INSTITUTUL
DE
MATEMATICĂ

INSTITUTUL NAȚIONAL
PENTRU CREAȚIE
ȘTIINȚIFICĂ ȘI TEHNICĂ

ISSN 0250 3638

INCOMPRESSIBLE FLUID FLOW THROUGH A NON HOMOGENEOUS AND ANISOTROPIC DAM

by

Ruxandra STAVRE and Bogdan VERNESCU

PREPRINT SERIES IN MATHEMATICS

No.1/1984

BUCUREȘTI

INCOMPRESSIBLE FLUID FLOW THROUGH A NON-HOMOGENEOUS AND ANISOTROPIC DAM

by

Ruxandra STAVRE and Bogdan VERNESCU*)

January 1984

*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd.Pacii 220, 79622 Bucharest,Romania

INCOMPRESSIBLE FLUID FLOW THROUGH A NON-HOMOGENEOUS AND ANISOTROPIC DAM

Ruxandra STAVRE

and

Bogdan VERNESCU

INTRODUCTION

In this paper we study the flow of an incompressible fluid through a non-homogeneous dam.

The problem for the homogeneous, rectangular dam was first considered by Baiocchi [1] and extended to the non-homogeneous case, in which the permeability coefficient has the form $k(x,y)=k_1(x)\cdot k_2(y)$ by Benci [2] and by Baiocchi, Friedman [3]. The existence and regularity of a solution for the non-homogeneous rectangular dam were proved by Baiocchi [4]. In all of them the so called "Baiocchi transformation" was used.

The homogeneous dam problem was also studied using another formulation by Brezis, Kinderlehrer, Stampacchia [5] that have proved the existence and regularity of a solution. In the same setting the uniqueness was studied by Carrillo, Chipot [6].

For the general form of $k(x,y)$ we employ non-linear variational inequalities using some ideas of [5].

In the second paragraph we study the variational formulation of the physical problem.

The next two paragraphs contain the proof of the existence and uniqueness of the solution using the additional assumption that

$\frac{\partial k}{\partial y} \geq 0$ (if this condition is not fulfiled some other flow models can occur e.g. Alt[7]).

In the fifth paragraph we extend the results to the non-homogeneous and anisotropic dam.

In the last paragraph we consider a layered dam. We prove that the free boundary is a subgraph in each of the layers even if the condition $\frac{\partial k}{\partial y} \geq 0$ is satisfied in every layer and not in the whole dam.

1. THE PHYSICAL PROBLEM

We denote by D the cross section of the dam, by $\Omega \subsetneq D$ the wet region and by $y=Y(x)$ the free boundary.

The boundary of D is formed by four disjoint parts: S_1 the impervious part, S_{2i} the parts in contact with the reservoirs, S_3 the wet part and S_4 the part in contact with the air.

We suppose that the projection of D on the x-axis is contained in the projection of S_1.

Let p be the pressure of the fluid and $k(x,y)$ the permeability coefficient, which is assumed to be a function bounded from below by a positive constant α.

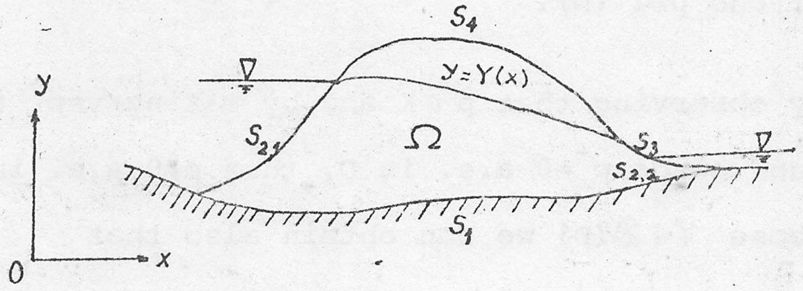
By Darcy's law, the continuity equation and the boundary conditions, we obtain the following problem:

Physical Problem (P.P)

$$
\begin{cases}
\operatorname{div}(k\nabla p) = -\dfrac{\partial k}{\partial y}, \; p>0 \quad \text{in } \Omega, \\
p = H_i - y \text{ on } S_{2i} \; ; \; p=0 \text{ on } S_3, \\
k\left(\dfrac{\partial p}{\partial n} + n_y\right) = 0 \text{ on } S_1 \\
p=0, \; k\left(\dfrac{\partial p}{\partial n} + n_y\right) = 0 \quad \text{on } y=Y(x),
\end{cases}
\qquad (1.1)
$$

where H_i represents the water-level in the i-reservoir and $n(n_x, n_y)$ the exterior unitary normal.

Other boundary condition on S_3 leads to another variational formulation (e.g. [5], [6]).



2. VARIATIONAL FORMULATION

Let:

$$\Gamma = \partial D - S_1 , \tag{2.1}$$

and $f : \Gamma \to \mathbb{R}$ defined by:

$$f = \begin{cases} H_i - y & \text{on } S_{2i} \\ 0 & \text{on } S_3 \cup S_4 . \end{cases} \tag{2.2}$$

We assume that:

$$k \in H^1(D) \cap L^\infty(D) \text{ and } \frac{\partial k}{\partial y} \geqslant 0 \text{ on } D. \tag{2.3}$$

We extend p by zero in $\bar{D} - \bar{\Omega}$ and we denote the extended function by p also.

Let H be the Heaviside function:

$$H(x) = \begin{cases} 1 , & \text{if } x > 0 \\ 0 & \text{if } x \leqslant 0 . \end{cases} \tag{2.4}$$

We define the following convex, closed subset of $H^1(D)$:

$$K = \{ v \in H^1(D) / v = f \text{ on } \Gamma \} \tag{2.5}$$

Let's consider the following variational formulation:

Variational Problem (V.P)

$$\begin{cases} \text{Find } p \in K, \\ \int_D k \nabla p . \nabla (v-p) \, dxdy + \int_D k H(p) \frac{\partial}{\partial y}(v-p) \, dxdy \geqslant 0, \ \forall v \in K. \end{cases} \tag{2.6}$$

THEOREM 2.1. If p is a solution of (V.P) then p is a weak solution of (P.P) and $p \in C^0(D)$.

Proof. By observing that $p^+ \in K$ and by making $v=p^+$ in (2.6) we obtain $\nabla p^- = 0$ and hence $p^- = 0$ a.e. in D, thus $p \geq 0$ a.e. in D.

If we choose $\varphi \in \mathcal{D}(D)$ we can obtain also that :

$$\operatorname{div}(k\nabla p) + [kH(p)]_y = 0 \text{ in } \mathcal{D}'(D), \tag{2.7}$$

and by regularity results [8] we deduce that $p \in C^0(D)$.

We define the following set:

$$\Omega = \{(x,y) \in D / p(x,y) > 0\}. \tag{2.8}$$

If $\varphi \in \mathcal{D}(\Omega)$, by taking $v = p \pm \varphi$ in (2.6) we obtain $(1.1)_1$ in the weak sense.

If $\varphi \in H^1(D)$, $\varphi = 0$ a.e. in $D - \Omega$, $\varphi = 0$ on $\partial\Omega - S_1$ by making $v = p \pm \varphi$ in (2.6) we get $(1.1)_3$ in $H^{1/2}(S_1)$.

Let the free boundary be defined by:

$$Y(x) = \sup\{y / (x,y) \in \Omega\} \cap D. \tag{2.9}$$

Considering $\varphi \in \mathcal{D}(D)$ and $v = p \pm \varphi$ in (2.6), $(1.1)_4$ results. Finally taking into account the definition (2.8) of Ω we conclude that $p=0$ on $y=Y(x)$.

3. EXISTENCE OF A SOLUTION

For proving the existence of a solution of the variational formulation we shall employ the same method as in [5]: we shall consider a sequence of variational inequalities with unique solutions, that will approximate the solution of (2.6).

Let $\varepsilon > 0$ and for each ε we define the function

$$H_\varepsilon(x) = \begin{cases} 1, & \text{if } x > \varepsilon \\ \dfrac{x}{\varepsilon}, & \text{if } 0 \leq x \leq \varepsilon, \\ 0, & \text{if } x < 0. \end{cases} \tag{3.1}$$

One can observe that H_ε is Lipschitz:

$$\|H_\varepsilon(u)-H_\varepsilon(v)\|_{L^2(D)} \leq \frac{1}{\varepsilon}\|u-v\|_{L^2(D)} \quad \text{for all } u,v\in L^2(D) \qquad (3.2)$$

We consider the following approximate problem:

Penalized Variational Problem $(P.V_\varepsilon)$

$$\left\{\begin{array}{l} \text{Find } p_\varepsilon\in K, \\ \int_D k\nabla p_\varepsilon . \nabla(v-p_\varepsilon)dxdy + \int_D kH_\varepsilon(p_\varepsilon)\frac{\partial}{\partial y}(v-p_\varepsilon)dxdy \geq 0 \quad \forall v\in K. \end{array}\right. \qquad (3.3)$$

THEOREM 3.1. There exists p_ε a solution of $(P.V_\varepsilon)$ and $p_\varepsilon \geq 0$ a.e. in D.

Proof. We define an operator $T:K\rightarrow K$ so that for all $q\in K$, $T(q)=p_\varepsilon$ where p_ε is the unique solution of the following variational inequality:

$$\left\{\begin{array}{l} \text{Find } p_\varepsilon\in K, \\ \int_D k\nabla p_\varepsilon . \nabla(v-p_\varepsilon)dxdy + \int_D kH_\varepsilon(q)\frac{\partial}{\partial y}(v-p_\varepsilon)dxdy \geq 0, \ \forall v\in K. \end{array}\right. \qquad (3.4)$$

The previous variational inequality has a unique solution, due to standard existence and uniqueness results for variational inequalities of the first type.

Let $q_1,q_2\in K$ and $p_\varepsilon^1=T(q_1)$, $p_\varepsilon^2=T(q_2)$. By adding the corresponding inequalities with $v=p_\varepsilon^2$ and $v=p_\varepsilon^1$ respectively and using (3.2), the Friedrichs' inequality and the definition of I, we get:

$$\|p_\varepsilon^1 - p_\varepsilon^2\|_{H^1(D)} \leq \frac{c}{\varepsilon}\|q_1-q_2\|_{L^2(D)} , \qquad (3.5)$$

and hence T is continuous in the weak topology of $H^1(D)$. K is a closed, convex set and hence compact in the weak topology. Thus by Schauder's theorem T has a fixed point which is a solution of (3.3).

For proving that $p_\varepsilon \geq 0$ a.e. we use the same method as in the first part of Theorem 2.1.

THEOREM 3.2. $(P.V_\varepsilon)$ has a unique solution.

Proof. Let us denote by:

$$K_0=\{v\in H^1(D)\,/\,v=0 \quad \text{on } \Gamma\}.$$

If $p_0\in K$ and $\bar{p}=p-p_0$ it results that $\bar{p}\in K_0$. From (3.3) we get:

$$\begin{cases} \bar{p}_\varepsilon \in K_0 \\ \int_D k\nabla\bar{p}_\varepsilon.\nabla v\,dxdy + \int_D k\; H_\varepsilon(\bar{p}_\varepsilon+p_0)\frac{\partial v}{\partial y}dxdy=\int_D k\nabla p_0.\nabla v\,dxdy, \forall v\in K_0 \end{cases} \tag{3.6}$$

Let p^1, p^2 be two solutions of (3.3) and $q=p^1-p^2=\bar{p}^1-\bar{p}^2$.

Hence:

$$\left|\int_D k\nabla q.\nabla v\,dxdy\right|\leqslant\frac{c}{\varepsilon}\int_D|q||\nabla v|\,dxdy \quad \forall v\in K_0. \tag{3.7}$$

If for $\delta>0$ we substitute v in (3.7) by $\frac{(q-\delta)^+}{q}\in K_0$, we have

$$\int_D\left|\nabla \ell n\left[1+\frac{(q-\delta)^+}{\delta}\right]\right|^2 dxdy\leqslant\frac{c^2}{\varepsilon^2}\cdot \text{mes}(D), \tag{3.8}$$

By Friedrichs' inequality it follows

$$\left\|\ell n\left[1+\frac{(q-\delta)^+}{\delta}\right]\right\|_{H^1(D)}\leqslant\frac{c}{\varepsilon^2}\;\text{mes}(D), \tag{3.9}$$

the constant being independent of δ. When δ tends to zero we obtain $q\leqslant 0$ a.e. in D, and hence $p_\varepsilon^1=p_\varepsilon^2$.

LEMMA 3.3. The sequence $\{p_\varepsilon\}_\varepsilon$ is bounded in $H^1(D)$.

Proof. From (3.3) we get:

$$\|p_\varepsilon\|^2_{H^1(D)}\leqslant\|k\|_{L^\infty(D)}\Big[\|p_\varepsilon\|_{H^1(D)}\|v\|_{H^1(D)}+\text{mes}(D)(\|v\|_{H^1(D)}+$$
$$+\|p_\varepsilon\|_{H^1(D)})\Big],\ \forall v\in K$$

and the conclusion is easely obtained.

THEOREM 3.4. There exists a solution of the variational problem (V.P).

Proof. The sequence $\{p_\varepsilon\}_\varepsilon$ being bounded, there exists a subsequence, denoted also by $\{p_\varepsilon\}_\varepsilon$ weakly convergent to $p\in K$.

The sequence $\{H_\varepsilon(p_\varepsilon)\}_\varepsilon$ is also bounded in $L^2(D)$ and therefore there exists $\tilde{H}\in L^2(D)$ so that:

$$H_\varepsilon(p_\varepsilon) \rightharpoonup \tilde{H} \text{ weakly in } L^2(D). \qquad (3.10)$$

and moreover $\tilde{H}=1$ when $p>0$.

We define:

$$\Omega_\varepsilon = \{(x,y)\in D / p_\varepsilon(x,y)<\varepsilon\}, \qquad (3.11)$$

which is an open set since we use a similar regularity result as in Theorem 2.1.

In order to pass to the inferior limit in (3.3) when ε tends to zero, we must compute the following term:

$$\int_{\Omega_\varepsilon} k\left(\frac{p_\varepsilon}{\varepsilon}-1\right)\frac{\partial p_\varepsilon}{\partial y}dxdy = \iint_{\Omega_\varepsilon}\left[\frac{1}{2\varepsilon}\frac{\partial}{\partial y}(kp_\varepsilon^2)-\frac{\partial}{\partial y}(kp_\varepsilon)-\frac{p_\varepsilon^2}{2\varepsilon}\frac{\partial k}{\partial y}+p_\varepsilon\frac{\partial k}{\partial y}\right]dxdy =$$

$$= \int_{\partial\Omega_\varepsilon} -\frac{k}{2}\varepsilon n_y ds + \int_{\Omega_\varepsilon}\frac{\partial k}{\partial y}\left(p_\varepsilon-\frac{p_\varepsilon^2}{2\varepsilon}\right)dxdy.$$

We obtain:

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} k\left(\frac{p_\varepsilon}{\varepsilon}-1\right)\frac{\partial p_\varepsilon}{\partial y}dxdy = 0.$$

Thus when ε tends to zero:

$$\int_D k\nabla p.\nabla(v-p)dxdy + \int_D k\tilde{H}\frac{\partial}{\partial y}(v-p)dxdy \geq 0 \quad \forall v\in K. \qquad (3.12)$$

By taking $v=p\pm\varphi$, with $\varphi\in\mathcal{D}(D)$ in (3.12) we obtain

$$\operatorname{div}(k\nabla p)+(k\tilde{H})_y = 0 \quad \text{in } \mathcal{D}'(D), \qquad (3.13)$$

and for $v=p\pm\min(p,\varepsilon\varphi)$, where $\varphi\in\mathcal{D}(D)$, $\varphi\geq 0$, $\varepsilon>0$:

$$\int_{\{p>\varepsilon\varphi\}} k\nabla p.\nabla\varphi dxdy + \frac{1}{\varepsilon}\int_{\{p\leq\varepsilon\varphi\}} k(\nabla p)^2 dxdy - \int_{\{p>\varepsilon\varphi\}}\frac{\partial k}{\partial y}\varphi dxdy -$$

$$-\frac{1}{\varepsilon}\int_{\{0<p\leq\varepsilon\varphi\}}\frac{\partial k}{\partial y}p dxdy = 0.$$

Since the second term is strictly positive and the last one tends to zero when ε tends to zero we get:

$$\int_D k\nabla p.\nabla\varphi dxdy \leq \int_D \frac{\partial k}{\partial y}\varphi dxdy, \quad \forall\varphi\in\mathcal{D}(D), \varphi\geq 0 \qquad (3.14)$$

From (3.13) and (3.14) we obtain

$$\frac{\partial}{\partial y}[k(\tilde{H}-1)] \leq 0 \text{ in } \mathcal{D}'(D). \qquad (3.15)$$

We have to prove next that H=0 when p=0.

LEMMA 3.5. Let p be a solution of (3.12). Then:

1. $p(x_o,y_o)>0$ implies that $p(x_o,y)>0$ for all $y \leq y_o$, $(x_o,y) \in D$,
2. $p(x_o,y_o)=0$ implies that $p(x_o,y)=0$ for all $y \geq y_o$, $(x_o,y) \in D$.

Proof. If $p(x_o,y_o)>0$, since from (3.13) p is continuous on D, there exists an open ball B, centered in (x_o,y_o) in which p is strictly positive and hence $\tilde{H}=1$ a.e.

From (3.15) $k(\tilde{H}-1)$ is a decreasing distribution of y. Because $k(\tilde{H}-1)$ is negative in D and zero in B it results that $k(\tilde{H}-1)$ is zero in the set $S=\{(x,y) \in D/(x,y_o) \in B,\ y \leq y_o\}$ and thus by (3.13):

$$\operatorname{div}(k\nabla p) = -\frac{\partial k}{\partial y} \leq 0 \text{ in } \mathcal{D}'(S). \qquad (3.16)$$

If we suppose that $p(x_o,y)=0$ for some $y<y_o$, $(x_o,y) \in D$, by the minimum principle ([8]) we obtain p=0 on S, but this contradicts the fact that $B \subset \{p>0\}$.

The second statement of the lemma is a simple consequence of the first one.

It can be proved as in [6] that the free boundary defined by (2.9) is lower semicontinuous and hence measurable.

LEMMA 3.6. If p is a solution of (3.12) then $\tilde{H}=\chi(\{p>0\})$.

Proof. Let $(x_o,y_o) \in D-\overline{\{p>0\}}$. This yields that there exists a square

$$S_\varepsilon = \{(x,y)/|x-x_o|<\varepsilon,\ |y-y_o|<\varepsilon\} \subset (D-\overline{\{p>0\}}),$$

in which p=0. By using the previous lemma we get that p=0 in

$$P_\varepsilon = \{(x,y)/\ |x-x_o|<\varepsilon,\ Y(x)<y<y_o+\varepsilon\} \subset (D-\overline{\{p>0\}}).$$

We denote by φ_δ a smooth function depending on x only which satisfies :

$$\varphi_\delta(x)=1 \quad \text{for } x\in[x_o-\varepsilon+\delta,\ x_o+\varepsilon-\delta]$$

$$\varphi_\delta(x)=0 \quad \text{for } x\notin[x_o-\varepsilon,\ x_o+\varepsilon],$$

$$0\leq \varphi_\delta(x)\leq 1,$$

and by:

$$R_\varepsilon = \{(x,y)\in D/\ |x-x_o|<\varepsilon,\ y<y_o+\varepsilon\}-P_\varepsilon .$$

Hence $p\pm\chi(P_\varepsilon\cup R_\varepsilon)\,\varphi_\delta(y_o+\varepsilon-y)$ is a test function for (V.P) and it follows:

$$0=\int_{R_\varepsilon} k\nabla p.\nabla[\varphi_\delta(y_o+\varepsilon-y)]+\int_{R_\varepsilon} k\,\frac{\partial}{\partial y}[\varphi_\delta(y_o+\varepsilon-y)]\ +\int_{P_\varepsilon} k\tilde{H}\frac{\partial}{\partial y}[\varphi_\delta(y_o+\varepsilon-y)]$$

The sum of the first two terms being zero we obtain

$$\int_{P_\varepsilon} k\tilde{H}\varphi_\delta=0$$

Making $\delta\rightarrow 0$ we get:

$$\int_{P_\varepsilon} k\tilde{H} = 0,$$

and therefore $\tilde{H}=0$ on P_ε and hence on S_ε.

We have proved that $\tilde{H}=0$ in a neighbourhood of every point of $D-\overline{\{p>0\}}$ and thus $\tilde{H}=0$ in $D-\overline{\{p>0\}}$ (e.g.[9]).

4. UNIQUENESS OF THE SOLUTION

For proving the uniqueness we shall use a result similar to lemma 5.1 of [6]:

LEMMA 4.1. If p_1 and p_2 are two solutions of (V.P) then

$$\int_D k[\nabla(p_i-p_o).\nabla\mathcal{J}+(H_i-H_o)\mathcal{J}_y]\,dxdy\leq\int_{D_i} k(x,Y_i(x))\,\mathcal{J}(x,Y_i(x))\,dx \quad (4.1)$$

for all $\mathcal{J}\in H^1(D)\cap C(\bar{D})$, $\mathcal{J}\geq 0$, and i=1,2, where

$p_o=\min(p_1,p_2)$, $H_o=\min(H_1,H_2)$, $H_1=H(p_1)$, $H_2=H(p_2)$,

$Y_o=\min(Y_1,Y_2)$, $A_o=\{p_o>0\}$, $D_i=\{x/Y_o(x)<Y_i(x)\}$

Proof. Let $\varepsilon>0$ and $\xi=\min(p_i-p_o,\varepsilon J)$, then $p\pm\xi$ is a test function for (V.P) and thus

$$\int_{\{p_i-p_o\leq\varepsilon J\}} k(\nabla(p_i-p_o))^2+\varepsilon\int_{\{p_i-p_o>\varepsilon J\}} k\nabla(p_i-p_o)\cdot\nabla J+$$

$$+\int_D k(H_i-H_o)[\min(p_i-p_o,\varepsilon J)]_y=0$$

The first integral is positive and by (2.3) it follows that

$$\int_{\{p_i-p_o>\varepsilon J\}} k\nabla(p_i-p_o).\nabla J+\int_D k(H_i-H_o)J_y\leq\int_{\{Y_o(x)\leq y\leq Y(x)\}}\frac{\partial}{\partial y}\left[k\left(J-\frac{p_i}{\varepsilon}\right)^+\right]$$

By making $\varepsilon\to0$ we obtain (4.1).

THEOREM 4.2. There exists only one solution p of (V.P) so that the boundary of each connected component of $\{p>0\}$ is in contact with at least a reservoir.

Proof. Let p_1 and p_2 be two solutions of (V.P) and A_o defined as in the previous lemma. We consider (x_o,y_o) belonging to a connected component A of A_o and (x_1,y_1) on the part of S_2 in contact with this component. There exists a polygonial line L that connects the two points considered. We denote by P an open set that contains the point (x_o,y_o) and satisfies $L\subset P\subset A$.

Next consider a function $\sigma\in H^1(P)\cap C(\bar{P})$ that is a solution of $\operatorname{div}(k\nabla\sigma)=0$ in P, $\sigma=1$ on $A\cap\partial P$, $0\leq\sigma\leq1$ on $\partial A\cap\partial P$, $\sigma\not\equiv1$ (4.2)
and extend it by 1 in D-P, the function obtained, denoted also by σ, belongs to $H^1(D)\cap C(\bar{D})$.

Integrating by parts for i=1,2 and taking into account that $g_i=g_o=1$ in P we get:

$$\int_{\partial(D\cap P)} k(p_i-p_o)\frac{\partial\sigma}{\partial n}=\int_D k[\nabla(p_i-p_o).\nabla\sigma+(g_i-g_o)\sigma_y]. \quad (4.3)$$

We denote by $(x_{f_i}, y_{f_i}) = \{(x,y)/y=Y_i(x)\} \cap \partial D$ for i=1,2, by $S \subset \partial D$ the set of points between (x_{f_1}, y_{f_1}) and (x_{f_2}, y_{f_2}) and by $S_\varepsilon = \{(x,y) \in D / d((x,y), (x_f, y_f)) < \varepsilon , (x_f, y_f) \in S\}$.

Let φ_ε be a smooth function, $0 \le \varphi_\varepsilon \le 1$, $\varphi_\varepsilon = 1$ in $A_o \cup S_\varepsilon$, $\varphi_\varepsilon = 0$ outside the ε-neighbourhood of $A \cup S_\varepsilon$.

From (3.13) written for p_i, i=1,2 we get:

$$\int_D k \nabla p_i \cdot \nabla[(1-\varphi_\varepsilon)\sigma] + (kH_i)[(1-\varphi_\varepsilon)\sigma]_y =$$

$$= \int_{\partial D} k\left(\frac{\partial p_i}{\partial n} + H_i n_y\right)(1-\varphi_\varepsilon)\sigma ds = 0.$$

Since in D either $p_o=0$ or $1-\varphi_\varepsilon=0$ it follows that

$$\int_D k\{\nabla p_o \cdot \nabla[(1-\varphi_\varepsilon)\sigma] + H_o[(1-\varphi_\varepsilon)\sigma]_y\} = 0 \qquad (4.5)$$

Subtracting (4.5) from (4.4) =>

$$\int_D k[\nabla(p_i-p_o)\cdot\nabla\sigma + (H_i-H_o)\sigma_y] = \int_D k[\nabla(p_i-p_o)\cdot\nabla(\varphi_\varepsilon\sigma) + (H_i-H_o)(\varphi_\varepsilon\sigma)_y]$$

and using Lemma 4.1 we obtain:

$$\int_D k[\nabla(p_i-p_o)\cdot\nabla\sigma + (H_i-H_o)\sigma_y] \le \int_{D_i} k(x,Y_i(x))\,\varphi_\varepsilon(x,Y_i(x))\cdot \sigma(x,Y_i(x))dx \qquad (4.6)$$

When $\varepsilon \to 0$ the right member of the above inequality vanishes and therefore by (4.3):

$$\int_{\partial(D\cap P)} k(p_i-p_o)\frac{\partial\sigma}{\partial n} ds \le 0 \quad \text{for } i=1,2 \qquad (4.7)$$

From the maximum principle $\frac{\partial\sigma}{\partial n} > 0$ on $D \cap \partial P$ and thus $p_i = p_o$ on $D \cap \partial P$ for i=1,2.

This yields that $p_1=p_2$ in P and hence in (x_o, y_o).

The equality of p_1 and p_2 in A_o and Lemma 3.5 imply that $\{p_1>0\}=\{p_2>0\}$ and therefore the uniqueness of the solution of (V.P) has been proved.

REMARK 4.3. If $k \in C^{\infty}(D)$ the proof of the uniqueness can use the same arguments as in Theorem 5.2[6].

REMARK 4.4. If we do not impose the condition that each connected component of the wet set is in contact with water reservoirs the (V.P) may have more solutions ([6], [10]).

5. THE CASE OF THE ANISOTROPIC MEDIUM

If the porous medium is also anisotropic, Darcy's law is written by means of a permeability tensor $k_{ij}(x_1,x_2)$ which is symmetric and positively defined.

Thus the physical problem is:

$$\begin{cases} \operatorname{div}(k\nabla p) = -\left(\dfrac{\partial k_{12}}{\partial x_1} + \dfrac{\partial k_{22}}{\partial x_2}\right), \; p>0 \quad \text{in } \Omega, \\ p = H_i - x_2 \text{ on } S_{2i}, \; p=0 \text{ on } S_3 \\ k_{ij}\dfrac{\partial p}{\partial x_j} n_i = -k_{i2} n_i \text{ on } S_1 \\ p=0, \; k_{ij}\dfrac{\partial p}{\partial x_j} n_i = -k_{i2} n_i \text{ on } x_2 = Y(x_1) \end{cases} \tag{5.1}$$

where we have used the summation convention for repeated indices and we have denoted x,y by x_1,x_2.

If (x_1,x_2) are the principal directions of k then $k_{12}=k_{21}=0$ and hence the variational formulation of the problem (4.1) is

$$\begin{cases} \text{Find } p \in K, \\ \displaystyle\int_D k_{ij}\frac{\partial p}{\partial x_j}\frac{\partial (v-p)}{\partial x_i} + \int_D H(p)k_{22}\frac{\partial (v-p)}{\partial x_2} \geq 0 \;\; \forall v \in K, \end{cases} \tag{5.2}$$

in which we suppose that $k_{ij} \in H^1(D) \cap L^{\infty}(D)$, for $i,j=1,2$.

THEOREM 5.1. There exists a unique solution $p \in H^1(D) \cap C(D)$ of (5.2).

The proof uses similar arguments with those of paragraphs 3 and 4 if we suppose that $\dfrac{\partial k_{22}}{\partial x_2} \geq 0$.

6. NON-HOMOGENEOUS DAM DISCONTINOUSLY STRATIFIED

In this paragraph we consider a dam formed by two layers D_1, D_2 with permeability coefficients $k_1=k_1(x,y)$, $k_2=k_2(x,y)$ respectively, where:

$$\begin{aligned} & k_1 \in H^1(D_1) \cap L^\infty(D_1), \quad k_2 \in H^1(D_2) \cap L^\infty(D_2), \\ & k_1, k_2 \geqslant \alpha > 0 \quad \text{a.e}, \quad \frac{\partial k_1}{\partial y} \geqslant 0, \quad \frac{\partial k_2}{\partial y} \geqslant 0 \end{aligned} \tag{6.1}$$

Therefore if we denote by:

$$k(x,y) = \begin{cases} k_1(x,y), & \text{if } (x,y) \in D_1 \\ k_2(x,y), & \text{if } (x,y) \in D_2 \end{cases} \tag{6.2}$$

we observe that the assumption of the first four paragraphs that $k \in H^1(D)$ is not fulfiled.

The variational formulation of the mechanical problem is given by (2.6) also.

Following the first part of the proof of Theorem 3.3 we obtain the existence of a solution of a weaker problem:

$$\begin{cases} \text{Find } p \in K, \ \tilde{H} \in L^\infty(D), \ 0 \leq \tilde{H} \leq 1, \\ \int_D k \nabla p . \nabla (v-p) + \int_D k \tilde{H} \frac{\partial}{\partial y}(v-p) \geqslant 0, \ \forall v \in K. \end{cases} \tag{6.3}$$

From (6.3) we get:

$$\operatorname{div}(k_1 \nabla p) + (k_1 \tilde{H})_y = 0 \quad \text{in } \mathcal{D}'(D_1) \tag{6.4}$$

$$\operatorname{div}(k_2 \nabla p) + (k_2 \tilde{H})_y = 0 \quad \text{in } \mathcal{D}'(D_2) \tag{6.5}$$

Using the same arguments as in Lemma 3.5 we obtain that $p(x_o,y_o)>0$, $(x_o,y_o) \in D_i$ implies $p(x_o,y)>0$ for all $y \leq y_o$, $(x,y) \in D_i$.

Thus the free boundary is a subgraph in each of the two layers D_1, D_2 and hence a free boundary configuration as the one obtained by Comincioli in [3] is not theoretically justified.

The results can be extended to the case of more layers.

ACKNOWLEDGEMENT. We would like to thank Dr.Horia Ene and Prof.Dr.V.Barbu for useful discutions and constant encouragement.

REFERENCES

1. BAIOCCHI C., Su un problema di frontiera libera connesso a questioni di idraulica, Ann.Mat.Pura Appl.92,107-127(1972)

2. BENCI V., On a filtration problem through a porous medium, Ann.Mat.Pura Appl. 100, 191-209 (1974).

3. BAIOCCHI C. & FRIEDMAN A., A filtration problem in a porous medium with variable permeability, Ann.Mat.Pura Appl.114, 377-393 (1977).

4. BAIOCCHI C., Problèmes à frontière libre en hydraulique: milieux non homogènes, Ann.Scuola Norm.Sup.Pisa Cl.Sci(4) 5, 429-453 (1978).

5. BREZIS H. & KINDERLEHRER D.& STAMPACCHIA G., Sur une nouvelle formulation du problème de l'écoulement à travers une digue, C.R.Acad.Sci.Paris Sér.A 287, 711-714(1978).

6. CARRILLO-MENENDEZ J.& CHIPOT M., On the dam problem,J.Differential Equations 45, 234-271(1982).

7. ALT H.W., Strömungen durch inhomogene poröse Medien mit freiem Rand, J.Reine Angew. Math.305, 89-115 (1979).

8. GILBARG D.& TRUDINGER N., Elliptic Partial Differential Equations of Second Order, Springer (1977).

9. VLADIMIROV V., Distributions en physique mathématique, MIR Moscou (1979).

10. BREZIS H., The dam problem revisited, Free boundary problems, Pitman (1983).