INSTITUTUL
DE
MATEMATICĂ

INSTITUTUL NAȚIONAL
PENTRU CREAȚIE
ȘTIINȚIFICĂ ȘI TEHNICĂ

ISSN 0250 3638

ON THE STRUCTURE OF NONCOMMUTATIVE COMPLETE LOCAL RINGS

by

Horia POP

PREPRINT SERIES IN MATHEMATICS

No. 24/1984

BUCUREȘTI

ON THE STRUCTURE OF NONCOMMUTATIVE COMPLETE LOCAL RINGS

by

Horia POP *)

April 1984

*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd. Pacii 220, 79622 Bucharest, Romania

ON THE STRUCTURE OF NONCOMMUTATIVE COMPLETE LOCAL RINGS

1. Introduction. For commutative rings the structure theorems of I.S.Cohen (see [5]) assert that a complete local ring is a homomorphic image of a ring of formal power series with coefficients in a field or in a p-adic (or truncated p-adic) ring. Some attempts to solve the noncommutative analogous problem were done in the case of algebras by Curtis [6], Hochschild [9] and [10] and more recently by McLean [12] for principal local rings which are finitely generated and separable over the center.

Bearing in mind the pattern of the ring of twisted formal power series in noncommutative variables, which appears in the problem of describing the structure of certain artinian rings with nonzero radical (see [15] and [16]), the purpose of this paper is to extend the structure theorems known in the commutative case to noncommutative complete local rings which are finitely generated and separable over the center. The example given by Vidal [17] shows that the demanded conditions are not only sufficient but, in a way, necessary (see also [15], Examples 1-3).

We assume throughout this paper that all rings have identity, all subrings contain the identity of the ring, all modules are unitary and all ring homomorphisms carry identity to identity. Unless specifically mentioned, all modules are considered as left modules. All algebras are finite dimensional.

2. Preliminaries. Let A be a ring. We shall denote by $J = J(A)$ its Jacobson radical and by $Z = Z(A)$ its center. In this paper A will always denote a ring which is finitely generated and separable over its center Z.

A is called a local ring if $\overline{A} = A/J$ is a division ring (which is equivalent to the fact that A has a unique left -or right- maximal ideal) and A is left noetherian.

We shall denote $m = Z \cap J$; then m is the unique maximal ideal of Z. By Eisenbud's theorem (see [8]), if A is left noetherian, then Z is noetherian, hence A is also right noetherian. It follows that $\bigcap_n m^n = 0$ and $\bigcap_n J^n = 0$ and, if A is complete in its J-adic topology, then Z is local and complete in its m-adic topology (see [6], Theorem 2).

We shall write $D = \overline{A} = A/J$ and $k = \overline{Z} = Z/m$.

The characteristic of A is 0 or p^r with p a prime number and $r \geqslant 1$. If A and $\overline{A}$ have the same characteristic (equal to 0 or p), then A is said to be an equal-characteristic ring; otherwise the characteristic of A is equal to 0 or p^r while the characteristic of $\overline{A}$ is p; in this case A is said to be an unequal-characteristic ring.

When k is a field of characteristic p, denote by W(k) the unique complete discrete valuation ring of residual field k (see [5], Lemma 13). Denote $W_r(k) = W(k)/p^{r+1}W(k)$. We shall denote eventually $W_\infty(k)$ for $W(k) = \text{proj lim } W_r(k)$.

If D is a separable division algebra over k, then there exists a unique complete discrete valuation ring W(D) of residual ring modulo its maximal ideal equal to D and such that the maximal ideal is generated by p (see [4] and [14]). We shall denote also $W_r(D) = W(D)/p^{r+1}W(D)$ and $W_\infty(D) = W(D) = \text{proj lim } W_r(D)$.

As a consequence of I.S.Cohen's Structure Theorems, Wedderburn Principal Theorem and Azumaya's Theorem (see [4]), we have the following:

<u>Theorem 1.</u> Let (A,J) be a complete local ring, finitely generated and separable over its center Z. Then:

1) in the equal characteristic case, Z contains a subring of representatives for k, isomorphic to k (this subring will be denoted again by k); A contains a k-subalgebra B isomorphic to D, such that $A = B + J$;

2) in the unequal characteristic case, let r be the exponent of p in char $A = p^r$, or else $r = \infty$ if char $A = 0$. Then Z contains a subring of representatives for k of the form $W_r(k)$, and A contains a $W_r(k)$-subalgebra $B = W_r(D)$ such that $A = B + J$.

Proof. See [15], theorems 1 and 2.

Definition 1. We say that the subring B of A, given by the above theorem, is a coefficient ring for A.

From now on, B will denote the coefficient ring of A, isomorphic to the division algebra D in the equal characteristic case, or to $W_r(D)$ in the unequal characteristic case.

Theorem 2. Let D be a separable division algebra over a field k and let B be as above. Then the number of classes of k-(respectively $W_r(k)$-)algebra homomorphisms $F : B \longrightarrow M_n(B)$, with respect to conjugation under an inner automorphism of $M_n(B)$, is finite.

Moreover, if the center Z(D) of D is a Galois extension of k, then any homomorphism F is conjugate to a diagonal mapping $\mathrm{diag}(s_1,\dots,s_n)$.

Proof. See [13], Theorems 2, 4, 5 and 6.

Lemma 1. Let $J = Ax_1 + \dots + Ax_n$, $x \in A$ and m a natural number. Then there exists a finite expansion such that

$$x - \sum_{i_1,\dots,i_k} b_{i_1\dots i_k} x_{i_1} \dots x_{i_k} \in J^{m+1} , \quad b_{i_1\dots i_k} \in B ,$$

where $0 \leq k \leq m$.

The proof is obvious.

From now on, we shall consider B-bimodules such that the actions of k (resp. of $W_r(k)$ in the unequal characteristic case) on the left and on the right coincide. We shall identify such B-bimodules with $B \otimes_k B^o$-(resp. $B \otimes_{W_r(k)} B^o$-)left modules.

Lemma 2. J is a B-bimodule.

The proof is immediate.

Lemma 3. If B = D, then there exists a B-bimodule T in J such that $J = T \oplus J^2$. Moreover, $J = T + T^2 + \dots + T^{m-1} + J^m$.

If $B = W_r(D)$, then there exists a B-bimodule T, contained in J, such that $J = T \oplus U$, where U is a B-bimodule contained in J^2 and $U/pU = J^2/pJ$. Moreover, $J = T + T^2 + \dots + T^{m-1} + J^m$.

In all cases, T is finitely generated as a left B-module.

Proof. See [15], Lemmas 3 and 4.

Lemma 4. Let M be a free left B-module of basis $\{x_1, \dots, x_n\}$ which is also a B-bimodule. Then the mapping $F: B \longrightarrow M_n(B)$ defined by $x_i b = \sum_{j=1}^{n} F_{ij}(b) x_j$ is a ring homomorphism.

The proof is obvious.

Definition 2. Given a ring homomorphism $F : B \longrightarrow M_n(B)$, we shall denote by $B[X_1, \dots, X_n, F]$ the ring of twisted noncommutative polynomials in $X_1, \dots, X_n$, that is, the ring of all formal finite expressions $\sum b_{i_1 \dots i_k} X_{i_1} \dots X_{i_k}$ with coefficients $b_{i_1 \dots i_k} \in B$ and noncommuting variables $X_1, \dots, X_n$ such that $X_i b = \sum_{j=1}^{n} F_{ij}(b) X_j$ for any $i = 1, \dots, n$.

If X is the column vector of components $X_1, \dots, X_n$ and if $F(b) = (F_{ij}(b))$, then one can write $Xb = F(b)X$.

In a similar manner, one defines the ring $B[[X_1, \dots, X_n, F]]$ of twisted noncommutative formal power series as the ring of formal (infinite) expressions $\sum b_{i_1 \ldots i_k} X_{i_1} \ldots X_{i_k}$ such that $Xb = F(b)X$.

In the particular case when F is a diagonal mapping, the ring of F-twisted polynomials (resp. power series) becomes the usual skew polynomial (resp. power series) ring.

The Example 4 in [15] shows that this notion is a true generalization, i.e. there exist twisted polynomial rings which are not skew polynomial rings.

Lemma 5. Let D be a separable division algebra over a field k and let $B = W_r(D)$ with $r \in N \cup \{\infty\}$. In the notations of Lemma 3, suppose that T can be decomposed by Köthe's Theorem as a direct sum of n cyclic left B-submodules. Then there exists a B-bimodule M, free with n generators as a left B-module, such that T is a homomorphic image of M as a B-bimodule.

The proof is given in [15], Proposition 2 for $n \in N$. For $n = \infty$ construct T as a projective limit.

3. Results. In order to extend the pattern obtained in the commutative case, once the existence of the coefficient field was established (Theorem 1), it remains to study the relations between the coefficient ring B and the radical J or, more explicitely, in what manner the generators of the left B-module T given by Lemma 3 commute with the coefficients from B.

The answer is given by the following :

Theorem 3. Let (A,J) be a complete local noncommutative ring such that A is finitely generated and separable over its center Z. Denote by B the coefficient ring of A. Then A is a homomorphic image of a F-twisted ring $B[[X_1, \ldots, X_n; F]]$ of formal power series in n noncommutative variables.

Proof. By Theorem 1 and Lemma 3, we have $A = B + T + T^2 + \ldots$
$\ldots + T^{m-1} + J^m$. But T is a free left B-module in the equal charac-

teristic case or is a homomorphic image of a free left D-module in the equal characteristic case (by Lemma 5), hence there exists (by Lemma 4) a twisting representation $F : B \longrightarrow M_n(B)$ such that $x_i b = \sum_{j=1}^{n} F_{ij}(b)x_j$, x_i being a system of generators of T over B. Using now the fact that A is complete, the theorem follows.

Theorem 4. Let (A,J) be a local ring such that A is finitely generated and separable over its center, and the center of $D = A/J$ is a Galois extension of $k = Z/m$. Let B be the coefficient ring of A. Then there exist some endomorphisms s_i of B such that A is a homomorphic image of a ring $B[[X_1,\ldots,X_n;s_1,\ldots,s_n]]$ of skew formal power series in noncommuting variables.

Proof. Use the same argument as in Theorems 3 and 2.

Theorem 5. There are only finitely many nonisomorphic twisted formal power series rings with a given ring of coefficients B and a given number of variables n.

Proof. This follows by Theorem 2, since there are only finitely many nonconjugate F's (keep in mind that if we change the basis $X_1,\ldots,X_n$ by $Y = CX$ with $C = (c_{ij})$, the corresponding F's are conjugate).

Department of Mathematics
National Institute for Scientific and Technical Creation
Bd.Păcii 220, 79622 Bucharest

REFERENCES

1. A.Albert, Structure of Algebras. Amer.Math.Soc.Colloq.Publ., XXIV, 1939.

2. E.Artin, C.Nesbitt, R.Thrall, Rings with Minimum Conditions. Univ.of Michigan, 1944.

3. M.Auslander, O.Goldman, The Brauer group of a commutative ring. Trans.Amer.Math.Soc. 97(1960), 367-409.

4. G.Azumaya, On maximally central algebras. Nagoya Math.J. 2(1951 119-150.

5. I.S.Cohen, On the structure and ideal theory of complete local rings. Trans.Amer.Math.Soc. 59(1946), 54-106.

6. Ch.Curtis, The structure of nonsemisimple algebras. Duke Math. J. 21(1954), 79-85.

7. F.DeMeyer, E.Ingraham, Separable Algebras over Commutative Rings Lecture Notes in Mathematics 181, Springer, Berlin, 1971.

8. D.Eisenbud, Subrings of artinian and noetherian rings. Math. Ann. 185(1970), 247-249.

9. G.Hochschild, On the structure of algebars with nonzero radical. Bull.Amer.Math.Soc. 53(1947), 369-377.

10. G.Hochschild, Note on maximal algebras. Proc.Amer.Math.Soc. 1 (1950), 11-14.

11. J.Lambeck, Lectures on Rings and Modules. Blaisdell Publ.Co., 1966.

12. K.R.McLean, Principal ideal rings and separability. Proc.London Math.Soc.(3), 45(1982), 300-318.

13. F.Pop, H.C.Pop, An extension of the Noether-Skolem theorem. Preprint INCREST 8/1983. (To appear in J.Pure Appl.Alg.)

14. H.C.Pop, Noncommutative p-adic rings and Witt vectors in connection with a theorem of Azumaya. Preprint INCREST 32/1983.

15. H.C.Pop, On the structure of artinian rings. Preprint INCREST 60/1983.

16. H.C.Pop, Ph.D.Thesis, Univ.of Bucharest, 1983.

17. R.Vidal, Contre-exemple noncommutatif dans la théorie des anneaux de Cohen. C.R.Acad.Sci.Paris, Série A, 284(1977) 791-794.