INSTITUTUL
DE
MATEMATICĂ

INSTITUTUL NAȚIONAL
PENTRU CREAȚIE
ȘTIINȚIFICĂ ȘI TEHNICĂ

ISSN 0250 3638

NUMERICAL APPROXIMATION OF AN INCOMPRESSIBLE BIPHASIC FLOW IN A POROUS MEDIUM

by

Mioara MELICESCU-RECEANU and Anca RADOSLOVESCU

PREPRINT SERIES IN MATHEMATICS

No.20/1984

BUCUREȘTI

NUMERICAL APPROXIMATION OF AN INCOMPRESSIBLE BIPHASIC FLOW IN A POROUS MEDIUM

by

Mioara MELICESCU-RECEANU*) and Anca RADOSLOVESCU**)

April 1984

*) *ICPITMUA, Bd. I.Ionescu de la Brad 6, 71592 Bucharest, Romania.*

**) *Computer Center CSP, Calea Victoriei 162, 71104 Bucharest, Romania.*

NUMERICAL APPROXIMATION OF AN INCOMPRESSIBLE BIPHASIC FLOW IN A POROUS MEDIUM

by

Mioara MELICESCU-RECEANU and Anca RADOSLOVESCU

Abstract. This paper presents the numerical approximation of an one-dimensional parabolic degenerated non-linear diffusion-transport equation, which models the non-stationary flow of twc imiscible, incompressible fluids through porous homogeneous media. In order to obtain the discrete solution of the problem, finite differences in time and finite elements in space are being used.

1. INTRODUCTION

In the general framework of the mechanics of oil recovery, the problem to dislocate oil by water from a deposit has known various approaches. One of these, due to G.Chavent, has been to consider the global pressure and the mobility of each fluid starting from the hypotheses of relative permeability and capillarity pressure, which leads to a parabolic diffusion-transport equation in order to describe the phenomenon [1].

Numerically, the problem was approached [2,3] by means

of discontinuous approximations for the saturation. The reason of such a choice was that for high water injection rate the parabolic diffusion-transport equation behaves like a first order hyperbolic equation, which can be solved following the ideas of Lesaint[4]. The way for solving the discrete problem obtained by the discontinuous finite elements, was to use a mixed formulation as in the paper of Raviart and Thomas[5].

In our paper we employ a numerical approximation method for first order parabolic degenerated non-linear diffusion - transport equation with unilateral boundary condition. In order to obtain the discrete solution of the problem we make use of finite differences in time and finite elements in space.

In the second section of the paper we give the mathematical model of the flow of two imiscible, incompressible fluids through porous homogeneous media.

The third section details the numerical approximation of the problem. First, we construct a semi-discrete Galerkin approximation of the problem by using the shape functions as a system of test functions. Then we discuss a number of problems connected with the numerical techniques employed, such as the type of finite elements and the numerical integration scheme. For time discretization we use semi-implicit scheme, which is described at the end of the section together with Richardson's scheme for automatic monitoring of the time step.

The last section is concerned with numerical results which are compared to experimental data available in the literature.

2. THE PHYSICAL PROBLEM

Let us consider a core sample of homogeneous porous material, with porosity ϕ, permeability K, length L and section σ, which we suppose to be initially filled with oil. We inject a Q(t) flow of water through its left-hand side and we observe the flow of oil and water recovered through the right-hand side of the sample. The lateral walls are supposed waterproof.

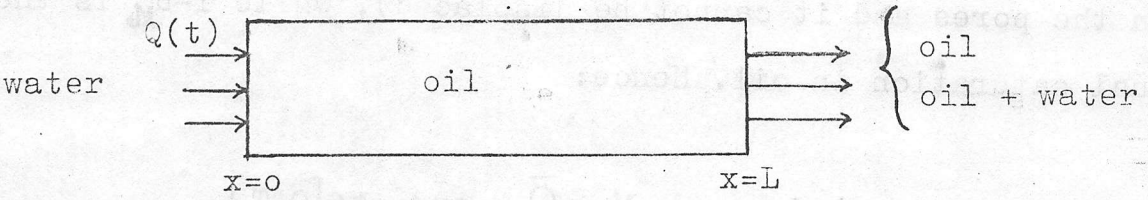

Fig. 1

Gravity effects are ignored. The axis of the sample is chosen as x-axis, while the variations of the saturation perpendicularly to this axis are neglected.

Denote by u(x,t) the water saturation at a point $x \in \overline{\Omega} = [0,L]$ at time $t \in [0,T]$, where T is the maximum time lapse during which the flow is being studied. Let $q(t) = Q(t)/(2\sigma)$ be the arithmetic mean of the seepage velocities of the two fluids.

The displacement of two imiscible fluids through the porous medium is governed by Darcy's law, which leads, by considering the hypotheses of relative permeabilities and cappillarity pressure and the continuity equation [1], to the following equation:

$$(1) \qquad \phi \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(K\, a(u) \frac{\partial u}{\partial x}) + q \frac{\partial b(u)}{\partial x} = 0 \quad \text{in } \Omega.$$

Here, the two real functions of saturation a and b are due to the relative mobility of each fluid and the capillarity

pressure, that are experimentally determined.

By definition the saturation satisfies the inequality:

$$0 \leq u(x,t) \leq 1 \quad , \quad \forall x \in \bar{\Omega} \quad \text{and} \quad t \in [0,T] .$$

Moreover, experimental data show that the saturation remains, as long as the displacement phenomenon is considered, between two values u_m and u_M, u_m being the residual saturation in water (for $u < u_m$ there is so little water that it is "trapped" within the pores and it cannot be displaced), while $1-u_M$ is the residual saturation in oil. Hence:

$$u_m \leq u(x,t) \leq u_M \quad , \quad \forall x \in \bar{\Omega} \quad \text{and} \quad t \in [0,T] .$$

At the right-hand side end of the sample the water does not appear ("well effect") as long as $u(L,t) < u_M$. The first time t^* for which $u(L,t^*)=u_M$ is interpreted as the water break-through time.

Let φ be the seepage velocity of water, which may be expressed [3] as:

$$\varphi(x,t) = (1+b(u))q - K\, a(u) \frac{\partial u}{\partial x} .$$

With these preliminary notations the initial and boundary conditions are:

$$(2) \qquad \begin{cases} u(x,o) = u_m & , \quad \forall x \in \Omega ; \\ u(o,t) = u_M & , \quad \forall t \in (o,T] ; \\ \varphi(L,t) = 0 & , \quad \forall t \in (o,t^*) ; \\ u(L,t) = u_M & , \quad \forall t \in [t^*,T] . \end{cases}$$

The mathematical investigation of the initial and boundary-value problem (1),(2) was made by Chavent [1], who has proved an existence theorem for solution of (1) in the degenerated case (a(0)=a(1)=0) with homogeneous Dirichlet boundary conditions and an uniqueness theorem for the one - dimensional, degenerated case. Later on he has given [6] an existence theorem for the solution of (1) for both the degenerated and non-degenerated case.

One of the approaches used in the literature to numerically solve this problem follows the ideas of Ravirt and Thomas [5] concerning first order parabolic degenerated non - linear equations, the problem being solved numerically [2,3] by means of a mixed formulation. Specifically, a new unknown function is introduced, while equation (1) becomes:

$$\begin{cases} r(u) = - K\, a(u) \dfrac{\partial u}{\partial x} , \\ \phi \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x} r(u) + q \dfrac{\partial b(u)}{\partial x} = 0 \end{cases}$$

Next the approximation is realized separatly for these two equations. Then, by using a Lesaint-Raviart scheme [4], the saturation u is approximated by discontinuous finite elements. This approach leads to a non-linear system both in the implicit diffusion case and in the explicit one. Obviously, to solve such a non-linear system, one must make use of an iterative method and this seems to be the main disadvantage of this approach. That is why we shall adopt in the following a different solving method which will be shown to have definite advantages with respect to this approach.

3. NUMERICAL ANALYSIS OF THE PARABOLIC EQUATION

3.1 Semi-discrete Galerkin approximation

In this section we will obtain an approximation of equation (1) by using finite elements in space and finite differences in time. First we construct a semi-discrete Galerkin approximation of (1) and then we make use of a semi-implicit scheme for the time discretization. This approach leads to a system of linear algebraic equations that allows to estimate the position of the saturation front and to determine the water break-through time.

The finite element discretization in terms of space elements of problem (1) suppose the division of the interval in M finite elements and to seek the saturation $u(x,t)$ in the form:

$$u(x,t) = \sum_{j=1}^{N} u_j(t)\, w_j(x) \quad , \tag{3}$$

where N is the number of global nodes in finite element mesh. The functions $w_1,\ldots,w_N$ form a system of N linearly independent shape functions, having the proprety:

$$w_i(x_j) = \delta_{ij} \quad , \quad i,j=1,\ldots,N \ ;$$

where $x_j \in \Omega$ is the abscissa of the global node j. We suppose $w_1,\ldots,w_N$ of class C^0 in $\overline{\Omega}$. The particular choice of these functions will be discussed in 3.2.

Since a continuous dependence on t is still assumed, $u(x,t)$ is referred to as a semi-discrete Galerkin approximation. Note that the coefficients u_j in (3) are functions of time t. They are the nodal values of the saturation.

We will use the Galerkin method [7], employing the shape functions $w_1,\ldots,w_N$ as a system of test functions. Thus, multiplying (1) by w_i, for $i=2,\ldots,P$, where $P=N$ if $t\in[0,t^*)$ and $P=N-1$ if $t\in[t^*,T]$, and integrating over Ω, we find:

$$\int_0^L \Phi\frac{\partial u}{\partial t} w_i \, dx - \int_0^L \frac{\partial}{\partial x}(K\cdot a(u)\frac{\partial u}{\partial x})w_i \, dx + \int_0^L q\,\frac{\partial}{\partial x}b(u)\cdot w_i \, dx = 0$$

from which, integrating by parts, we obtain:

$$\int_0^L \Phi\frac{\partial u}{\partial t}w_i\,dx + \int_0^L Ka(u)\frac{\partial u}{\partial x}\frac{\partial w_i}{\partial x}\,dx - \int_0^L q\; b(u)\frac{\partial w_i}{\partial x}\,dx +$$

(4)

$$+\left[(q\; b(u)-Ka(u)\frac{\partial u}{\partial x})w_i\right]_{x=L} = 0 \quad , \quad i=2,\ldots,P.$$

Now, introducing (3) into (4) and taking into account the initial condition $(2)_1$, we obtain:

$$(5)\qquad \begin{cases} \displaystyle\sum_{j=1}^{N} \dot{u}_j(t)\; m_{ij} + \sum_{j=1}^{N} u_j(t)\; s_{ij}(t) = c_i(t) \quad , \quad i=\overline{2,P} \; , \\ u_1(0)=u_M \quad , \quad u_j(0)=u_m \qquad \text{for } j=\overline{2,N} \; . \end{cases}$$

where $\dot{u}_j(t)$ denotes the time derivative of the nodal saturation u_j at time t and

$$(6)\qquad m_{ij}=\Phi\int_0^L w_i w_j\,dx \; ,$$

$$(7)\qquad s_{ij}(t)=\int_0^L K\; a(u)\;\frac{\partial w_i}{\partial x}\cdot\frac{\partial w_j}{\partial x}\,dx \; ,$$

$$(8)\qquad c_i(t)=\int_0^L q\; b(u)\;\frac{\partial w_i}{\partial x}\,dx-\left[(q\; b(u)-Ka(u)\sum_{j=1}^{N} u_j(t)\frac{\partial w_j}{\partial x})w_i\right]_{x=L}$$

It should be also noticed that a and b depend on x and t through

the saturation.

Assume now that the nodal saturation are known at time t and that we want to estimate their values at time $t+\Delta t$ when Δt is a small time step. We assume also that both t and $t+\Delta t$ belcng to the same interval $[0,t^*]$ or $[t^*,T]$ and that the functions a, b and q preserve their values at time t during the whole time interval $[t,t+\Delta t)$. Consequently, the functions $s_{ij}(t)$ and $c_i(t)$ have the same proprety.

For the determination of the nodal saturations at time $t+\Delta t$ from system (5), we may use various schemes of time integration. If we chose an explicit scheme:

$$u_j(t+\Delta t)=u_j(t)+\dot{u}_j(t)\,\Delta t$$

the time step required for the numerical stability (see [7]) would be very small. Therefore, we will use the semi-implicit scheme:

$$\dot{u}_j(t+\theta\Delta t) = \frac{1}{\Delta t}(u_j(t+\Delta t)-u_j(t)) \tag{8}$$

and we choose $\theta\in[\frac{1}{2},1]$ for assuring an unconditional numerical stability. As shown in [7], taking $\theta=\frac{1}{2}$, which correspons to the Crank-Nicolson scheme, leads to the highest accuracy of the solution, but can result in some spurious oscillatory effects. On the other side, taking $\theta=1$, which correspons to the fully implicit scheme, requires the use of smaller time steps and hence increases the computing time. Consequently, we will choose $\theta\in(\frac{1}{2},1)$.

Writing system (5) at time $t+\theta\Delta t$, taking into account the approximation (8) and using the linear approximation:

$$u_j(t+\theta\Delta t)=(1-\theta)u_j(t)+\theta u_j(t+\Delta t)\ ,$$

we obtain:

$$(9) \qquad \sum_{j=2}^{P} e_{ij}(t,\Delta t)\, u_j(t+\Delta t) = f_i(t,\Delta t)\ ,\ i=\overline{2,P}\ ,$$

where:

$$(10) \qquad e_{ij}(t,\Delta t)=\frac{m_{ij}}{\Delta t}+\theta s_{ij}(t)-K\theta\, a(u_N(t))\,\frac{\partial w_j}{\partial x}(L)\,\delta_{iN}\ ,$$

$$(11) \qquad f_i(t,\Delta t)=\sum_{i=2}^{P} u_j(t)\left[e_{ij}(t,\Delta t)-s_{ij}(t)\right]+c_i(t)-$$

$$-(s_{i1}+s_{iN}\,\delta_{P\,N-1})u_N\ ,$$

$$c_i(t)=\int_0^L q\, b(u)\,\frac{\partial w_i}{\partial x}\,dx+q\,\delta_{iN}\ .$$

Thus, at each time t we have to solve a system of linear algebraic equations (9). The initial conditions at time t=0 are:

$$\begin{cases} u_1(0)=u_M\ , \\ u_j(0)=u_m\ ,\ j=\overline{2,N}\ . \end{cases}$$

3.2 Details of the numerical techniques employed

We use Lagrangian three-noded finite elements. Denoting by e_j, $j=\overline{1,M}$, the finite elements into which the interval $[0,L]$ is divided and by k_i^j, $i=1,2,3$, the local nodes of the element e_j, we can write:

$$x_{2j-2+i} \equiv k_i^j\ ,\ i=1,2,3,\ j=\overline{1,M}\ ,$$

where x_k, $k=\overline{1,2M+1}$ are the global nodes of the finite element mesh.

Let $I = [-1,1]$ be the canonical interval. Then, the interval I_j, occupied by the element e_j, is represented by:

$$x = \frac{h}{2}\hat{x} + x_{2j} \ , \quad \hat{x} \in I \ , \quad x \in \bar{I}_j ,$$

where $h=L/(M-1)$.

We choose the local shape functions on the canonical interval I as Lagrangian interpolation polynomials of second order defined on I:

$$\varphi_1(\hat{x}) = \frac{1}{2}\hat{x}(\hat{x}-1) \ , \quad \varphi_2(\hat{x}) = 1-\hat{x}^2 \ , \quad \varphi_3(\hat{x}) = \frac{1}{2}\hat{x}(\hat{x}+1) \ .$$

Then, the local shape functions w_i^j associated with the local nodes k_i^j may be written:

$$w_i^j(x) = \varphi_i(\hat{x}) \quad \text{for} \quad x = \frac{h}{2}\hat{x} + x_{2j}, \ \hat{x} \in I \ , \ x \in \bar{I}_j \ ,$$

and the global shape functions w_k associated with the global nodes x_k are:

$$w_k(x) = \varphi_i\left(\frac{2}{h}x - \frac{2}{h}x_{2j}\right) \quad \text{for } i=1,2,3, \ j=\overline{1,M} \ , \ x \in \bar{I}_j \ ,$$

where $k=2j-2+1$. Consequently, we have:

$$\int_{x_{2j-1}}^{x_{2j+1}} w_{2j-2+i}(x)\,dx = \frac{h}{2}\int_{-1}^{1} \varphi_i(\hat{x})\,d\hat{x} \ ,$$

and

$$w'_{2j-2+i}(x) = \frac{2}{h}\varphi'_i\left(\frac{2}{h}x - \frac{2}{h}x_{2j}\right) \ , \ i=1,2,3, \ j=\overline{1,M} \ , \ x \in \bar{I}_j ,$$

from which we obtain:

$$\int_{x_{2j-1}}^{x_{2j+1}} w'_{2j-2+i}(\xi)\,d\xi = \int_{-1}^{1} \varphi'_i(\hat{\xi})\,d\hat{\xi}\ ,\quad i=1,2,3,\ j=\overline{1,M}\ ,$$

$$\int_{x_{2j-1}}^{x_{2j+1}} w'_{2j-2+i}(\xi)w'_{2j-2+k}(\xi)d\xi=\frac{2}{h}\int_{-1}^{1}\varphi'_i(\hat{\xi})\varphi'_k(\hat{\xi})d\hat{\xi},\quad \begin{matrix} i,k=1,2,3,\\ j=\overline{1,M}\ . \end{matrix}$$

The relations mentioned above will be employed for assembling the matices s_{ij}, m_{ij} and c_i.

We shall also use the Gauss-Legendre numerical integration formula for the canonical interval $I=[-1,1]$, namely:

$$\int_{-1}^{1} f(\xi)\,d\xi = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right)+8f(0)+5f\left(\sqrt{\frac{3}{5}}\right)\right]$$

where f is one of the functions φ_1, φ_2 or φ_3.

Appling the Gauss-Legendre scheme in (7) in order to calculate $s_{ij}(t)$, we need the nodal values of the function $a(u)$ at the integration points. Unfortunately, we know from the literature [3], only some tabulated values of the functions $\alpha(u)$ and $b(u)$, where $\alpha'(u)=a(u)$. For this reason we make use of the following formula of numerical derivation:

$$(13)\qquad a(u)=\frac{1}{(\Delta x)^{n-1}}\sum_{i=1}^{n}\frac{(-1)^{n-i}y_i}{(i-1)!(n-i)!}\sum_{\substack{j=1\\ j\neq i}}^{n}\prod_{\substack{k=1\\ k\neq i,j}}^{n}(u-u_k)$$

where $y_i=\alpha(u_i)$, $i=\overline{1,n}$; $\alpha(u_1),\ldots,\alpha(u_n)$ are the known discrete values of the function $\alpha(u)$ for some equally spaced values $u_1,\ldots,u_n$ of the saturation, while $\Delta x=u_i-u_{i-1}$ for $i=\overline{2,n}$.

The values of the functions b and a at the integration points are obtained by linear interpolation from their given values, respectively from their values computed by (13) for $u_1,\ldots,u_n$.

3.3 Richardson's extrapolation

The system of the linear algebraic equations (9) allows the determination of the nodal saturation at time $t+\Delta t$ from their values at time t. It is essential that at any given time the integration errors do not exceed a preset limit. The optimum choice of the time steps is an important problem: if they are too large the results become erroneous because of the integration errors and if the time step are too small, then the computing time becomes too large. For this reason an automated time step monitoring is highly desirable. The Richardson's scheme, which we will describe in the following, allows to automatically choose the optimum time step Δt in function of the admissible error per step (see e.g.[8] or [9]).

It is known that the semi-implicit time integration scheme (8) gives an error of order $(\Delta t)^2$ per step. Let $u(t+\Delta t)$ be the exact solution of system (5) at time $t+\Delta t$ and denote by $u^1(t+\Delta t)$ and $u^2(t+\Delta t)$ the estimations obtained by using (9) for $u(t+\Delta t)$ proceeding from t to $t+\Delta t$ in one step of magnitude Δt and, respectively, in two steps, each of magnitude $\Delta t/2$.

Let $(\Delta t)^2\Delta_i$ be the computation error of $u_i(t+\Delta t)$, $i=\overline{1,N}$, in passing from t to $t+\Delta t$ in one step. We assume that Δ_i varies slowly with time and is nearly independent of Δt when Δt is sufficiently small. Then we can write approximately:

$$(14) \qquad \begin{cases} u_i(t+\Delta t) = u_i^1(t+\Delta t) + (\Delta t)^2\Delta_i \\ u_i(t+\Delta t) = u_i^2(t+\Delta t) + 2\left(\frac{\Delta t}{2}\right)^2\Delta_i \end{cases},$$

from which it follows that:

$$(15) \qquad \Delta_i = \frac{2}{(\Delta t)^2}\left[u_i^2(t+\Delta t) - u_i^1(t+\Delta t)\right] \quad .$$

The advancement of the solution from t to $t+\Delta t$ is considered successful if the relative error, corresponding to the step Δt, does not exceed a prescribed limit, i.e.

$$(16) \qquad \|\Delta\|(\Delta t)^2 \leq \varepsilon \|u(t)\| \quad ,$$

where $\|\cdot\|$ is an adequate norm on R^N. In our computations we choose:

$$(17) \qquad \|\Delta\| = \left(\frac{1}{N}\sum_{i=1}^{N} \Delta_i^2\right)^{\frac{1}{2}} \quad ,$$

which corresponds to a global caracterization of the error.

The largest value $\overline{\Delta t}$ of t satisfying (16) is:

$$\overline{\Delta t} = \left(\frac{\varepsilon \|u(t)\|}{\|\Delta\|}\right)^{\frac{1}{2}} \quad .$$

If $\Delta t > \overline{\Delta t}$ (the advancement of the solution from t to $t+\Delta t$ does not satisfy the error requirement), then we repeat the estimations for u_i^1 and u_i^2 with $t = \overline{\Delta t}$. It may also be desirable to reduce the number of tentatives by shortening the new time interval, taking $t = \gamma\overline{\Delta t}$, with $\gamma \in [0.6, 1]$.

The process stops when $\Delta t \leq \overline{\Delta t}$ or when a prescribed number of tentatives is reached. Then, the Richardson's extrapolation for $u_i(t+\Delta t)$ can be update by (14) and (15). Thus:

$$u_i(t+\Delta t) = 2 \cdot u_i^2(t+\Delta t) - u_i^1(t+\Delta t) \quad .$$

Therefore, the time integration algorithm can be described as follows:

I. Choose $\varepsilon, \theta, \gamma, n_{max}, k_{max}, \overline{\Delta t}$.

Assemble the global matrix m_{ij} by using (6) and take t=0.

Initialize the nodal saturations:

$$u_1(t)=u_{\mathcal{M}}\ ,\quad u_i(t)=u_m\ ,\quad i=\overline{2,N-1}\ .$$

II. Compute $\|u(t)\|$ by (17).

III. Assemble the global matrix $s_{ij}(t)$, by using (7) and (13).

Take $\Delta t=\gamma\overline{\Delta t}$.

Compute $e_{ij}(t,\Delta t)$ and $f_i(t,\Delta t)$ by using (10) and (11).

Solve the system:

$$\sum_{j=2}^{P} e_{ij}(t,\Delta t\ )\ u_i^1(t+\Delta t) = f_i(t,\Delta t)\ ,\quad i=\overline{2,P}.$$

Compute $e_{ij}(t,\Delta t/2)$ and $f_i(t,\Delta t/2)$ also by (10) and (11), and solve the system:

$$\sum_{j=2}^{P} e_{ij}(t,\Delta t/2)\ u_i^1(t+\Delta t/2) = f_i(t,\Delta t/2)\ ,\quad i=\overline{2,P}\ .$$

Compute $s_{ij}(t+\Delta t/2)$, $e_{ij}(t+\Delta t/2,\Delta t/2)$ and $f_i(t+\Delta t/2,\Delta t/2)$ from $u^1(t+\Delta t/2)$, by using (7),(10) and (11). Solve the system:

$$\sum_{j=2}^{P} e_{ij}(t+\Delta t/2,\Delta t/2)\ u_i^2(t+\Delta t) = f_i(t+\Delta t/2,\Delta t/2),\quad i=\overline{2,P}\ .$$

Compute $\|\Delta\|$ by:

$$\|\Delta\| = \frac{2}{(\Delta t)^2}\left[\frac{1}{N}\sum_{i=1}^{N}(u_i^2(t+\Delta t)-u_i^1(t+\Delta t))^2\right]^{\frac{1}{2}}$$

and take $\overline{\Delta t} = \left[\varepsilon\|u(t)\|/\|\Delta\|\right]^{\frac{1}{2}}$.

IV. Repeat step III until $\Delta t \leq \overline{\Delta t}$ or the number of tentatives reaches k_{max}.

V. Take $u_i(t+\Delta t)=2u_i^2(t+\Delta t)-u_i^1(t+\Delta t)\ ,\ i=\overline{1,N}$, and $t=t+\Delta t$.

If $u_N(t)=u_{\mathcal{M}}$ then take $t^*=t$.

VI. Repeat steps II to V until $t \geq T$ or the number of time steps reaches n_{max}.

4. NUMERICAL RESULTS

The numerical method has been applied using the following values of the parameters: L=24 cm, T=2400 s, Φ=0.156 , K=1.3 10^{-8}, σ=18 cm^2, q=4.629 10^{-4}cm/s.

These data are the same as those used by Chavent and Cohen [2] and Forges [3] and correspond to the rather high water injection rate of 1 cm^3/min, and a stiff water saturation front is expected.

The minimum and the maximum values of the saturation are: u_m=0.15 and u_M=0.63.

We take into account for α and b the estimations given in [3] for 14 arbitrary values of saturation between 0.15 and 0.63. In order to obtain the values of a(u) we have applied the approximation (13).

Using approximations by polynomials of second order the discrete solution leaves the interval $[u_m, u_M]$, so that it was neccesary to extrapolate the functions a and b outside of this interval. We take:

$$b(u) = \begin{cases} -1 & \text{if } u \leq u_m \\ 1 & \text{if } u \geq u_M \end{cases}.$$

For $u \leq u_m$, a(u)=0 and for $u > u_M$, a(u) was linearly extrapolated.

Fig.2 shows succesive positions of the saturation front. The two curves at time t=1882 s and t=1880 s correspond respectively to the water break-through time and to the last curve plotted before that time.

The break-through time t =1882 s was obtained for

the following choice of the parameters: $\theta = \frac{2}{3}$, $\gamma = 0.8$, $\varepsilon = 10^{-2}$.

The use of Richardson's extrapolation assures to obtain the break-through time by a relatively small number of steps, the computing error per step being smaller than ε.

A numerical analysis of the results obtained shows also that estimation obtained for the break-through time is approximatively independent of the choice of θ and γ, unlike the estimation obtained by discontinuous finite elements [2], which depends on the time step. Table 1 gives the number of iterations (IT) and the estimated break-through time (BT), corresponding to various selections of ε, θ and γ.

ε	θ	γ	IT	BT
10^{-3}	0.6	0.8	1224	1880
10^{-2}	0.6	0.6	542	1880
		0.8	413	1882
		0.9	364	1882
		1	324	1882
10^{-2}	0.8	0.8	413	1881
	0.9	0.8	416	1881
10^{-1}	0.6	0.9	106	1891
		1	94	1944

Table 1.

It comes out that for $\varepsilon = 10^{-2}$ the minimum number of iterations, IT=324, is obtained for $\theta = 0.6$ and $\gamma = 1$. That seems to be the optimum selection for θ and γ as far as the computation time is concerned.

The numerical results are in good agreement with those obtained by Chavent and Cohen [2] and Forges [3]. These

authors have shown that by using a discontinuous finite element method one obtains good results for break-through time and front of saturation, but only for the transport equation. For the complete equation however they conclude that the implicit approximation for the diffusion is practically unapplicable because of the great computation time (18 min. for a time step 30 sec.) and an explicit approximation requires to choose a very small time step in order to assure the numerical stability.

The analysis of our results shows that the drawbacks of the discontinuous finite element or finite differences method in the complete equation (diffusion+transport) are removed by using the present method.

In conclusion it can be said that by using the semi - discrete Galerkin method together with the unilateral condition at the right-hand side of the sample, gives a good approximation of the saturation front and of the water break-through time.

Acknolegements. We are much indebted to Dr.H.I.ENE for suggesting this problem and for his interest and help in realize it.

We are also deeply grateful to Dr.E.SOOS and Dr.C. TEODOSIU whose suggestions were extremely helpful in preparing the final version of this paper.

REFERENCES

[1] G.CHAVENT A ormulation of diphasic incompressible flows in p edia, Lecture Notes in Mathematics, vol. 503, C er,(1976).

[2] G.CHAVENT and EN Numerical approximation and identifica n a 1-d parabolic degenerated non-linear diffusion a ansport equation, LABORIA Report No.270.

[3] F.FORGES Ap hation d'une équation diffusion-transport par des méth d'éléments finis espace-temps, Thèse, (1980).

[4] P.LESAINT and VIART On a finite element method for solving the on transport equation, Symposium on Mathematical cts of finite elements in P.D.E. Math. Research Cen University of Visconsin, Madison, (1974).

[5] P.A.RAVIART an THOMAS A mixed finite element method for second o elliptic problems.

[6] G.CHAVENT Un ème d'existence dans une inéquation variationnel rabolique dégénérée modélisant le déplacement fluide par un autre non miscible, LABORIA Repo .260.

[7] O.C.ZIENKIEWICZ e finite element method, McGraw-Hill Book Company, (19

[8] W.C.CARPENTER an A.T.GILL Automated solutions of time-depende oblems, The Mathematics of Finite Elements and fications, J.Whiteman, Ed.,Academic Press, (1973).

[9] C.TEODOSIU,E.SO ROSU A finite-element model of the heat flow in ets during the transfer and after upsetting in extrusion container, (to appear).

0.63
0.59
0.55
0.51
0.47
0.43
0.39
0.35
0.31
0.27
0.23
0.19
0.15
U
0
4
8
12
16
20
24
X
302
599
899
1200
1498
1800
1882

Fig. 2