INSTITUTUL
DE
MATEMATICĂ

INSTITUTUL NAȚIONAL
PENTRU CREAȚIE
ȘTIINȚIFICĂ ȘI TEHNICĂ

ISSN 0250 3638

PRINCIPAL MODULES OF LINEAR MAPS AND THEIR APPLICATIONS

by

Dan VUZA

PREPRINT SERIES IN MATHEMATICS

No. 21/1984

BUCUREȘTI

PRINCIPAL MODULES OF LINEAR MAPS AND THEIR APPLICATIONS

by

Dan VUZA*)

April 1984

*) The National Institute for Scientific and Tehnical Creation, Department of Mathematics, Bd. Pacii 220, 79622 Bucharest, Romania

PRINCIPAL MODULES OF LINEAR MAPS AND THEIR APPLICATIONS

Dan VUZA

The National Institute for Scientific and Technical Creation

Bucharest, Romania

The theory of principal modules was developed in our papers [9], [10]. An application of this theory was already given in our lecture delivered at the First Romanian-GDR Seminar on Banach Space Theory held at Bucharest in 1981 (see [11]). The purpose of this paper is to present some further applications. Our methods work also in the non-locally convex case; this possibility allows us to derive A.R.Schep's result on kernel operators as a consequence of principal modules theory.

1. Principal modules

In this section we recall some basic definitions and results from [9] and [10].

For a vector lattice E we shall use the standard notations:

$$E_+ = \{x \mid x \in E,\ x \geqslant 0\},$$
$$[x,y] = \{z \mid z \in E,\ x \leq z \leq y\},$$
$$E(x) = \{y \mid y \in E,\ (\exists)\ a,\ a \in R_+,\ |y| \leq a|x|\}.$$

A lattice-ordered algebra with unit is a vector lattice A with a strong unit e endowed with a bilinear multiplication $(a,b) \longmapsto ab$ such that $ae = ea = a$ and $|ab| = |a||b|$ for every $a,b \in A$.

Let A be a lattice-ordered algebra with unit. By an A-module we shall mean a vector lattice E which is an algebraic module over A such that $A_+E_+ \subset E_+$. For E Archimedian the definition implies that $|ax| = |a||x|$ for every $a \in A$, $x \in E$. By a submodule of E we shall mean an algebraic submodule. A set $M \subset E_+$ will be called A_+-stable if $A_+M \subset M$.

Let A,B be Archimedian lattice-ordered algebras with units e_1, e_2 and let $A \bar{\otimes} B$ be the tensor product in the sense of D.H.Fremlin ([4]). There is an unique structure of lattice-ordered algebra with unit on $A \bar{\otimes} B$ such that $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2$ for every $a_1, a_2 \in A$,

$b \in B$; its unit is $e_1 \otimes e_2$. Suppose that E is an Archimedian A-module and also a B-module. Then we can give an unique structure of $A \bar{\otimes} B$-module on E such that $(a \otimes b)x = a(bx)$ for every $a \in A$, $b \in B$, $x \in E$.

A principal A-module is an A-module E endowed with a locally solid topology such that $\{ax \mid a \in A\}$ is dense in the principal order ideal $E(x)$ generated by x for every $x \in E$.

Let E be an A-module and let F be a B-module. Suppose that E and F are principal, that F is order complete and its topology is order continuous. Denote by $L_r'(E,F)$ the vector lattice of all continuous order bounded linear maps from E to F. Define structures of A-module and B-module on $L_r'(E,F)$ by

$$(aU)(x) = U(ax),$$

$$(bU)(x) = bU(x).$$

Then $L_r'(E,F)$ becomes an $A \bar{\otimes} B$-module. The solid strong topology on $L_r'(E,F)$ has as a basis of neighborhoods of 0 the sets $\{U \mid U \in L_r'(E, F), |U|(x) \in V\}$ for every $x \in E_+$ and every neighborhood V of 0 in F.

Theorem 1.1 ([10]). With respect to the solid strong topology, $L_r'(E,F)$ is a principal $A \bar{\otimes} B$-module.

For the applications we shall need the following lemmas.

Lemma 1.1 Let E be a principal A-module and let F be a closed submodule. Then F is an order ideal.

Lemma 1.2 Let E be a principal A-module and let $M \subset E_+$ be A_+-stable and closed. If $x \in M$ then $[0,x] \subset M$.

Lemma 1.3 Suppose that E is an A-module and a B-module endowed with a locally solid topology and let F be a closed A-submodule and B-submodule. Then F is an $A \bar{\otimes} B$-submodule.

Lemma 1.4 Under the same hypothesis as in lemma 1.3 let $M \subset E_+$ be a closed A_+-stable and B_+-stable subset such that $M + M \subset M$. Then M is $(A \bar{\otimes} B)_+$-stable.

2. Kernel operators

Theorem 2.1 Let E,F be A-modules such that E is order complete, F is principal and its topology is separated and Fatou (see [5], ch. 2 for definition). Let $U : E \longrightarrow F$ be an order continuous positive A-linear map. Then $U([0,x]) = [0,U(x)]$ for every $x \in E_+$.

Proof. Let $\mathcal{R}$ be a family of Fatou pseudonorms defining the topology τ of F. We introduce the topology τ_1 on E by the pseudonorms $x \longmapsto \rho(U(|x|))$ for every $\rho \in \mathcal{R}$. τ_1 is Fatou and U is continuous for

τ_1 and τ. Let $x \in E_+$ and $y \in [0, U(x)]$. As F is principal there is a net $(a_\gamma) \subset A_+$ such that $a_\gamma \leq e$ and $a_\gamma U(x) \longrightarrow y$ for τ. The sequence $a_\gamma x$ is τ_1-Cauchy; indeed, as F is Archimedian we have

$$U(|a_\gamma x - a_{\gamma'} x|) \leq U(|a_\gamma - a_{\gamma'}| x) = |a_\gamma - a_{\gamma'}| U(x) = |a_\gamma U(x) - a_{\gamma'} U(x)|.$$

As $0 \leq a_\gamma x \leq x$ it follows by Nakano's theorem ([5], ch.2) that there is $x_1 \in [0, x]$ such that $a_\gamma x \longrightarrow x_1$ for τ_1. As U is continuous and τ is separated we have $y = U(x_1)$.

As an application we prove A.R.Schep's result from [7]. Let $X_i = (S_i, \sum_i, \mu_i)$, $i=1,2$ be σ-finite measure spaces and let $X_1 \times X_2$ be the product measure space. Let $L_0(X_i)$ be the vector lattice of equivalence classes of measurable functions on S_i and let E_i be an order ideal in $L_0(X_i)$. We denote by $I(E_1, E_2)$ the order ideal of those $f \in L_0(X_1 \times X_2)$ such that the class of the function $t \longmapsto \int |f(s, t) h(s)| d\mu_1(s)$ belongs to E_2 for every $h \in E_1$. Every $f \in I(E_1, E_2)$ defines a kernel operator $K(f) \in L_r(E_1, E_2)$ (the vector lattice of all order bounded linear maps from E_1 to E_2). In this way we have defined an order continuous positive linear map $K : I(E_1, E_2) \longrightarrow L_r(E_1, E_2)$.

Let $A(X_i)$ be the lattice-ordered algebra with unit of all simple functions in $L_0(X_i)$. Let A be the vector subspace of $L_0(X_1 \times X_2)$ generated by the characteristic functions of the sets $M_1 \times M_2$ with $M_i \in \sum_i$. We have $A(X_1) \otimes A(X_2) = A(X_1) \otimes A(X_2)$ and there is a canonical isomorphism $\alpha : A(X_1) \otimes A(X_2) \longrightarrow A$. By using the usual pointwise product we define structures of $A(X_i)$-module on E_i and of A-module on $I(E_1, E_2)$; by using the isomorphism α, $I(E_1, E_2)$ becomes also an $A(X_1) \otimes A(X_2)$-module. The construction described in section 1 shows that $L_r(E_1, E_2)$ is an $A(X_1) \otimes A(X_2)$-module; it is easily seen that K is $A(X_1) \otimes A(X_2)$-linear.

Define the order continuous topology on E_2 by the pseudonorms

$$h \longmapsto \int_M |h| \wedge \chi_{S_2} d\mu_2 , \quad M \in \sum{}_2 , \ \mu_2(M) < \infty ;$$

define a topology on E_1 by the pseudonorms

$$g \longmapsto \int_M U(|g|) \wedge \chi_{S_2} d\mu_2, \quad M \in \sum{}_2, \ \mu_2(M) < \infty , \ U \in L_r(E_1, E_2)_+ .$$

In this way E_1 and E_2 become principal modules and $L_r'(E_1, E_2) = L_r(E_1, E_2)$. Hence $L_r(E_1, E_2)$ is a principal $A(X_1) \otimes A(X_2)$-module in the strong solid topology. By taking in theorem 2.1 $E = I(E_1, E_2)$, $F = L_r(E_1, E_2)$, $U = K$ we obtain the following result which is the principal step in proving that the kernel operators form a band in $L_r(E_1, E_2)$.

Theorem 2.2 $K([0, f]) = [0, K(f)]$ for every $f \in I(E_1, E_2)$.

3. Principal modules of linear operators on Banach lattices

For an order complete vector lattice E let $\mathcal{A}(E)$ be the vector subspace of $L_r(E,E)$ generated by the projections on bands; $\mathcal{A}(E)$ is a lattice-ordered algebra with unit and the map $(A,x) \longmapsto A(x)$ defines a structure of $\mathcal{A}(E)$-module on E. This module is principal for every locally solid topology on E. If E,F are order complete vector lattices then $\mathcal{A}(E)\bar{\otimes}\mathcal{A}(F) = \mathcal{A}(E)\otimes\mathcal{A}(F)$.

Let E, F be Banach lattices such that F has order continuous norm. Let $\hat{E}$ be the order ideal generated by E in E". We define a structure of $\mathcal{A}(\hat{E})\otimes\mathcal{A}(F)$-module on $L_r(E,F)$ in the following way: if $a\in\mathcal{A}(\hat{E})\otimes\mathcal{A}(F)$ and $U\in L_r(E,F)$ let aU be the restriction to E of the map aU".

The proofs of the following theorems follow from theorem 1.1.

Theorem 3.1 If E,F are order complete Banach lattices and F has order continuous norm then $L_r(E,F)$ is an $\mathcal{A}(E)\otimes\mathcal{A}(F)$-principal module in the solid strong topology.

Theorem 3.2. If F has order continuous norm then $L_r(E,F)$ is an $\mathcal{A}(\hat{E})\otimes\mathcal{A}(F)$-principal module in the solid strong topology.

Let E,F be Banach lattices such that F is order complete. The solid uniform topology on $L_r(E,F)$ is given by the norm $U \longmapsto \||U|\|$. A subset M of E is called L-bounded if for every $\varepsilon>0$ there is $y\in E_+$ such that $\|(|x|-y)_+\|\leq\varepsilon$ for every $x\in M$.

Let E,F be Banach lattices. We denote by LW(E,F) the set of those linear maps $U : E \longrightarrow F$ such that the image of the unit ball in E by U is L-bounded; if F is order complete we denote by $LW_r(E,F)$ the order ideal of those $U\in L_r(E,F)$ such that $|U|\in LW(E,F)$. The solid L-bounded topology on $L_r(E,F)$ is given by the seminorms $U\mapsto \sup_{x\in K}\||U|(x)\|$ for every L-bounded set $K\subset E_+$.

Theorem 3.3 Suppose that E' and F have order continuous norms, that E is a principal A-module and that F is a principal B-module. Then $LW_r(E,F)$ is an $A\bar{\otimes}B$-principal module in the solid uniform topology.

Theorem 3.4 Suppose that F has order continuous norm, that E is a principal A-module and that F is a principal B-module. Then $L_r(E,F)$ is an $A\bar{\otimes}B$-principal module in the solid L-bounded topology.

Theorem 3.5 Suppose that E' and F have order continuous norms. Then $LW_r(E,F)$ is an $\mathcal{A}(\hat{E})\otimes\mathcal{A}(F)$-principal module in the solid uniform topology.

Theorem 3.6 Suppose that F has order continuous norm. Then $L_r(E,F)$ is an $\mathcal{A}(\hat{E})\otimes\mathcal{A}(F)$ - principal module in the solid L-bounded topology.

These theorems are derived from theorem 1.1, theorem 3.2 and the following lemmas:

Lemma 3.1 Suppose that E' and F have order continuous norms. Let $U \in LW_r(E,F)_+$. Then the solid strong topology and the solid uniform topology coincide on $[-U, U]$.

Lemma 3.2 Let E, F be Banach lattices such that F is order complete and let $U \in L_r(E,F)_+$. Then the solid strong topology and the solid L-bounded topology coincide on $[-U, U]$.

Theorem 3.7 Let E be a Banach lattice having order continuous norm, let $\mathcal{U}$ be an algebraic two-sided closed ideal in the algebra of all continuous linear maps from E to E and let $U \in \mathcal{U} \cap L_r(E,E)$, $V \in L_r(E,E)$ and $W \in LW(E,E)$ be such that $|V| \leq |U|$. Then $VW \in \mathcal{U}$.

Proof. By theorems 3.1 and 3.4 there is a net $a_\delta \in \mathcal{A}(E)\otimes\mathcal{A}(E)$ such that $a_\delta U \longrightarrow V$ for the solid L-bounded topology. As $\mathcal{U}$ is a two-sided ideal we have $(a_\delta U)W \in \mathcal{U}$. As $W \in LW(E,E)$ it follows that $(a_\delta U)W \longrightarrow VW$ in the uniform topology; hence $VW \in \mathcal{U}$.

Taking $\mathcal{U}$ to be ideal of compact operators and $V = W$ we find a result of C.D. Aliprantis and O.Burkinshaw ([1]).

4. The approximation of linear operators by finite-rank operators

Let E,F be Banach spaces. We denote by $\mathcal{F}(E,F)$ the set of all continuous finite-rank operators from E to F. A linear map $U : E \longrightarrow F$ is called approximable if it belongs to the closure $A(E,F)$ of $\mathcal{F}(E,F)$ in the uniform topology.

Theorem 4.1 Let E,F be Banach lattices such that E',F have order continuous norms. Then $LW_r(E,F) \cap A(E,F)$ is a band in $LW_r(E,F)$.

Proof. It is easily seen that $LW_r(E,F) \cap A(E,F)$ is an $\mathcal{A}(\hat{E})\otimes\mathcal{A}(F)$ -submodule of $LW_r(E,F)$. By theorem 3.5 and lemma 1.1 it is an order ideal; by lemma 3.1 is a band.

If E,F are normed vector lattices let P(E,F) be the set of all linear maps $U : E \longrightarrow F$ such that there are $f_1,\ldots,f_n \in E'_+$ and $y_1,\ldots$ $\ldots,y_n \in F_+$ such that $U(x) = \sum_{i=1}^{n} f_i(x)y_i$ for every $x \in E$. A linear map $U : E \longrightarrow F$ is called positively approximable if it belongs to the uniform closure of P(E,F).

Theorem 4.2 Let E,F be Banach lattices such that E', F have order continuous norms. Then the set of those $U \in LW_r(E,F)$ such that $|U|$ is positively approximable is a band in $LW_r(E,F)$.

Proof. Same proof as for theorem 4.1 using theorem 3.5, lemma 1.2 and lemma 3.1.

Let E,F be normed vector spaces. The compact topology on the space $L(E,F)$ of all linear continuous maps from E to F is given by the seminorms $U \longmapsto \sup_{x \in K} \|U(x)\|$ for every precompact set K. We denote by $CA(E,F)$ the closure of $\mathcal{F}(E,F)$ with respect to the compact topology. The example of A. Szankowski (see [6], ch. 1g) shows that there is a reflexive Banach lattice E such that $1_E \notin CA(E,E)$. Let E,F be normed vector lattices. We denote by $CA_+(E,F)$ the closure of $P(E,F)$ with respect to the compact topology.

Theorem 4.3 Let E,F be Banach lattices such that F has order continuous norm . The following are true:

i) $CA(E,F) \cap L_r(E,F)$ is a band in $L_r(E,F)$.

ii) $\{U \mid U \in L_r(E,F),\ |U| \in CA_+(E,F)\}$ is a band in $L_r(E,F)$.

Proof.

i) It suffices to prove that $CA(E,F) \cap L_r(E,F)$ is an order ideal because an application of lemma 3.2 will show that it is a band.

To this purpose it suffices to prove that $CA(E(x),F) \cap L_r(E(x),F)$ is an order ideal for every $x \in E_+$. This will follow from theorem 3.4 and lemma 1.1 if we construct a lattice-ordered algebra A with unit such that $E(x)$ is a principal A-module and $CA(E(x),F)$ is an $A \bar{\otimes} \mathcal{A}(F)$-submodule. As $E(x)$ is isomorphic to a space $C(X)$ we may take $A = C(X)$; as $CA(E(x),F)$ is an A-submodule and an $\mathcal{A}(F)$-submodule it is an $A \bar{\otimes} \mathcal{A}(F)$-submodule by lemma 1.3.

ii) Same proof as for i) using lemmas 1.2 and 1.4.

Let E,F be Banach lattices such that F is order complete. We denote by $SLA(E,F)$ (respectively by $SSA(E,F)$) the closure of $\mathcal{F}(E,F)$ in the solid L-bounded topology (respectively the solid strong topology). We denote by $SLA_+(E,F)$ (respectively by $SSA_+(E,F)$) the closure of $P(E,F)$ in the solid L-bounded topology (respectively the solid strong topology).

Lemma 4.1 Let E,F be Banach lattices such that F has order continuous norm. Let $U,V \in L_r(E,F)$ be such that $0 \le U \le V$ and $V \in SLA_+(E,F)$. Then $U \in SLA_+(E,F)$.

Proof. Apply theorem 3.6 and lemma 1.2.

We denote by $\mathcal{J}(E,F)$ the band generated by $\mathcal{F}(E,F)$ in $L_r(E,F)$.

Theorem 4.4 Let E,F be Banach lattices such that F has order continuous norm and let $U \in L_r(E,F)$. The following are equivalent:

i) $U \in \mathcal{J}(E,F)$.

ii) $U \in SLA(E,F)$.

iii) $U \in SSA(E,F)$.

iv) U_+, $U_- \in SLA_+(E,F)$.

v) U_+, $U_- \in SSA_+(E,F)$.

Proof.

ii) $\Rightarrow$ iv) It suffices to prove that U_+, $U_- \in SLA_+(E,F)$ for every $U \in \mathcal{F}(E,F)$. We have $U(x) = \sum_{i=1}^{n} f_i(x)y_i$ hence $|U| \leq V$ where $V(x) = \sum_{i=1}^{n} |f_i|(x)|y_i|$. By lemma 4.1 the result follows.

iii) $\Rightarrow$ i) Let P be the projection on the band orthogonal to $\mathcal{J}(E,F)$ and let $U \in SSA(E,F)$. From the relation $|P(U)| = P(|U-V|) \leq |U-V|$ which is true for every $V \in \mathcal{F}(E,F)$ it follows that $P(U) = 0$; hence $U \in \mathcal{J}(E,F)$.

i) $\Rightarrow$ ii) From lemma 4.1 and the equivalence ii) $\Leftrightarrow$ iv) it follows that $SLA(E,F)$ is an order ideal; by lemma 3.2 it is a band. As $\mathcal{F}(E,F) \subset SLA(E,F)$ the result follows.

Let E,F be Banach lattices such that F is order complete. We denote by $RA(E,F)$ the closure of $\mathcal{F}(E,F)$ in the solid uniform topology and by $RA_+(E,F)$ the closure of $P(E,F)$ in the solid uniform topology.

Lemma 4.2 Let E,F be Banach lattices such that E' and F have order continuous norms. Let $U,V \in L_r(E,F)$ be such that $0 \leq U \leq V$ and $V \in RA_+(E,F)$. Then $U \in RA_+(E,F)$.

Proof. Same proof as for lemma 4.2 using theorem 3.5 and lemma 1.2.

Theorem 4.5 Let E,F be Banach lattices such that E' and F have order continuous norms and let $U \in L_r(E,F)$. The following are equivalent:

i) $U \in \mathcal{J}(E,F)$ and $|U|$ is compact.

ii) $U \in RA(E,F)$.

iii) U_+, $U_- \in RA_+(E,F)$.

Proof.

ii) $\Rightarrow$ iii) Similar to the proof of ii) $\Rightarrow$ iv) in theorem 4.4, using lemma 4.3.

ii) $\Rightarrow$ i) Obviously $|U|$ is compact and $U \in SSA(E,F)$; by theorem 4.4, $U \in \mathcal{J}(E,F)$.

i) $\Rightarrow$ ii) Let $\mathcal{V} = \{V \mid V \in L_r(E,F),\ |V| \wedge |U| \in RA_+(E,F)\}$.

By lemma 4.2 $\mathcal{V}$ is an order ideal; by lemma 3.1 it is a band. From lemma 4.2 we have that $\mathcal{F}(E,F) \subset \mathcal{V}$; hence $\mathcal{J}(E,F) \subset \mathcal{V}$. In particular $|U| \in RA_+(E,F)$; as U_+, $U_- \leq |U|$ it follows from lemma 4.2 that $U \in RA(E,F)$.

Theorems 4.4 and 4.5 improve the following results of U.

Schlotterbeck (see [8]):

If F has order continuous norm and $U \in \mathcal{J}(E,F)$ then $U \in CA(E,F)$.

If E,F are reflexive, U is compact positive and $U \in \mathcal{J}(E,F)$ then $U \in RA_+(E,F)$.

5. M-tensor products

Let E,F be Banach lattices having order continuous norms. We denote by $L_m(E',F)$ the Banach lattice of those linear maps $U : E' \longrightarrow F$ such that the image of the unit ball by U is order bounded. The norm on $L_m(E',F)$ is given $\|U\|_m = \|\sup \{|U(f)| ; f \in E', \|f\| \le 1\}\|$. Let $\mathcal{F}_*(E',F)$ be the subspace of $\mathcal{F}(E',F)$ generated by the maps $f \longmapsto f(x)y$ with $x \in E$, $y \in F$ and let $M_*(E',F)$ be the closure of $\mathcal{F}_*(E',F)$ in $L_m(E',F)$ with respect to the norm $\| \ \|_m$. It is known that $M_*(E',F)$ is a sublattice of $L_m(E',F)$ isometric and order isomorphic to the M-tensor product $E \hat{\otimes}_M F$ ([2]).

Theorem 5.1. $M_*(E',F)$ is an order ideal in $L_r(E',F)$.

The proof goes along the same lines as for the previous results.

REFERENCES

1. C.D.ALIPRANTIS, O.BURKINSHAW, Positive compact operators on Banach lattices, Math.Z. 174 (1980), 289-298.
2. J.CHANEY, Banach lattices of compact maps, Math.Z.129 (1972), 1-19.
3. P.DODDS, D.H.FREMLIN, Compact operators in Banach lattices, Israel J.Math. 34, 4 (1979), 287-320.
4. D.H.FREMLIN, Tensor products of Archimedian vector lattices, American J.Math. 94 (1972), 777-798.
5. D.H.FREMLIN, Topological Riesz Spaces and Measure Theory, Cambridge, the University Press (1974).
6. J.LINDENSTRAUSS, L.TZAFRIRI, Classical Banach Spaces II, Springer (1979).
7. A.R.SCHEP, Kernel operators, Indag.Math. 41, 1 (1975), 39-53.
8. U.SCHLOTTERBECK, Tensorprodukte von Banachverbanden und positive Operatoren, Habilitation, Tübingen (1977).
9. D.VUZA, Strongly lattice-ordered modules over function algebras, Preprint Series Math. INCREST, Bucureşti, 62 (1980).
10. D.VUZA, Extension theorems for strongly lattice-ordered modules and applications to linear operators, Preprint Series Math. INCREST, Bucureşti, 100 (1981).

11. D.VUZA, The perfect M-tensor product of perfect Banach lattices, Proceedings of the First Romanian-GDR Seminar, Lecture Notes in Math. 991, Springer (1983).

11. D.VUZA. The perfect M-tensor product of perfect Banach lattices, Proceedings of the First Romanian-GDR Seminar, Lecture Notes in Math. 991, Springer (1983).