (canonically embedded in $\mathbb{C}^n$), then such an implication is always true. Moreover, if $K_1, \ldots, K_m$ are arbitrary compact subsets of $\mathbb{R}^n$ and $K = K_1 \cup \ldots \cup K_m$, then the following decomposition holds:

$$(1.1) \qquad A'(K) = A'(K_1) + \ldots + A'(K_m).$$

This important result, which is a substitute for the existence of partitions of unity in the theory of hyperfunctions, has been proved by Martineau [10] (see also [16], [9]).

Equation (1.1) suggests the following question: Is the decomposition (1.1) a spectral decomposition? In other words, if $K \subset \mathbb{R}^n$ is a fixed compact set, is there any decomposable system of commuting linear operators on $A'(K)$ whose spectral decompositions are described by (1.1)? (We use the terminology from [20], but full details will be given in the next sections.) To give an answer to this question, an obvious candidate is the system of operators associated with the multiplications by the coordinate functions (see Theorem 5.3).

Let us note from the very beginning that the general theory of spectral decompositions of commuting systems of linear operators on Fréchet spaces, as developed for instance in [20], cannot be directly applied. Indeed, if K_1 , K_2 are compact sets , $K_1 \subset K_2$, then $A'(K_1) \subset A'(K_2)$ and the inclusion is continuous. Nevertheless, the space $A'(K_1)$ is not, in general, a closed subspace of $A'(K_2)$. This follows, roughly speaking, in the following way. If $A'(K_1)$ were a closed subspace of $A'(K_2)$, then, by duality, the restriction mapping from the algebra of germs of analytic functions in neighbourhoods of K_2 into the algebra of germs of analytic functions in neighbourhoods of K_1 would be surjective, and this fact is not always true. Since the theory of spectral decompositions usually requires that the "involved" subspaces be closed, the above remark shows that it is necessary to make some adjustments to the general theory if one wants to include this important case.

The aim of this paper is to analyze the decomposition theorem for analytic functionals, more generally for analytic operators (see Definition 2.1), from the point of view of the theory of spectral decompositions of commuting systems of linear operators. To this end, we need a few abstract results concerning certain commuting systems of linear operators in Fréchet spaces, which might be of interest for their own sake. Specifically, we work with spectral capacities more general than the current ones, and therefore with a more general concept of decomposability (see Definitions 4.1 and 4.2). We show that decomposable systems of linear operators (in this sense) still have some of the standard properties (for instance, the single valued extension property).

The space of all analytic operators that are carried by a real compact set K provides a universal extension (see Definition 3.4) with "good" spectral properties for all commuting systems of linear operators (in a fixed Fréchet space) whose joint spectrum is contained in K . One of the consequences of this fact will be presented in the last section.

2. ANALYTIC OPERATORS

Let X be an arbitrary (complex) Fréchet space, whose topology is defined by the family of seminorms $\{\|x\|_m\}_{m=1}^{\infty}$, $x \in X$. If Y is another Fréchet space, we denote by $\mathcal{L}(X,Y)$ the space of all linear and continuous operators from X into Y . (The symbol $\mathcal{L}(X,Y)$ will have the same meaning for arbitrary topological vector spaces; we write $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$.)

Let $A = A(\mathbb{C}^n)$ be the space of all entire analytic functions on $\mathbb{C}^n$, endowed with its natural Fréchet space structure. The analogy with the case of analytic functionals leads to the following :

2.1. DEFINITION. The elements of $\mathcal{L}(A,X)$ will be called <u>analytic operators</u>. If $K \subset \mathbb{C}^n$ is a compact set, an analytic operator u is said to be <u>carried</u> by K if for every neighbourhood V of K and each integer $m \geq 1$ there exists a constant $C_{m,V} \geq 0$ such that

$$(2.1) \qquad \|u(f)\|_m \leqslant C_{m,V} \|f\|_V \ , \quad f \in A \ ,$$

where $\|f\|_V = \sup\{|f(z)| \ ; \ z \in V\}$. We denote by $\mathcal{L}_K(A,X)$ the set of all analytic operators $u \in \mathcal{L}(A,X)$ that are carried by K ; this is a Fréchet space whose topology is induced by the collection of seminorms

(2.2) $\|u\|_{m,V} = \sup\left\{\|u(f)\|_m \; ; \; f \in A \; , \; \|f\|_V \leq 1\right\}$,

where $m=1,2,3,\ldots$, and V runs a (countable) fundamental system of neighbourhoods of K (see [12] for a slightly less general concept, when restricted to Fréchet spaces).

Let us note that if $X=\mathbb{C}$, then $\mathcal{L}_K(A,X)=A'(K)$. If K_1 and K_2 are compact subsets of $\mathbb{C}^n$, $K_1 \subset K_2$, then $\mathcal{L}_{K_1}(A,X) \subset \mathcal{L}_{K_2}(A,X)$ and the inclusion is continuous. If $K \subset \mathbb{R}^n$ (more generally, if K is polynomially convex), then $\mathcal{L}_K(A,X)$ is equal to the space $\mathcal{L}(A(K),X)$, where $A(K)$ is the algebra of germs of analytic functions in neighbourhoods of K , which is an inductive limit of Fréchet spaces. This follows from the density of A in $A(K)$ (see, for instance, [9], Prop. 9.1.2).

In this section we shall show that for each analytic operator that is carried by a real compact set one can define an appropriate concept of support. Moreover, a decomposition similar to (1.1) still holds. By using the already known scalar case (as expounded in [9]), combined with some tensor product techniques, our task will not be too difficult.

For a fixed compact set $K \subset \mathbb{R}^n$, let us denote by Ω_K the open set $\mathbb{R}^{n+1} \setminus (K \times \{0\})$. Let $\mathcal{H}(\Omega_K)$ be the space of all harmonic functions on Ω_K , endowed with its natural Fréchet space structure. If $s'=(s,s_{n+1})$ is the variable of the space $\mathbb{R}^{n+1}$, where $s=(s_1,\ldots,s_n)$ is the variable of $\mathbb{R}^n$, then we define the subspace $\mathcal{H}_1(\Omega_K)$ as the set of those functions $h \in \mathcal{H}(\Omega_K)$ such that

(2.3) $h(s,-s_{n+1}) = -\,h(s,s_{n+1})$.

Plainly, $\mathcal{H}_1(\Omega_K)$ is a closed subspace of $\mathcal{H}(\Omega_K)$. We also define the subspace $\mathcal{H}_0(\Omega_K)$ as the family of those functions $h\in\mathcal{H}_1(\Omega_K)$ such that

$$(2.4)\qquad \lim_{\|s'\|\to\infty} h(s') = 0 ,$$

where $\|s'\|^2=|s_1|^2+\dots+|s_{n+1}|^2$. The space $\mathcal{H}_0(\Omega_K)$ has a Fréchet space structure (the uniform convergence on compact subsets of Ω_K and on the sets of the form $\{s'\in\mathbb{R}^{n+1} ; \|s'\|\geq r\}$ for large $r\geq 0$).

Let us consider the function

$$(2.5)\qquad P(s') = s_{n+1}\, c_{n+1}^{-1}\, \|s\|^{-n-1} , \quad s\neq 0 ,$$

where c_{n+1} is the area of the unit sphere in $\mathbb{R}^{n+1}$.

2.2. THEOREM [9]. There exists a linear and continuous mapping $A'(K)\ni u\mapsto h_u\in\mathcal{H}_0(\Omega_K)$ given by the equality

$$(2.6)\qquad h_u(s') = u_z(P(s'-(z,0))) , \quad s'\in\Omega_K .$$

Conversely, to every $h\in\mathcal{H}_1(\Omega_K)$ there corresponds a unique $u\in A'(K)$, such that

$$(2.7)\qquad u((\partial g/\partial s_{n+1})\big|_{s_{n+1}=0}) = -\int h(s')\,\Delta(\theta g)(s')\, ds' ,$$

where $\theta\in C_0^\infty(\mathbb{R}^{n+1})$, $\theta=1$ in a neighbourhood of $K\times\{0\}$, g is any harmonic function in $\mathbb{R}^{n+1}$ and Δ is the Laplace operator. The function $h - h_u$ is harmonic in $\mathbb{R}^{n+1}$ and it vanishes identically if and only if $h\in\mathcal{H}_0(\Omega_K)$.

Details concerning the proof of this statement can be found in [9], Propositions 9.1.3 and 9.1.5.

2.3. REMARK. The spaces $A'(K)$, $\mathcal{H}_0(\Omega_K)$, $\mathcal{H}_1(\Omega_K)$ and $\mathcal{H}(\Omega_K)$ are nuclear. Indeed, the fact that $A'(K)$ and $\mathcal{H}(\Omega_K)$ are nuclear is well known (see [16] and resp. [11]). The space $\mathcal{H}_1(\Omega_K)$ is nuclear as a closed subspace of $\mathcal{H}(\Omega_K)$ [11].

Since (2.6) is a Fréchet space isomorphism, the space $\mathcal{H}_0(\Omega_K)$ is also nuclear. In particular, if $\mathcal{F}$ is any of the above spaces, then the completion of the algebraic tensor product $\mathcal{F}\otimes X$ with respect to either the injective topology or with the projective one is the same, and it will be denoted by $\mathcal{F}\hat{\otimes} X$.

Let $\mathcal{H}(\Omega_K,X)$ be the Fréchet space of all X-valued harmonic functions in Ω_K. We denote by $\mathcal{H}_0(\Omega_K,X)$ ($\mathcal{H}_1(\Omega_K,X)$) the subspace of those functions $h\in\mathcal{H}(\Omega_K,X)$ that satisfy (2.3) and (2.4) (resp. (2.3)).

2.4. LEMMA. <u>We have the following identifications</u> :

$$\mathcal{H}(\Omega_K,X)=\mathcal{H}(\Omega_K)\hat{\otimes} X,$$

$$\mathcal{H}_j(\Omega_K,X)=\mathcal{H}_j(\Omega_K)\hat{\otimes} X \quad (j=0,1), \tag{2.8}$$

$$\mathcal{L}(A(K),X)=A'(K)\hat{\otimes} X.$$

<u>Proof</u>. According to [7], Theorem II.3.13, if $\mathcal{F}(S)$ is a nuclear Fréchet space of scalar functions defined on the set S, whose topology is stronger than the pointwise convergence, then we have the identification

$$\mathcal{F}(S)\hat{\otimes} X=\{f:S\longrightarrow X\ ;\ x'\circ f\in\mathcal{F}(S)\ ,\ x'\in X'\}, \tag{2.9}$$

where X' is the strong dual of X. The equalities $\mathcal{H}(\Omega_K,X)=\mathcal{H}(\Omega_K)\hat{\otimes} X$ and $\mathcal{H}_1(\Omega_K,X)=\mathcal{H}_1(\Omega_K)\hat{\otimes} X$ are simple consequences of (2.9).

Let us prove the equality $\mathcal{H}_0(\Omega_K,X)=\mathcal{H}_0(\Omega_K)\hat{\otimes} X$. From (2.9) it follows that $\mathcal{H}_0(\Omega_K)\hat{\otimes} X\supset\mathcal{H}_0(\Omega_K,X)$. Conversely, we consider the locally compact space $S=\Omega_K\cup\{\infty\}$, with its natural topology (the neighbourhoods of ∞ are the sets $\{\|s'\|>r\}$ for large $r\geqslant 0$). If $C(S,X)$ is the space of all X-valued continuous functions on S, then we have

$$\mathcal{H}_0(\Omega_K)\hat{\otimes} X \subset C(S,\mathbb{C})\hat{\otimes}_\varepsilon X = C(S,X)$$

(see [2], Section 1.3), where ε designates the injective tensor product. In particular, the functions from $\mathcal{H}_0(\Omega_K)\hat{\otimes} X$ have a limit at infinity, which must be null by (2.9).

Let us obtain the last equality from (2.8). The space $A'(K)$ is precisely the dual of $A(K)$ [16]. Therefore, $A'(K)$ can be regarded as a space of functions on the set $S = A(K)$, and its topology is stronger than the pointwise convergence. From (2.9) we then deduce that $A'(K)\hat{\otimes} X \supset \mathcal{L}(A(K),X)$. Conversely, if u belongs to $A'(K)\otimes X$, then we fix a seminorm $\|x\|_m$ on X and a neighbourhood V of K. If X_m is the normed space that is associated with the seminorm $\|x\|_m$ and $j_m: X \longrightarrow X_m$ is the canonical mapping, we derive from (2.9) that

$$\sup\{|x'\circ j_m\circ u(f)| \ ; \ f\in A \ , \ \|f\|_V \leq 1\} < \infty$$

for each $x' \in X_m'$. By the uniform boundedness principle (applied on the Banach space X_m'), there exists a constant $C_{m,V} \geq 0$ such that (2.1) is fulfilled. Hence $u \in \mathcal{L}(A(K),X)$.

We can now give the vector version of Theorem 2.2.

2.5. THEOREM. There exists a linear and continuous mapping $\mathcal{L}(A(K),X) \ni u \rightarrow h_u \in \mathcal{H}_0(\Omega_K,X)$ that is given by (2.6).

Conversely, to every $h \in \mathcal{H}_1(\Omega_K,X)$ there corresponds a unique $u \in \mathcal{L}(A(K),X)$ such that (2.7) holds. The function $h - h_u$ is harmonic in $\mathbb{R}^{n+1}$ and it vanishes identically if and only if $h \in \mathcal{H}_0(\Omega_K,X)$.

Proof. Theorem 2.2 provides the exact sequence of nuclear Fréchet spaces

$$0 \longrightarrow \mathcal{H}_1(\mathbb{R}^{n+1}) \xrightarrow{\ \rho\ } \mathcal{H}_1(\Omega_K) \xrightarrow{\ \lambda\ } A'(K) \longrightarrow 0 \ ,$$

where ρ is the restriction and λ is the mapping (2.7), which is continuous (see the proof of Proposition 9.1.5 from [9]). In virtue of (2.8) and [2], Section 3.3, the sequence

$$0 \longrightarrow \mathcal{H}_1(\mathbb{R}^{n+1}, X) \xrightarrow{\rho\hat{\otimes}1} \mathcal{H}_1(\Omega_K, X) \xrightarrow{\lambda\hat{\otimes}1} \mathcal{L}(A(K), X) \longrightarrow 0$$

is also exact. Let χ be the mapping (2.6) in the scalar case. Then χ is an isomorphisms of $A'(K)$ onto $\mathcal{H}_0(\Omega_K)$, by Theorem 2.2. Hence the mapping $\chi\hat{\otimes}1$ is an isomorphism of $\mathcal{L}(A(K), X)$ onto $\mathcal{H}_0(\Omega_K, X)$, by (2.8) and [2]. Since $\lambda|\mathcal{H}_0(\Omega_K)$ is the inverse of χ, and $\chi\hat{\otimes}1$, $\lambda\hat{\otimes}1$ are the vector versions of (2.6) and (2.7), our assertions hold.

From this technical result, we obtain (as in [9]) the existence of the support for analytic operators that are carried by real compact sets (see [12] for a different approach).

2.6. THEOREM. Let $\mathcal{L}_{\mathbb{R}^n}(A, X)$ be the set of all analytic operators that are carried by compact sets in $\mathbb{R}^n$. For every $u \in \mathcal{L}_{\mathbb{R}^n}(A, X)$ there exists a smallest compact set $\sigma(u) \subset \mathbb{R}^n$ such that $u \in \mathcal{L}_{\sigma(u)}(A, X)$. One has $\sigma(u) = \emptyset$ if and only if $u = 0$. The set $\sigma(u)$ will be called the support of u.

The proof is similar to that of Theorem 9.1.6 from [9], so that we omit it. (The only difference is that one uses Theorem 2.5 instead of Theorem 2.2.) We just note that if $\sigma(u) = \emptyset$, then h_u is harmonic in $\mathbb{R}^{n+1}$ and bounded. Therefore $h_u = 0$, so that $u = 0$. In other words, one has $\mathcal{L}_{\emptyset}(A, X) = 0$ for every Fréchet space X. From this theorem we also derive the equality

$$(2.10) \qquad \mathcal{L}_K(A, X) = \{u \in \mathcal{L}_{\mathbb{R}^n}(A, X) \; ; \; \sigma(u) \subset K\},$$

for each $K \subset \mathbb{R}^n$.

2.7. THEOREM. Let $K_1, \dots, K_m$ be arbitrary compact subsets in $\mathbb{R}^n$ and let K be their union. Then we have the decomposition

$$(2.11) \qquad \mathcal{L}_K(A,X) = \mathcal{L}_{K_1}(A,X) + \ldots + \mathcal{L}_{K_m}(A,X) .$$

Proof. It is sufficient to prove the assertion for $m=2$. The general case then follows by induction. Note that the mapping

$$A'(K_1) \oplus A'(K_2) \ni u_1 \oplus u_2 \longrightarrow u_1 + u_2 \in A'(K)$$

is surjective, by (1.1). Using (2.8), we obtain that the mapping

$$\mathcal{L}_{K_1}(A,X) \oplus \mathcal{L}_{K_2}(A,X) \ni u_1 \oplus u_2 \longrightarrow u_1 + u_2 \in \mathcal{L}_K(A,X)$$

is also surjective [2], and the proof is complete.

Theorems 2.6 and 2.7 provide a wider range of examples for our spectral theory. In addition, they will effectively be used in connection with other aspects of our work.

Finally, let us remark that for a suitably defined concept of vector-valued hyperfunction (as an element of the quotient $\mathcal{L}_{\overline{U}}(A,X)/\mathcal{L}_{\partial U}(A,X)$, where $U \subset \mathbb{R}^n$ is a bounded open set) one can derive from Theorems 2.6 and 2.7 other more specific consequences, as in [9], Chapter IX.

3. ANALYTIC FUNCTIONAL CALCULUS

We need some facts concerning the analytic functional calculus for commuting systems of linear operators on Fréchet spaces [20] (see also [17], [18], [19], [13], [14]).

Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a system of indeterminates. If L is a linear space, we denote by $\Lambda^p[\sigma, L]$ the space of all homogeneous L-valued exterior forms of degree p in $\sigma_1, \ldots, \sigma_n$ $(0 \leqslant p \leqslant n)$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a system of commuting endomorphisms of L, then we can consider the Koszul complex

$$K(L,\alpha) : 0 \longrightarrow \Lambda^0[\sigma, L] \xrightarrow{\delta_\alpha^0} \ldots \xrightarrow{\delta_\alpha^{n-1}} \Lambda^n[\sigma, L] \longrightarrow 0 ,$$

where $\delta_\alpha^p\xi=(\alpha_1\otimes\sigma_1+\ldots+\alpha_n\otimes\sigma_n)\wedge\xi$ for all $\xi\in\Lambda^p[\sigma,L]=$ $=L\otimes\Lambda^p[\sigma,\mathbb{C}]$. The cohomology spaces of the Koszul complex $K(L,\alpha)$ will be denoted by $H^p(L,\alpha)$ $(0\leqslant p\leqslant n)$.

The system of endomorphisms α is said to be <u>nonsingular</u> (or <u>singular</u>) on L if the cohomology of the complex $K(L,\alpha)$ is trivial (or non-trivial).

Now, let X be a Fréchet space. We shall work with <u>regular commuting systems</u> of operators $a=(a_1,\ldots,a_n)\subset\mathcal{L}(X)$, i.e. commuting n-tuples of operators whose spectrum $\sigma(a_j,X)$ (in the sense of [22]) is a compact subset of $\mathbb{C}$ for all $j=1,\ldots,n$.

The joint spectrum of the regular commuting system $a=(a_1,\ldots,a_n)\subset\mathcal{L}(X)$ is the complement in $\mathbb{C}^n$ of the set of those points $w\in\mathbb{C}^n$ for which there exists an open polydisc $D\ni w$ such that the system α_a is nonsingular on $A(D,X)$, where

$$\alpha_a(z)=(z_1-a_1,\ldots,z_n-a_n)\ ,\ z=(z_1,\ldots,z_n)\in\mathbb{C}^n$$

and $A(U,X)$ is the space of all X-valued analytic functions in the open set $U\subset\mathbb{C}^n$. We note that this definition of the joint spectrum in Fréchet spaces is equivalent to that from [20] (or [13]), via some sheaf theory arguments (see the proof of Theorem 1.2 from [4]).

For every regular commuting system $a=(a_1,\ldots,a_n)\subset\mathcal{L}(X)$ there exists a unital algebra homomorphism

$$A(\sigma(a,X))\ni f\longrightarrow f(a)\in\mathcal{L}(X) \tag{3.1}$$

such that $\mu_j(a)=a_j$, where $\mu_j(z)=z_j$ are the coordinate functions and $\sigma(a,X)$ is the joint spectrum of the system a. In addition, for every integer $m\geqslant 1$ and each m-tuple $f=(f_1,\ldots,f_m)\subset A(\sigma(a,X))$ one has

(3.2) $\quad \sigma(f(a),X) = f(\sigma(a,X))$,

where $f(a) = (f_1(a),\dots,f_m(a))$.

The mapping (3.1) is continuous in the following sense: The bilinear mapping

(C) $\quad A(\sigma(a,X)) \times X \ni (f,x) \longrightarrow f(a)x \in X$

is continuous; in other words, for every integer $m \geq 1$ and each neighbourhood V of $\sigma(a,X)$ there are a constant $C = C_{m,V} \geq 0$ and an integer $m' \geq 1$ such that $\|f(a)x\|_m \leq C \|f\|_V \|x\|_{m'}$. (This property of continuity follows, from instance, from [20], Proposition III.8.13.)

The homomorphism (3.1) with the properties (3.2) (the "spectral mapping theorem") and (C) is called the <u>analytic functional calculus</u> of the regular commuting system $a = (a_1,\dots,a_n)$. Moreover, the analytic functional calculus is uniquely determined by these properties, as shown in [15]. (There are at least two ways to construct the analytic functional calculus, originating in [18] and [19], but they lead to the same homomorphism [15].)

The set $\sigma(a,X)$ is not necessarily the minimal carrier of the analytic functional calculus. It follows from [4] and [5] that there exist a Banach space X and a commuting system $a = (a_1,\dots,a_n) \subset \mathcal{L}(X)$ $(n \geq 2)$ such that the analytic functional calculus naturally extends to a continuous homomorphism of $A(K)$ into $\mathcal{L}(X)$, where K is a proper subset of $\sigma(a,X)$.

3.1. PROBLEM. Let $\mathcal{K}$ be a family of compact subsets of $\mathbb{C}^n$ and let $\{K_j\}_{j\in J} \subset \mathcal{K}$ have the following property: For every $j \in J$ there exists a unital algebra homomorphism $u_j : A(K_j) \to \mathcal{L}(X)$ satisfying (C) (written for u_j) such that $u_{j'}(f) = u_{j''}(f)$ for all $f \in A$ and $j', j'' \in J$. Find conditions on $\mathcal{K}$ which insure the existence of an algebra homomorphism $u:A(K)\longrightarrow\mathcal{L}(X)$ satisfying (C) (written for u) such that $u|A(K_j)=u_j$ for all $j\in J$, where K is the intersection of the family $\{K_j\}_{j\in J}$, for every such a family $\{K_j\}_{j\in J}$ in $\mathcal{K}$.

If $\mathcal{K}$ is the family of all compact subsets of $\mathbb{C}^n$, then the answer to Problem 3.1 is negative, as shown by an example due to M. Putinar (with whom the author had many discussions on the subject). Namely, consider the compact sets in $\mathbb{C}^2$

$$K_1=\{(z_1,z_2)\ ;\ 1/2\leqslant\max\{|z_1|,|z_2|\}\leqslant 1\},$$

$$K_2=\{(z_1,z_2)\ ;\ z_1=z_2\ ,\ |z_1|\leqslant 1\}.$$

We define the character $u(f)=f(0)$ on both $A(K_1)$ (where it makes sense by canonical extension) and $A(K_2)$, but we do not have $u\in A'(K_1\cap K_2)$.

We think that an exhaustive treatment of Problem 3.1 (or rather a version of it) should take into consideration compact subsets in Stein manifolds.

3.2. PROPOSITION. *The family $\mathcal{K}$ of all compact subsets of $\mathbb{R}^n$ provides a solution to Problem 3.1.*

Proof. Since $\mathcal{L}(X)$ is not, in general, a Fréchet space, the assertion is not a direct consequence of Theorem 2.6.

We shall constantly use the density of A in $A(K)$, as well as the fact that the vector version of (2.6) is an isomoprhism (Theorem 2.5).

For every $x\in X$ we define the mapping $u_{jx}(f)=u_j(f)x$, $f\in A$. From the assumed continuity of u_j's, ~~it~~ follows that $u_{jx}\in\mathcal{L}(A(K_j),X)$ for each x. Since the members of the family $\{u_j\}_j$ agree on A, there exists $u_x\in\mathcal{L}(A(K),X)$ which extends every u_{jx} ,on account of Theorem 2.6.Moreover, $u_x(f)$ makes sense for every $f\in A(K)$ and the mapping $f\to u_x(f)$ is continuous on $A(K)$ for each $x\in X$ .We want to show that the mapping $u(f)x=u_x(f)$ $(x\in X)$ is linear and continuous for each $f\in A(K)$,and that the assignment $(f,x)\mapsto u(f)x$ of $A(K)\times X$ into X is continuous.

Let $h_{jx}\in\mathcal{H}_0(\Omega_{K_j},X)$, $h_x\in\mathcal{H}_0(\Omega_K,X)$ be the functions given by (2.6) for u_{jx} and u_x ,respectively.Then we have $h_x(s')=h_{jx}(s')$ whenever $s'\in\Omega_{K_j}$ .In particular,the mapping $x\to h_x$ is linear,so that the mapping $x\to u_x$ is linear.

Let us deal with the continuity of the mapping $(f,x)\mapsto u(f)x$. Let L be a compact subset of Ω_K. Therefore $L\cap(K\times\{0\})=\emptyset$, and we can find a finite family $S_p=K_{j_p}\times\{0\}$ $(p=1,\dots,\nu)$ such that $L\cap S_1\cap\dots\cap S_\nu=\emptyset$ .It follows easily that we have a decomposition $L=L_1\cup\dots\cup L_\nu$ such that each L_p is compact and $L_p\cap S_p=\emptyset$ .If $s'\in L_p$,then $h_x(s')=h_{j_px}(s')$ .Consequently

$$\sup_{s'\in L}\|h_x(s')\|_m\leqslant\max_{1\leqslant p\leqslant\nu}\ \sup_{s'\in L_p}\|h_{j_px}(s')\|_m$$

for every integer $m\geqslant 1$ .Using this estimate and the isomorphism (2.6),we deduce that for every integer $m\geqslant 1$ and each neighbourhood V of K there are integers $m_p\geqslant 1$,neighbourhoods V_p of K_{j_p} $(p=1,\dots,\nu)$ and a constant $C\geqslant 0$ such that

$$\|u_x\|_{m,V}\leqslant C\max_{1\leqslant p\leqslant\nu}\|u_{j_px}\|_{m_p,V_p}\ .$$

From the assumed continuity of the assignments $(f,x)\to u_j(f)x$ from $A(K_j)\times X$ into X ,we infer the existence of some integers $m_p'\geqslant 1$ and constants $C_p\geqslant 0$ such that if $f\in A$,then

$$\|u(f)x\|_m \leqslant \|u_x\|_{m,V} \|f\|_V \leqslant$$
$$\leqslant C \max_{1\leqslant p\leqslant \nu} \sup_{\|f_p\|_{V_p}\leqslant 1} \|u_{j_p}(f_p)x\|_{m_p} \|f\|_V \leqslant$$
$$\leqslant C(\max_{1\leqslant p\leqslant \nu} C_p \|x\|_{m'_p}) \|f\|_V .$$

From this calculation we deduce that $u(f)\in\mathcal{L}(X)$ for each $f\in A(K)$ and that the mapping $(f,x)\longrightarrow u(f)x$ is continuous.

Finally, the fact that u is an algebra homomorphism follows from the corresponding property of u restricted to A, the above continuity and the density of A in $A(K)$.

We shall present another connection between the analytic functional calculus and analytic operators. Let us note that the space $\mathcal{L}_K(A,X)$ ($K\subset\mathbb{C}^n$ a compact set) is both an $\mathcal{L}(X)$-module and an A-module, in a natural way. In particular, the system of the coordinate functions $\mu=(\mu_1,\dots,\mu_n)$ induces in $\mathcal{L}_K(A,X)$ a (commuting) system of multiplication operators, also denoted by $\mu=(\mu_1,\dots,\mu_n)$.

3.3. PROPOSITION. Let $K\subset\mathbb{C}^n$ be a compact subset. Then for every regular commuting system $a=(a_1,\dots,a_n)\subset\mathcal{L}(X)$ with $\sigma(a,X)\subset K$ the space X can be identified with a closed subspace X_a of $\mathcal{L}_K(A,X)$, which has a natural A-module structure. Moreover, in this identification, the commuting system $(a_1,\dots,a_n)$ is the restriction of the commuting system $\mu=(\mu_1,\dots,\mu_n)$ to the space X_a.

Proof. We define $X_a=\{u_x\,;\,x\in X\}$, where $u_x(f)=f(a)x$ for all $f\in A$ and $x\in X$. Since $\sigma(a,X)\subset K$, it is clear that $u_x\in\mathcal{L}_K(A,X)$ and that $x\longrightarrow u_x$ is linear, injective (because of the equality $u_x(1)=x$) and continuous. The equality $u_x(1)=x$

also implies that the mapping $x \to u_x$ has closed range. Hence X and X_a are isomorphic as Fréchet spaces. Let us show that X_a has an A-module structure. Indeed, if $g \in A$, then $gu_x(f) = u_x(gf) = (gf)(a)x = g(a)f(a)x = u_{g(a)x}(f)$ for all $f \in A$, so that X_a is an A-module. In particular, $\mu_j u_x(f) = a_j u_x(f)$ for every $f \in A$, and hence the commuting system $\mu = (\mu_1, \ldots, \mu_n)$ extends the commuting system $a = (a_1, \ldots, a_n)$, when acting on X_a.

3.4. DEFINITION. The pair $(\mathcal{L}_K(A,X), \mu)$ is called the <u>universal extension</u> of the regular commuting systems $a = (a_1, \ldots, a_n) \subset \mathcal{L}(X)$ with $\sigma(a,X) \subset K$.

As one might expect, the universal extension has some interesting properties for compact sets $K \subset \mathbb{R}^n$, and therefore for commuting systems of operators with real joint spectrum (see the fifth section).

In the remainder of this section we shall give some technical results concerning the analytic functional calculus, which are needed in the next sections.

Let X be a fixed Fréchet space and let $\mathcal{S}_0(X)$ be the family of all subspaces $Y \subset X$ such that Y has a Fréchet space structure of its own and the inclusion $Y \subset X$ is continuous.

Let $a = (a_1, \ldots, a_n) \subset \mathcal{L}(X)$ be a regular commuting system. We denote by $\mathrm{Lat}(a)$ the family of those subspaces $Y \in \mathcal{S}_0(X)$ that are invariant under the action of $a_1, \ldots, a_n$, such that the restriction $a_Y = (a_1|Y, \ldots, a_n|Y)$ is also regular.

If $Y \in \mathrm{Lat}(a)$ and $Z \in \mathrm{Lat}(a_Y)$, then the system a acts naturally in the quotient Y/Z. We denote by $a_{Y/Z}$ the system of endomorphisms of Y/Z that is induced by a. We also set

$$\sigma(a;Y,Z) = \sigma(a,Y)\cup\sigma(a,Z) .$$

We shall be interested to give some results concerning the "spectral theory" of systems of endomorphisms induced by regular commuting systems of operators in quotients of the previous type, when no topology is involved.(The importance of such quotients is pointed out, for instance, in [23].)

3.5. LEMMA. <u>Let $a=(a_1,\dots,a_n)\subset\mathcal{L}(X)$ be a regular commuting system, let $Y\in \mathrm{Lat}(a)$ and let $Z\in \mathrm{Lat}(a_Y)$ .Then the quotient Y/Z has a natural $A(\sigma(a;Y,Z))$-module structure.</u>

<u>Proof</u>. Let $\tau : Z\to Y$ be the inclusion, which is continuous. If $b_j=a_j|Y$ and $c_j=a_j|Z$ $(j=1,\dots,n)$, then $\tau c_j=b_j\tau$ for all j .Therefore $\tau f(c)=f(b)\tau$ by virtue of Corollary III.9.11 from [20], where $f\in A(\sigma(a;Y,Z))$, $b=(b_1,\dots,b_n)$, $c=(c_1,\dots,c_n)$. In particular, $f(a_Y)(Z)\subset Z$,so that Y/Z has an $A(\sigma(a;Y,Z))$ - -module structure that is defined in the following way :

$$(3.3)\qquad f(a_{Y/Z})(y+Z) = f(a_Y)y+Z \ ,\ f\in A(\sigma(a;Y,Z)) \ ,\ y\in Y .$$

For the next proof we need some details concerning the construction of the analytic functional calculus by means of the Cauchy-Weil integral [18],[13],[20].Let $a=(a_1,\dots,a_n)\subset\mathcal{L}(X)$ be a regular commuting system.Let $U\supset\sigma(a,X)$ be an open set in $\mathbb{C}^n$, let $f\in A(U)$ $(=A(U,\mathbb{C}))$ and let $x\in X$ .We consider the exterior form $\eta(z)=f(z)x\,\sigma_1\wedge\dots\wedge\sigma_n$,which satisfies the equation $(\delta_{\alpha_a}+\bar\partial)\eta=0$ in U (for the notation, see the beginning of this section).If $V=U\setminus\sigma(a,X)$,then there exists an exterior form $\xi\in\Lambda^{n-1}[(\sigma,\bar\zeta),C^\infty(V,X)]$ such that $(\delta_{\alpha_a}+\bar\partial)\xi=\eta$ in V , where $\bar\zeta$ is the system of differentials $(d\bar z_1,\dots,d\bar z_n)$ .Then consider a function $\varphi\in C^\infty(U)$ such that $\varphi=0$ in a neighbourhood of $\sigma(a,X)$ and the support of $1-\varphi$ is compact. If P_σ is the mapping that annihilates every monomial that contains one of $\sigma_1,\dots,\sigma_n$ and leaves the other terms invariant, then we have

$$(3.4) \qquad f(a)x = q_n \int (\bar{\partial}\varphi P_\sigma \xi)(z)\wedge dz ,$$

where $dz = dz_1\wedge\dots\wedge dz_n$ and $q_n = (-1)^{n+1}(2\pi i)^{-n}$. The integral (3.4) does not depend on the particular choice of φ and ξ, and defines the homomorphism (3.1) with the properties (3.2) and (C) (details can be found in [20], Chapter III).

3.6. PROPOSITION. Let $a=(a_1,\dots,a_n)\subset\mathcal{L}(X)$ be a regular commuting system and let $Y,Z,W\in\mathrm{Lat}(a)$ be such that $X=Y+Z$ and $Y\cap Z\in\mathrm{Lat}(a_Y)\cap\mathrm{Lat}(a_Z)$. Then there exists a natural mapping $\theta : W\longrightarrow Z/(Y\cap Z)$ such that

$$(3.5) \qquad \theta f(a_W) = f(a_{Z/(Y\cap Z)})\theta ,$$

for every $f\in A(\sigma(a,W))\cup\sigma(a;Z,Y\cap Z))$.

Proof. Every $x\in X$ has a decomposition of the form $x=x_Y+x_Z$, with $x_Y\in Y$ and $x_Z\in Z$. If $x=x'_Y+x'_Z$ is another decomposition of the same type, then $x_Y - x'_Y = x'_Z - x_Z\in Y\cap Z$, so that the mapping

$$(3.6) \qquad X\ni x\longrightarrow x_Z+Y\cap Z\in Z/(Y\cap Z)$$

is correctly defined. Let θ be the restriction of (3.6) to the space W. Notice that the equation (3.5) is not a direct consequence of Corollary III.9.11 from [20], since the quotient $Z/(Y\cap Z)$ is not, in general, a Fréchet space. Therefore we need a direct argument.

Let $U\supset\sigma(a,W)\cup\sigma(a;Z,Y\cap Z)$ be an open set, let $f\in A(U)$ and let $x\in W$ be fixed, $x=x_Y+x_Z$ as above. Then the form $\eta(z)=f(z)x\,\sigma_1\wedge\dots\wedge\sigma_n$ satisfies the equation $(\delta_{\alpha_a}+\bar{\partial})\eta=0$

in U .Then we can find a form $\xi \in \Lambda^{n-1}[(\sigma,\bar{\zeta}),C^{\infty}(V_1,W)]$ such that $(\delta_{\alpha_a}+\bar{\partial})\xi = \eta$ in $V_1 = U\setminus\sigma(a,W)$.Since $X = Y+Z$ and $Y\cap Z \in \mathrm{Lat}(a_Y)\cap \mathrm{Lat}\,(a_Z)$,we deduce that the sequence of Fréchet spaces

$$0 \longrightarrow Y\cap Z \xrightarrow{\ u\ } Y\oplus Z \xrightarrow{\ v\ } X \longrightarrow 0$$

is exact,where $u(x) = x\oplus(-x)$ and $v(x\oplus y) = x+y$ .Since $C^{\infty}(V_1)$ is nuclear,the sequence

$$0 \to C^{\infty}(V_1,Y\cap Z) \xrightarrow{1\hat{\otimes} u} C^{\infty}(V_1,Y)\oplus C^{\infty}(V_1,Z) \longrightarrow$$
$$\xrightarrow{1\hat{\otimes} v} C^{\infty}(V_1,X) \longrightarrow 0$$

is also exact [2] .In particular,the mapping

$$\Lambda^{n-1}[(\sigma,\bar{\zeta}),C^{\infty}(V_1,Y)] \oplus \Lambda^{n-1}[(\sigma,\bar{\zeta}),C^{\infty}(V_1,Z)] \longrightarrow$$
$$\longrightarrow \Lambda^{n-1}[(\sigma,\bar{\zeta}),C^{\infty}(V_1,X)] \longrightarrow 0$$

given by $\xi_1\oplus\xi_2 \longrightarrow \xi_1+\xi_2$ is surjective (the latter sequence is a direct sum of the last spaces of the former).Thus we can decompose $\xi = \xi_Y+\xi_Z$,where the coefficients of ξ_Y are Y-valued and the coefficients of ξ_Z are Z-valued.

Now,consider the form

$$\lambda = (\delta_{\alpha_a}+\bar{\partial})\xi_Z - \eta_Z = \eta_Y - (\delta_{\alpha_a}+\bar{\partial})\xi_Y$$

where $\eta_Z = fx_Z\,\sigma_1\wedge\ldots\wedge\sigma_n$ and $\eta_Y = fx_Y\,\sigma_1\wedge\ldots\wedge\sigma_n$ .Note that the coefficients of the form λ have values in $Y\cap Z$ .Indeed, the space $C^{\infty}(V_1,Y)\cap C^{\infty}(V_1,Z)$ is (algebraically) isomorphic to the kernel of $1\hat{\otimes}v$,and the latter is isomorphic to $C^{\infty}(V_1,Y\cap Z)$ .Moreover, $(\delta_{\alpha_a}+\bar{\partial})\lambda = 0$ in V_1 .Then there exists a form $\rho \in \Lambda^{n-1}[(\sigma,\bar{\zeta}),C^{\infty}(V_2,Y\cap Z)]$ such that $(\delta_{\alpha_a}+\bar{\partial})\rho = \lambda$ in $V_2 = U\setminus(\sigma(a,W)\cup\sigma(a,Y\cap Z))$.Take a function $\varphi \in C^{\infty}(U)$ such that $\varphi = 0$ in a neighbourhood of $\sigma(a,W)\cup\sigma(a;Z,Y\cap Z)$ and the support of $1-\varphi$ is compact.Then,by (3.4) we have

$$f(a_W)x = q_n \int (\bar{\partial}\varphi P_\sigma \xi)(z) \wedge dz =$$

$$(3.7) \qquad = q_n \int (\bar{\partial}\varphi P_\sigma(\xi_Z - \rho))(z) \wedge dz +$$
$$+ q_n \int (\bar{\partial}\varphi P_\sigma(\xi_Y + \rho))(z) \wedge dz \,.$$

Notice that $\eta_Z = (\delta_{\alpha_a} + \bar{\partial})(\xi_Z - \rho)$ in V_2 ,and hence

$$f(a_Z)x_Z = q_n \int (\bar{\partial}\varphi P_\sigma(\xi_Z - \rho))(z) \wedge dz \quad ,$$

by (3.4) .We also note that

$$\int (\bar{\partial}\varphi P_\sigma(\xi_Y + \rho))(z) \wedge dz \in Y \quad .$$

In this way (3.7) is a decomposition of the element $f(a_W)x$ into a sum from $Z+Y$,and the first term of this sum is $f(a_Z)x_Z$ .Consequently,according to (3.3),

$$\theta f(a_W)x = f(a_Z)x_Z + Y \cap Z = f(a_{Z/(Y\cap Z)})\theta x \quad ,$$

and the proof is complete.

We need also a version of Lemma IV.2.2 from [20].

3.7. LEMMA. <u>Let</u> $a=(a_1,\dots,a_n) \subset \mathcal{L}(X)$ <u>be a regular commuting system,let</u> $Y \in \mathrm{Lat}(a)$ <u>and let</u> $D \subset \mathbb{C}^n$ <u>be an open polydisc such that</u> $D \supset \sigma(a;X,Y)$ <u>.Then we have</u>

$$H^p(A(D,X)/A(D,Y),\alpha_a) = 0 \quad , \quad 0 \leqslant p \leqslant n-1 \,.$$

<u>Proof.</u> We apply Lemma I.2.6 from [20].Let $\alpha_p(z) = z_p - a_p$ $(p=1,\dots,n)$ and let $C = A(D,X)/A(D,Y)$ .Also set $\alpha_0 = 0$ .The kernel of α_p $(p \geqslant 1)$ in $C/(\alpha_1 C + \dots + \alpha_{p-1}C)$ is null iff from every relation of the form $\alpha_p F_p = \alpha_1 F_1 + \dots + \alpha_{p-1}F_{p-1}$,with $F_1,\dots,F_p \in C$,we deduce that $F_p = \alpha_1 G_1 + \dots + \alpha_{p-1}G_{p-1}$,where $G_1,\dots,G_{p-1} \in C$ .Let $f_j \in F_j$ $(j=1,\dots,n)$.Then we have $\alpha_p f_p = \alpha_1 f_1 + \dots + \alpha_{p-1}f_{p-1} + h_p$,where $h_p \in A(D,Y)$ .We now proceed as in the proof of Lemma IV.2.2 from [20].Namely,we get a

representation of the form $f_p = \alpha_1 g_1 + \ldots + \alpha_{p-1} g_{p-1} + v_p$,where

$$g_j(z) =$$

$$= (2\pi i)^{-1} \int_{\Gamma_p} (w_p - z_p)^{-1} (w_p - a_p)^{-1} f_j(z_1, \ldots, w_p, \ldots, z_n)\, dw_p ,$$

$$v_p(z) =$$

$$= (2\pi i)^{-1} \int_{\Gamma_p} (w_p - z_p)^{-1} (w_p - a_p)^{-1} h_p(z_1, \ldots, w_p, \ldots, z_n)\, dw_p .$$

Here Γ_p is a smooth contour that surrounds $\sigma(a_p, X) \cup \sigma(a_p, Y)$ in the corresponding projection of D .Plainly,the functions g_j and v_p can be extended to the whole D .Moreover,since $(w_p - a_p)^{-1} y = (w_p - a_p | Y)^{-1} y$ for all $w_p \in \Gamma_p$ and $y \in Y$,we deduce that $v_p \in A(D,Y)$ .Therefore,if we put $G_j = g_j + A(D,Y)$,then $F_p = \alpha_1 G_1 + \ldots + \alpha_{p-1} G_{p-1}$ .According to Lemma I.2.6 from [20], we then have $H^p(C, \alpha_a) = 0$ for $0 \leqslant p \leqslant n-1$.

Lemma 3.7 gives no information about the last cohomology space.It happens that this space also has a notable property. To state it,let us remark that the integral (3.4) makes sense for every $f \in A(U,X)$ .By abuse of notation,the symbol $f(a)$ will designate the value of the integral (3.4) for such a function f ,which is a vector of X .

3.8. LEMMA. Let $a = (a_1, \ldots, a_n) \subset \mathcal{L}(X)$ be a regular commuting system,let $Y \in \mathrm{Lat}(a)$,let $D \supset \sigma(a;X,Y)$ be an open polydisc and let $f \in A(D,X)$ .The form $\eta = (f + A(D,Y))\, \sigma_1 \wedge \ldots \wedge \sigma_n$ is in the range of the operator

$$\delta_{\alpha_a} : \Lambda^{n-1}[\sigma, A(D,X)/A(D,Y)] \longrightarrow \Lambda^n[\sigma, A(D,X)/A(D,Y)]$$

if and only if $f(a) \in Y$.

Since this result is not needed for further development, we omit its proof (see [6], Theorem 1.3.3 for a particular case).

4. SPECTRAL CAPACITIES

As in the previous section, we denote by $\mathcal{S}_0(X)$ the family of all Fréchet spaces $Y \subset X$ for which the inclusion is continuous.

Let Ω be a topological space and let $C\ell(\Omega)$ be the family of all closed subsets of Ω.

4.1. DEFINITION. A mapping $C\ell(\Omega) \ni F \longrightarrow X(F) \in \mathcal{S}_0(X)$ is said to be a <u>spectral capacity</u> if it has the properties:

(1) $X(\emptyset) = 0$, $X(\Omega) = X$;

(2) If $\{F_j\}_{j=1}^{\infty} \subset C\ell(\Omega)$ and $F = \cap\{F_j\ ;\ j \geq 1\}$, then $X(F) = \cap\{X(F_j)\ ;\ j \geq 1\}$ and $X(F) \in \mathcal{S}_0(X(F_j))$ for all j;

(3) For each finite open covering $\{G_k\}_{k=1}^{m}$ of Ω we have the decomposition $X = X(\overline{G}_1) + \ldots + X(\overline{G}_m)$.

Definition 4.1 extends the usual concept (see [1], [6], [20] etc.) to the case when the values of the mapping are not necessarily closed subspaces of the given space. Besides the customary examples [20], it is easily seen that the mapping

$$C\ell(K) \ni F \longrightarrow \mathcal{L}_F(A,X) \in \mathcal{S}_0(\mathcal{L}_K(A,X))$$

and, in particular, the mapping

$$C\ell(K) \ni F \longrightarrow A'(F) \in \mathcal{S}_0(A'(K))$$

are spectral capacities in the sense of Definition 4.1, where K is a fixed compact set in $\mathbb{R}^n$, by virtue of the decompositions (2.11) and (1.1), respectively.

Using Definition 4.1 we can introduce a corresponding concept of decomposability (see also [3], [6], [20] etc.).

4.2. DEFINITION. A regular commuting system $a = (a_1, \ldots, a_n) \subset \mathcal{L}(X)$ is said to be <u>decomposable</u> if there exists a spectral capacity $\mathcal{C}\ell(\sigma(a,X)) \ni F \longrightarrow X(F) \in \mathcal{S}_0(X)$ such that $X(F) \in \mathrm{Lat}(a)$ and $\sigma(a,X(F)) \subset F$ for all closed sets $F \subset \sigma(a,X)$.

Let us mention that spectral decompositions with respect to subspaces that are ranges of operators have been proposed many years ago by C.Foiaş (for single operators in Banach spaces) but the lack of significant examples has prevented the development of such a theory.We shall see in the next section that the system of multiplication operators $\mu=(\mu_1,\ldots,\mu_n)$ is decomposable on the space $\mathcal{L}_K(A,X)$ (K compact in $\mathbb{R}^n$),in the sense of Definition 4.2.

4.3. LEMMA. <u>Let</u> $a=(a_1,\ldots,a_n) \subset \mathcal{L}(X)$ <u>be a decomposable system and let</u> $Y \in \mathrm{Lat}(a)$ <u>.If</u> $F_Y = \sigma(a,Y) \cap \sigma(a,X)$ <u>,then</u> $Y \subset X(F_Y)$, <u>where</u> $F \longrightarrow X(F)$ <u>is a spectral capacity of the system</u> a.

<u>Proof.</u> Let G_1, G_2 be open sets in $\mathbb{C}^n$ such that $G_1 \supset F_Y$, $\bar{G}_2 \cap F_Y = \emptyset$ and $G_1 \cup G_2 \supset \sigma(a,X)$ .If $F_j = \bar{G}_j \cap \sigma(a,X)$,then $X = X_1 + X_2$,where $X_j = X(F_j)$ $(j=1,2)$,by the decomposability of the system a .Let θ be the natural mapping from Y into X_2/X_{12}, given by Proposition 3.6,with $X_{12} = X_1 \cap X_2 = X(F_1 \cap F_2)$. Let us remark that

$$\sigma(a,Y) \cap \sigma(a;X_2,X_{12}) \subset \sigma(a,Y) \cap (F_2 \cup (F_1 \cap F_2)) = \emptyset ,$$

so that there exists a function $f \in A(\sigma(a,Y) \cup \sigma(a;X_2,X_{12}))$ such that $f=1$ in a neighbourhood of $\sigma(a,Y)$ and $f=0$ in a neighbourhood of $\sigma(a;X_2,X_{12})$.Then $f(a_{X_2/X_{12}})=0$ by (3.3) , and hence $\theta f(a_Y) = \theta \circ 1_Y = 0$ (with 1_Y the identity on Y),in virtue of (3.5).From the definition of the mapping θ ,we then derive that $Y \subset X_1 = X(F_1)$.

Now,we can consider a sequence $\{F_{1,k}\}_{k=1}^{\infty}$ such that every set $F_{1,k}$ has the properties of F_1 and $\cap\{F_{1,k}\ ;\ k \geqslant 1\} = F_Y$.

Then, by Definition 4.1,

$$Y \subset \bigcap_{k=1}^{\infty} X(F_{1,k}) = X(F_Y) ,$$

and the proof of the lemma is complete.

As in the usual case (see Theorem IV.1.9 from [20]), the previous lemma leads easily to the following uniqueness result.

4.4. THEOREM. Let $a=(a_1,\dots,a_n)\subset\mathcal{L}(X)$ be a decomposable system. Then there exists only one spectral capacity corresponding to this system.

Theorem 4.4 insures the uniqueness of the spaces that form the spactral capacity but it doesn't say anything about their topologies. A natural question whose answer is not known by the author of this text is the following : Are the topologies of the spaces that form the spectral capacity of a given decomposable system uniquely determined?

Anyway, we can speak about the spectral capacity associated with a decomposable system $a=(a_1,\dots,a_n)\subset\mathcal{L}(X)$, which will be denoted by $X_a(F)$ for each closed $F\subset\sigma(a,X)$.

An important consequence of the decomposability of a system of operators is the so-called single valued extension property (see, for instance, [20], Chapt. IV, Section 2). We still have this property with our more general conditions.

We recall that a regular commuting system $a=(a_1,\dots,a_n)\subset$ $\subset\mathcal{L}(X)$ is said to have the single valued extension property if for every point $w\in\mathbb{C}^n$ there exists an open polydisc $D\ni w$ such that $H^p(A(D,X),\alpha_a)=0$ for $0\leqslant p\leqslant n-1$.

4.5. THEOREM. If $a=(a_1,\dots,a_n)\subset\mathcal{L}(X)$ is a decomposable system, then it has the single valued extension property.

Proof. Let D_0 and D be open polydiscs such that $D_0 \subset \overline{D}_0 \subset D$. From the decomposability of the system a we infer the existence of two closed sets F_1, F_2 in $\sigma(a,X)$ such that $F_1 \subset D$, $F_2 \cap D_0 = \emptyset$ and $X = X_1 + X_2$, where $X_j = X_a(F_j)$ $(j=1,2)$. In particular, the sequence of Fréchet spaces

$$0 \longrightarrow X_1 \cap X_2 \xrightarrow{u} X_1 \oplus X_2 \xrightarrow{v} X \longrightarrow 0$$

is exact, where $u(x) = x \oplus (-x)$ and $v(x_1 \oplus x_2) = x_1 + x_2$. Since $A(D)$ is nuclear, the tensor multiplication of the above sequence with $A(D)$ leads to the exact sequence

$$(4.1) \qquad 0 \longrightarrow A(D, X_1 \cap X_2) \xrightarrow{1 \hat{\otimes} u} A(D, X_1) \oplus A(D, X_2) \xrightarrow{1 \hat{\otimes} v} A(D,X) \longrightarrow 0 .$$

Consequently, there exists a natural mapping

$$\theta : A(D,X) \longrightarrow A(D,X_1)/A(D,X_1 \cap X_2)$$

given by $\theta f = f_1 + A(D, X_1 \cap X_2)$, where $f = f_1 + f_2$, with $f_j \in A(D,X_j)$ (which corresponds to (3.6)), because the kernel of $1 \hat{\otimes} v$ is isomorphic to $A(D, X_1 \cap X_2)$.

Now, let $\eta \in \Lambda^p[\sigma, A(D,X)]$, where $0 \leqslant p \leqslant n-1$, p fixed, be such that $\delta_{\alpha_a}\eta = 0$ in D. From (4.1) we deduce that $\eta = \eta_1 + \eta_2$, with $\eta_j \in \Lambda^p[\sigma, A(D,X_j)]$. Let us observe that $0 = \theta\delta_{\alpha_a}\eta = \delta_{\alpha_a}\theta\eta = \delta_{\alpha_a}\theta\eta_1$. Since $\sigma(a,X_1) \subset F_1 \subset D$ and $\sigma(a, X_1 \cap X_2) \subset \subset F_1 \cap F_2 \subset D$, according to Lemma 3.7 there exists a form

$$\zeta \in \Lambda^{p-1}[\sigma, A(D,X_1)/A(D,X_1 \cap X_2)]$$

such that $\theta\eta_1 = \delta_{\alpha_a}\zeta$. Hence $\eta_1 = \delta_{\alpha_a}\xi_1 + \eta_1'$, where $\xi_1 \in \Lambda^{p-1}[\sigma, A(D,X_1)]$ and $\eta_1' \in \Lambda^p[\sigma, A(D, X_1 \cap X_2)]$. In this way, $\eta = \eta_1 + \eta_2 = \delta_{\alpha_a}\xi_1 + \eta_2'$, with $\eta_2' = \eta_2 + \eta_1' \in \Lambda^p[\sigma, A(D,X_2)]$. Since $\delta_{\alpha_a}\eta_2' = 0$ and $F_2 \cap D_0 = \emptyset$, by Proposition III.8.3 from

[20] we infer that $\eta_2' = \delta_{\alpha_a}\xi_2$, where $\xi_2 \in \Lambda^{p-1}[\sigma, A(D_0, X_2)]$. Thus the equation $\delta_{\alpha_a}\xi = \eta$ has a solution $\xi = \xi_1 + \xi_2 \in \Lambda^{p-1}[\sigma, A(D_0, X)]$ for every polydisc $D_0 \subset \bar{D}_0 \subset D$.

Finally, we either proceed as in the last part of the proof of Proposition IV.2.5 from [20] to get a global solution, or apply Theorem 1.2 from [4] . Both ways imply that the system a has the single valued extension property.

The <u>analytic local spectrum</u> of a regular commuting system $a = (a_1, \ldots, a_n) \subset \mathcal{L}(X)$ at the point $x \in X$ is the complement in $\mathbb{C}^n$ of the set of those points $w \in \mathbb{C}^n$ for which there exist an open set $U \ni w$ and functions $u_1, \ldots, u_n$ in $A(U, X)$ such that $(z_1 - a_1)u_1(z) + \ldots + (z_n - a_n)u_n(z) = x$ for all $z \in U$ (see [20], Chapter IV, Section 2 for other details). If $a = (a_1, \ldots, a_n) \subset \mathcal{L}(X)$ is a decomposable system and $\gamma_a(x)$ is the analytic local spectrum of the system a at x , then we have the equality

$$(4.2) \qquad \gamma_a(x) = \cap\{F \in C\ell(\sigma(a, X)) \; ; x \in X_a(F)\} \; .$$

Indeed, if $\sigma_a(x)$ is the right hand side of (4.2), then $x \in X_a(\sigma_a(x))$, and hence $\gamma_a(x) \subset \sigma_a(x)$. The converse inclusion follows from an argument that is similar to the proof of Proposition 3.6, so that we only outline it. Let $\{G_1, G_2\}$ be an open covering of $\sigma(a, X)$ such that $G_1 \supset \gamma_a(x)$ and $\bar{G}_2 \cap \gamma_a(x) = \emptyset$. Let $F_j = \bar{G}_j \cap \sigma(a, X)$ $(j = 1, 2)$, and set $Y = X_a(F_1)$, $Z = X_a(F_2)$. Thus $X = Y + Z$. Let also U_1 , U_2 be open sets such that $\gamma_a(x) \subset \subset U_1$, $F_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$. We now proceed as in the proof of Propsition 3.6, with $U = U_1 \cup U_2$, $f \in A(U)$ equal to one in U_1 and equal to zero in U_2 , and $\sigma(a, W)$ replaced by $\gamma_a(x)$. The other notation is similar. We observe that if $\eta(z) = f(z)x\,\sigma_1 \wedge \cdots \wedge \sigma$

then there is a form $\xi \in \Lambda^{n-1}[(\sigma,\bar{\zeta}), C^{\infty}(V_1,X)]$ such that $(\delta_{\alpha_a}+\bar{\partial})\xi=\eta$ in $V_1=U\setminus\gamma_a(x)$, where $\xi|U_1$ is given by Lemma IV.2.7 from [20] and $\xi|U_2=0$ by hypothesis. With the symbols as in the proof of Proposition 3.6, equation (3.7) has the form

$$\begin{aligned} x &= q_n\int(\bar{\partial}\varphi P_{\sigma}\xi)(z)\wedge dz = \\ &= q_n\int(\bar{\partial}\varphi P_{\sigma}(\xi_Z-\rho))(z)\wedge dz + \\ &+ q_n\int(\bar{\partial}\varphi P_{\sigma}(\xi_Y+\rho))(z)\wedge dz \end{aligned}$$

(see also Lemma IV.2.9, especially formula (2.5), from [20]).

Since $\eta_Z=(\delta_{\alpha_a}+\bar{\partial})(\xi_Z-\rho)$ is equal to zero in a neighbourhood of $\sigma(a,Z)\subset F_2$, it follows that

$$\int(\bar{\partial}\varphi P_{\sigma}(\xi_Z-\rho))(z)\wedge dz=0\ ,$$

by (3.4) (since $f(a_Z)=0$). Therefore

$$x=q_n\int(\bar{\partial}\varphi P_{\sigma}(\xi_Y+\rho))(z)\wedge dz\in Y=X_a(F_1)\ .$$

Finally, we have $F_1=\bar{G}_1\cap\sigma(a,X)\supset\gamma_a(x)$, where G_1 is arbitrary. Therefore $x\in X_a(\gamma_a(x))$, i.e. the desired conclusion.

The last result of this section is a version, valid for Fréchet spaces, of Proposition IV.2.11 from [20] (see also [21]).

4.6. THEOREM. <u>Let $a=(a_1,\dots,a_n)\subset\mathcal{L}(X)$ be a regular commuting system with the single valued extension property. Then the subset $\{x\in X\ ;\ \gamma_a(x)=\sigma(a,X)\}$ is of the second category in</u> X.

<u>Proof.</u> We first prove the following <u>statement</u>:

Let $D\subset\mathbb{C}^n$ be an open polydisc such that $D\cap\sigma(a,X)\neq\emptyset$.

Then the subset $(\alpha_1 A(D,X)+\ldots+\alpha_n A(D,X))\cap X$ is of the first category in X, where $\alpha_j(z)=z_j-a_j$ $(j=1,\ldots,n)$.

Indeed, let us consider the continuous and linear mapping

$$T_a:\ \bigoplus_{j=1}^{n} A(D,X)\longrightarrow A(D,X)\quad,\quad T_a\Big(\bigoplus_{j=1}^{n} f_j\Big)=\sum_{j=1}^{n}\alpha_j f_j\ .$$

Since X can be regarded as a closed subspace of $A(D,X)$, then $Y_a=T_a^{-1}(X)$ is a closed subspace of $A(D,X)\oplus\ldots\oplus A(D,X)$.

Let $S_a=T_a|Y_a$. Then we have

$$S_a(Y_a)=(\alpha_1 A(D,X)+\ldots+\alpha_n A(D,X))\cap X\ .$$

Assume now that $S_a(Y_a)$ is a set of the second category in X. Since Y_a and X are Fréchet spaces, then S_a is surjective. Thus the mapping

$$Y_a\hat{\otimes}A(D)\xrightarrow{S_a\hat{\otimes}1}X\hat{\otimes}A(D)=A(D,X)$$

is also surjective. Since Y_a is a closed subspace of $A(D,X)\oplus\ldots\oplus A(D,X)$ and $A(D)$ is nuclear, then $Y_a\hat{\otimes}A(D)$ can be identified with a closed subspace of the space

$$\Big(\bigoplus_{j=1}^{n} A(D,X)\Big)\hat{\otimes}A(D)=\bigoplus_{j=1}^{n} A(D\times D,X)\ .$$

Moreover, the mapping $S_a\hat{\otimes}1$ is the restriction to the previous subspace of the mapping (with values in $A(D\times D,X)$)

$$(T_a\hat{\otimes}1)\Big(\bigoplus_{j=1}^{n} f_j\Big)(z,w)=\sum_{j=1}^{n}\alpha_j(z)f_j(z,w)\quad,\ z,w\in D\ .$$

Since $S_a\hat{\otimes}1$ is surjective, every element $f\in A(D,X)$ can be written as

$$f(w)=\alpha_1(z)g_1(z,w)+\ldots+\alpha_n(z)g_n(z,w)\quad,\ z,w\in D\ ,$$

with $g_1,\ldots,g_n$ in $A(D\times D,X)$. This shows, for $w=z$, that the mapping T_a is surjective. However, such a conclusion is a contradiction. Indeed, in this case $H^n(A(D,X),\alpha_a)=0$, whereas, automatically, $H^p(A(D,X),\alpha_a)=0$ for $0\leqslant p\leqslant n-1$ (since the system a has the single valued extension property). Then $D\cap\sigma(a,X)=\emptyset$, which contradicts our assumption.

Now, let $\{w_k\}_{k=1}^{\infty}$ be a dense subset of $\sigma(a,X)$. Let $\{D_{kq}\}_{q=1}^{\infty}$ be a family of open polydiscs such that the center of D_{kq} is w_k and the polyradius of D_{kq} tends to zero as $q\longrightarrow\infty$, for each q. Then the set

$$X_0=\bigcup_{k,q=1}^{\infty}(\alpha_1 A(D_{kq},X)+\ldots+\alpha_n A(D_{kq},X))\cap X$$

is of the first category in X, by the statement at the beginning of this proof. Let $x_0\notin X_0$ and assume that $\sigma(a,X)\setminus\gamma_a(x_0)\neq\emptyset$. By the density of the set $\{w_k\}_k$ in $\sigma(a,X)$, the fact that the polyradius of D_{kq} tends to zero as $q\longrightarrow\infty$ and the definition of $\gamma_a(x_0)$, we can find a pair (k,q) such that

$$x_0=\alpha_1(z)u_1(z)+\ldots+\alpha_n(z)u_n(z)\ ,\quad z\in D_{kq}\ ,$$

where $u_j\in A(D_{kq},X)$ for $j=1,\ldots,n$. Then $x_0\in X_0$, which is a contradiction. Consequently $\gamma_a(x_0)=\sigma(a,X)$, so that the set

$$\{x\in X\ ;\ \gamma_a(x)=\sigma(a,X)\}\supset X\setminus X_0$$

is of the second category in X, and the proof is complete.

It is beyond our scope to recapture here all of the properties that are known for decomposable systems, as in [20], Chapt. IV. We only mention that one might consider spectral capacities on pseudorings of closed sets and get the corresponding assertions for them.

5. CONSEQUENCES FOR THE UNIVERSAL EXTENSION

In this section we shall derive some consequences of the previous results in the case of the universal extensions (see

Definition 3.4).

Let X be a Fréchet space and let $S \subset \mathbb{R}^n$ be a fixed compact set. For the sake of simplicity, the space $\mathcal{L}_S(A,X)$ will be denoted by $X^A(S)$. We shall show that the system of multiplication operators $\mu=(\mu_1,\dots,\mu_n)$ is decomposable on $X^A(S)$.

5.1. LEMMA. Let $K \subset \mathbb{R}^n$ be a compact set and let $f=(f_1,\dots,f_m) \subset A(K)$. Then for every $w_0 \in \mathbb{C}^m \setminus f(K)$ there are open neighbourhoods $U \ni w_0$ in $\mathbb{C}^m$ and $V \supset K$ in $\mathbb{C}^n$, and functions $g_1,\dots,g_m$ in $A(U \times V)$ such that

$$(5.1) \qquad \sum_{j=1}^{m} (w_j - f_j(z)) g_j(w,z) = 1 \ , \ w \in U \ , \ z \in V \ .$$

Proof. Let $U \ni w_0$ be an open polydisc such that $\bar{U} \cap f(K) = \emptyset$. Since $K \subset \mathbb{R}^n$, by a well known theorem of Grauert the set K has a fundamental system of neighbourhoods which are Stein manifolds. Therefore we can choose a neighbourhood V of K, V a Stein manifold, such that $f(V) \cap U = \emptyset$. The existence of the functions $g_1,\dots,g_m$ then follows from Theorem B of Cartan (see [8] for details).

5.2. THEOREM. The analytic functional calculus of the commuting system $\mu=(\mu_1,\dots,\mu_n)$ on $X^A(S)$ is given by

$$(5.2) \qquad (f(\mu)u)(g) = u(fg) \ , \ f \in A(S) \ , \ g \in A \ , \ u \in X^A(S) \ .$$

Moreover, $\sigma(\mu, X^A(S)) = S$.

Proof. The right side of (5.2) makes sense for $f \in A(S)$, since every $u \in X^A(S)$ can be extended by continuity to $A(S)$ (because of the density of A in $A(S)$). It is plain that (5.2) is a unital algebra homomorphism satisfying the condition (C) from the third section. Therefore, it is the functional calculus of μ provided that (3.2) is fulfilled.

Let us prove that $\sigma(f(\mu),X^A(S))=f(S)$ for all m-tuples $f=(f_1,\dots,f_m)\subset A(S)$. Indeed, if $w_0\notin f(S)$, in virtue of Lemma 5.1 there are open sets $U\ni w_0$, $V\supset S$ and functions $g_1,\dots,g_m$ in $A(U\times V)$ such that (5.1) holds. If $u\in A(U,X^A(S))$, then we can define the continuous endomorphism β_j of $A(U,X^A(S))$ by the equations

$$((\beta_j u)(w))(h)=u(w)_z(g_j(w,z)h(z)),\quad w\in U,\ h\in A$$

for all $j=1,\dots,m$. Similarly, if we define the endomorphisms

$$((\alpha_j u)(w))(h)=w_j u(w)(h)-u(w)(f_j h),$$

then (5.1) shows that $\alpha_1\beta_1+\dots+\alpha_m\beta_m=1$. Therefore, by virtue of Lemma I.2.5 form [20], the system of endomorphisms $\alpha_{f(\mu)}(w)=(w_1-f_1(\mu),\dots,w_m-f_m(\mu))$ is nonsingular on $A(D,X^A(S))$, so that $w_0\notin\sigma(f(\mu),X^A(S))$. Hence $\sigma(f(\mu),X^A(S))\subset f(S)$.

Assume now that $f(S)\setminus\sigma(f(\mu),X^A(S))\neq\emptyset$. Then we can find a compact nonempty subset $K\subset S$ such that $f(K)\cap\sigma(f(\mu),X^A(S))=\emptyset$. By the first part of the proof, we have $\sigma(f(\mu),X^A(K))\subset f(K)$ (where $X^A(K)=\mathcal{L}_K(A,X)$), and hence $\sigma(f(\mu),X^A(K))$ does not intersect $\sigma(f(\mu),X^A(S))$. We shall apply Proposition 3.6 with $X=Z=X^A(S)$, $Y=0$ and $W=X^A(K)$. Then θ is just the inclusion $X^A(K)\subset X^A(S)$. If the function from (3.5) is supposed to be equal to 1 in a neighbourhood of $\sigma(f(\mu),X^A(K))$ and equal to 0 in a neighbourhood of $\sigma(f(\mu),X^A(S))$, then (3.5) shows that $X^A(K)=A'(K)\hat{\otimes}X=0$, which is a contradiction when $X\neq0$ (which is the case that interests us). Consequently, $\sigma(f(\mu),X^A(S))=f(S)$.

In particular, if f is just the n-tuple of the coordinate functions, the above result shows that $\sigma(\mu,X^A(S))=S$. We have shown in this way that (3.2) holds, and the proof is complete.

5.3. THEOREM. The commuting system $\mu=(\mu_1,\ldots,\mu_n)$ is decomposable on $X^A(S)$ and its spectral capacity is given by

$$C\ell(S)\ni F \longrightarrow X^A(F)=\mathcal{L}_F(A,X)=\{u\in X^A(S)\ ;\ \sigma(u)\subset F\}.$$

Proof. As we have already noticed in the previous section, the mapping $F\longrightarrow X^A(F)$ $(F\in C\ell(S))$ is a spectral capacity. Moreover, $\sigma(\mu,X^A(F))=F$, as follows from the preceding theorem. Hence the system μ is decomposable (in the sense of Definition 4.2). The last equality of the statement follows from (2.10).

5.4. REMARK. The commuting system $\mu=(\mu_1,\ldots,\mu_n)$ on $X^A(S)$ is a little bit more than decomposable. Namely, it has the property that its restriction to each subspace $X^A(F)$ $(F=\overline{F}\subset S)$ is still decomposable. Therefore, we can say that the system μ is strongly decomposable on $X^A(S)$, extending naturally the corresponding one-dimensional concept [1].

5.5. COROLLARY. For every $u\in X^A(S)$ one has $\sigma(u)=\gamma_\mu(u)$.

Proof. Indeed,

$$\sigma(u)=\cap\{F\Subset\mathbb{R}^n\ ;\ u\in X^A(F)\},$$

and the equality follows via Theorems 4.5, 5.3 and the equality (4.2).

5.6. COROLLARY. The set of those $u\in\mathcal{L}_S(A,X)$ such that $\sigma(u)=S$ is of the second category in $\mathcal{L}_S(A,X)$.

Proof. We have $\sigma(\mu,X^A(S))=S$, by Theorem 5.2. Therefore the set $\{u\in X^A(S)\ ;\ \sigma(u)=S\}$ is of the second category, in virtue of Theorem 4.6 and Corollary 5.5.

5.7. THEOREM. Let $a=(a_1,\ldots,a_n)=\mathcal{L}(X)$ be a regular commuting system such that $\sigma(a,X)\subset\mathbb{R}^n$. Then this system is the restriction of a (strongly) decomposable system.

Proof. Indeed, the universal extension of the system a (see Definition 3.4) is a (strongly) decomposable system, by virtue of Theorem 5.3 (and Remark 5.4).

We close this section with an application of Theorem 5.7.

5.8. THEOREM. Let $a=(a_1,\dots,a_n)\subset\mathcal{L}(X)$ be a regular commuting system such that $\sigma(a,X)\subset\mathbb{R}^n$. Then the system a has the single valued extension property.

Proof. Let $S=\sigma(a,X)$. By Proposition 3.3, the space X can be identified with a closed subspace X_a of $X^A(S)$. Then we have the exact sequence of Fréchet spaces

$$0\longrightarrow X_a\longrightarrow X^A(S)\longrightarrow X^A(S)/X_a\longrightarrow 0\ ,$$

from which we derive the exactness of the long cohomology sequence

$$\text{(5.3)}\qquad \begin{aligned}&\dots\longrightarrow H^p(A(D,X_a),\alpha_a)\longrightarrow H^p(A(D,X^A(S)),\alpha_\mu)\longrightarrow\\ &\longrightarrow H^p(A(D,X^A(S)/X_a),\alpha_\mu)\longrightarrow\dots,\end{aligned}$$

for every open polydisc $D\subset\mathbb{C}^n$ (see [20], Theorem I.2.1). Since the system μ has the single valued extension property, we have

$$\text{(5.4)}\qquad H^p(A(D,X^A(S)),\alpha_\mu)=0\ ,\ 0\leqslant p\leqslant n-1\ .$$

We shall show by induction that

$$(I_p)\qquad H^p(A(D,X),\alpha_a)=0\quad,\quad 0\leqslant p\leqslant n-1\ .$$

whenever $\sigma(a,X)\subset\mathbb{R}^n$ (for arbitrary X and a).

The property (I_0) is an easy consequence of (5.3). If we admit that (I_p) holds for a certain $p<n-1$, then we also have

$$\text{(5.5)}\qquad H^p(A(D,X^A(S)/X_a),\alpha_\mu)=0\ .$$

Indeed, $X^A(S)/X_a$ is a Fréchet space and

$$\sigma(\mu,X^A(S)/X_a)\subset\sigma(\mu,X^A(S))\cup\sigma(\mu,X_a)=S\subset\mathbb{R}^n$$

(see Remark IV.1.7 from [20]). Then (5.5) holds by (I_p). From (5.3), (5.4) and (5.5) we then obtain that $H^{p+1}(A(D,X_a),\alpha_a) = 0$, so that the system has the single valued extension property. The proof of the theorem is complete.

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