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INTRODUCTION

Let $X^r_d \subset \mathbb{P}^{r+s}_{\mathbb{C}}$ be a smooth connected, nondegenerate, complex projective variety of dimension r and degree d. Hartshorne (see [Ha$_1$]) and Barth and Van de Ven (see [Ba]) proved that if d is "small" with respect to r, X must be a complete intersection. The main result of this note gives a (sharp) bound such that if d is "small" with respect to s, both the abstract and the embedded structure of X could be determined. More precisely we shall prove the following:

Theorem I. Assume $X^r_d \subset \mathbb{P}^{r+s}_{\mathbb{C}}$ is as above, with $r \geq 2$; let H denote its general hyperplane section.

- If $d \leq 2s+1$, X is one of the following:

a) A (birationally) ruled surface ;

b) A scroll (see (0.1)) whose base is a curve or a (birationally) ruled surface;

c) A hyperquadric fibration (see (0.2));

d) A variety with $\omega_X^{-1} \simeq \mathcal{O}_X(H)^{r-1}$; these are completely classified (see [I$_s$], [F$_2$], [F$_3$]);

e) $\mathbb{P}^r$, $H \in |\mathcal{O}(1)|$, or Q^r (the hyperquadric), $H \in |\mathcal{O}(1)|$;

f) There is a reduction (X', H') (see (1.1)) such that $X' \simeq \mathbb{P}^3$, $H' \in |\mathcal{O}(3)|$, or $X' \simeq Q^3$, $H' \in |\mathcal{O}(2)|$, or $X' \simeq \mathbb{P}^4$, $H' \in |\mathcal{O}(2)|$, or X' is a $\mathbb{P}^2$ - bundle over a smooth curve such that $\mathcal{O}_{X'}(H')$ induces $\mathcal{O}(2)$ on each fibre;

- *If* $d=2s+2$ *and* X *is not as above, then either:*

g) X *is a K3 surface, or*

h) $r\geq 3$, X *satisfies* $\omega_X^{-1}\simeq\mathcal{O}_X(H)^{r-2}$.

Some additional information in case f) is contained in (1.6) a). The example of hypersurfaces of degree 5 shows that the bound in Theorem I is sharp.

The above result "explains" the possibility of listing varieties of small degree. In [Io] we have treated in great detail the cases when $d\leq 7$.

The proof of Theorem I requires the study of the adjunction mapping (see [S_1], [VdV], [Io]), whose roots are classical. The following result, which is of interest in itself, is an important step in the proof of Theorem I:

Theorem II. Let X *be a smooth, connected, complex projective three-fold and* H *a smooth very ample divisor on it. Then,* H *is a (birationally) ruled surface if and only if* X *is of one of the following types:*

a) *A scroll* (see (0.1)) *over a curve or over a* (birationally) *ruled surface;*

b) *A hyperquadric fibration* (see (0.2));

c) *A three-fold with* $\omega_X^{-1}\simeq\mathcal{O}_X(H)^2$; *these are classified in* [Is];

d) $\mathbb{P}^3$, $H\in|\mathcal{O}(1)|$ *or* Q^3, $H\in|\mathcal{O}(1)|$;

e) *There is a reduction* (X',H') (see (1.1)) *such that* $X'\simeq\mathbb{P}^3$, $H'\in|\mathcal{O}(3)|$ *or* $X'\simeq Q^3$, $H'\in|\mathcal{O}(2)|$, *or* X' *is a* $\mathbb{P}^2$*-bundle over a smooth curve such that* $\mathcal{O}_{X'}(H')$ *induces* $\mathcal{O}(2)$ *on each fibre.*

Theorem II can be viewed as a refinement, in case H is a very ample divisor, of a result proved differently by Sommese (see [S_3] Th. (2.4)) in case of ample divisors. See Remark (1.5) for a more precise discussion of the relation between Sommese's result and Theorem II above. Besides the use of the adjunction mapping, our proof of Theorem II depends on the results about three-folds whose canonical bundles are not numerically effective due to S.Mori (see [Mo]), and on a theorem due to L.Bădescu (see [B_1],[B_2], [B_3]) which describes all smooth projective three-folds supporting a geometrically ruled surface as an ample divisor.

I am indebted to L.Bădescu for some helpful conversation.

§0. In this section we fix our notation and terminology and recall some useful results.

The notation (X^r,H) always means that X is a smooth, connected, complex projective algebraic variety of dimension $r \geq 2$ and H is a smooth very ample divisor on it. We mention a few standard notation we shall employ:

- K_X or ω_X - canonical divisor or sheaf
- $q(X)=h^1(X,\mathcal{O}_X)$-irregularity
- $p_g(X)=h^0(X,\omega_X)$-geometric genus
- $g(C)$-genus of a curve C
- $p_m(X)=h^0(X,\omega_X^m)$, $m\geq 1$ - m-th plurigenus
- $\alpha_X:X\longrightarrow Alb(X)$-Albanese mapping
- Q^r-smooth r-dimensional hyperquadric.

We use [Ha_2] and [G-H] as general references.

(0.1) If (X^r,H) is as above we call such a pair a <u>scroll</u> if there is a morphism $f:X\longrightarrow Y$ onto a smooth t-dimensional variety Y, with $1\leq t\leq r-1$, such that all fibres of f are linear (r-t)-dimensional varieties via the restriction of $\mathcal{O}_X(H)$ (in [Io] these were called "linear fibrations", the term "scroll" being reserved for the case t=1).

(0.2) A pair (X^r,H) as above is called a <u>hyperquadric fibration</u> if there is a morphism $f:X\longrightarrow C$ onto a smooth curve such that the (closed) fibres of f are hyperquadrics via $\mathcal{O}_X(H)$. In this case the singular fibres of f are ordinary cones, see [Io] , Lemma (0.6).

The following theorem plays an important role in this paper:

(0.3) If (X^r,H) is as above, the linear system $|K+(r-1)H|$ is base-points free if and only if (X^r,H) is not one of the following: $\mathbb{P}^r$, $H\in|\mathcal{O}(1)|$, or Q^r, $H\in|\mathcal{O}(1)|$, or $\mathbb{P}^2$, $H\in|\mathcal{O}(2)|$, or a scroll over a curve.

For a proof in the case of surfaces (which is the main one), see [VdV] Th.II or [S_1] Prop.(1.5). The result follows by induction on r, starting with r=2, see [Io] Th.(1.4).

The map $\varphi=\varphi_{|K+(r-1)H|}$, when defined, is called the adjunction mapping.

The following result is not difficult (see for instance [Io] Prop.(1.11); see also [S_3] Th.(1.2) and Th.(1.3) for generalizations).

(0.4) Let (X^r,H) be as above. If the map $\varphi=\varphi_{|K+(r-1)H|}$ is defined, we have one of the following:

a) dim $\varphi(X)=0$, or equivalently, $\omega_X^{-1}\simeq\mathcal{O}_X(H)^{r-1}$;

b) dim $\varphi(X)=1$; in this case (X^r,H) is a hyperquadric fibration;

c) $r\geq 3$, dim $\varphi(X)=2$; in this case (X^r,H) is a scroll over a surface;

d) dim $\varphi(X)=r$.

We need the following elementary lemma (see for instance [S_3] (0.10) or [Io] (0.7)).

(0.5) If $f:X\to\mathbb{P}^n$ is a morphism and $D\subset X$ an ample effective divisor, we have $f(D)=f(X)$ or dim $f(D)=$dim D.

The following lemma was noticed by Sommese (see [S_1] (2.3.3)). Since the argument is simple, we include it for convenience.

(0.6) *Let* (X^2,H) *be as above. Assume the map* $\varphi=\varphi_{|K+H|}$ *is generically finite. Then, any one-dimensional connected component* C *of a (closed) fibre of* φ *satisfies:* $(H\cdot C)=1$, $(C^2)=(C\cdot K)=-1$.

Proof. We may replace φ by its Stein factorization. Let E be an irreducible one-dimensional component of a fibre of φ, given the reduced structure. We have:

$0=(H+K\cdot E)$, so $(K\cdot E)\leqslant -1$, and $(E^2)+(E\cdot K)\geqslant -2$ by genus formula. Since $(E^2)<0$ (see [Mu]) it follows $(E^2)=(E\cdot K)=-1$ and $(H\cdot E)=1$. Assume there would be another component E' (with reduced structure) meeting E. It follows: $0>(E+E')^2=-2+2(E\cdot E')\geqslant 0$ - a contradiction. Since φ factors through the contraction of E, it follows that the fibre is also reduced and we are done.

The following result is a version of the classical "sweeping-out" method. In case $r=3$ it appears in [Is] Prop.(5.1); see also $[S_2]$ Th.(1.2).

(0.7). *Let* $(X^r,H), r\geqslant 3$, *be as above and let* $f:X\longrightarrow Y$ *be a generically finite morphism. Let* $E\subset H$ *be a connected component of a fibre of the restriction* $f|_H$ *and assume that* $E\simeq \mathbb{P}^{r-2}$, $\mathcal{O}_E(E)\simeq\mathcal{O}(-1)$, $\mathcal{O}_X(H)\otimes\mathcal{O}_E\simeq\mathcal{O}(1)$. *Then, there is a divisor* $F\subset X$ *such that* $F\cap H=E$ *(so that* $F\simeq\mathbb{P}^{r-1}$, $\mathcal{O}_F(F)\simeq\mathcal{O}(-1)$, $\mathcal{O}_X(H)\otimes\mathcal{O}_F\simeq\mathcal{O}(1)$, $f(F)=$ *one point).*

We sketch the proof for reader's convenience.

Proof. The normal bundle of E in X is given by an extension:

$$0\to\mathcal{O}_E(-1)\to N_{E/X}\longrightarrow\mathcal{O}_E(1)\longrightarrow 0.$$

Since $H^1(\mathcal{O}_E(-1))=H^1(\mathcal{O}_E(1))=0$, we get $H^1(N_{E/X})=0$ and $h^0(N_{E/X})=r-1$. Therefore, the connected component of the Hilbert scheme of X containing E has dimension r-1. Let $F \subset X$ be the union of all deformations of E in X; we have $F \neq X$ since $\dim f(X)=r$. An easy dimension count shows that through a general point of F there passes a (r-2)- dimensional family of linear varieties of dimension r-2, contained in F. It follows easily that $F \simeq \mathbb{P}^{r-1}$ and the rest is clear.

We will need the following result of Bădescu, see $[B_1]$ Th.5, $[B_2]$ Th.1 and Th.3, and $[B_3]$:

(0.8). If X is a smooth complex projective three-fold supporting a geometrically ruled surface D as an ample divisor, then: either X is a $\mathbb{P}^2$-bundle over a smooth curve and $\mathcal{O}_X(D)$ induces $\mathcal{O}(1)$ on each fibre, or, in case $D \simeq \mathbb{P}^1 \times \mathbb{P}^1$, there are two further possibilities: $X \simeq \mathbb{P}^3$, $D \in |\mathcal{O}(2)|$ or $X \simeq Q^3$, $D \in |\mathcal{O}(1)|$.

§1. Let (X^r,H) be as in §0. An effective divisor $E \subset X$ is called exceptional if $E \simeq \mathbb{P}^{r-1}$, $\mathcal{O}_E(E) \simeq \mathcal{O}(-1)$ and $\mathcal{O}_X(H) \otimes \mathcal{O}_E \simeq \mathcal{O}(1)$. Note that the set of exceptional divisors contained in X is finite and, in case $r \geq 3$, any two exceptional divisors are disjoint.

(1.1) Following Sommese (see $[S_3]$) we shall call a pair (X',H') the reduction of (X^r,H), $r \geq 3$, if there is a morphism $\sigma: X \longrightarrow X'$ which is the contraction of all exceptional divisors $E_1,\dots,E_k$ contained in X and H' is the (unique) divisor on X' such that $H = \sigma^*(H') - E_1 - \dots - E_k$; in this case H' is ample on X', see for instance $[F_1]$, Lemma (5.7).

(1.2) Lemma. Let (X^3,H) be as in §0. Assume that the map $\varphi = \varphi_{|K+2H|}$ is generically finite. Then, the reduction (X',H') of (X,H) has the property that $\varphi' = \varphi_{|K'+2H'|}$ is a finite morphism (here K' stands for $K_{X'}$). Moreover, keeping the notations of

(1.1), <u>we have</u> $\varphi=\varphi'\circ\sigma$.

<u>Proof</u>. Assume there is some exceptional divisor contained in X (otherwise, let $(X',H')=(X,H)$). Let $\sigma : X\longrightarrow X'$ be the contraction of all such exceptional divisors $E_1,\dots,E_k$. It follows by adjunction formula that φ contracts to a point any exceptional divisor. Therefore, we get a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \varphi(X) \\ \sigma\downarrow & \nearrow_{\tilde\varphi} & \\ X' & & \end{array}$$

If H' is such that $H=\sigma^*(H')-E_1-\dots-E_k$, it follows easily that the map $\tilde\varphi$ identifies to $\varphi'=\varphi_{|K'+2H'|}$ (where K' denotes $K_{X'}$). Moreover, as we remarked in (1.1), H' is ample on X'. We only have to prove that φ' is finite. Assuming the contrary, there is some integral curve C on X' such that $(2H'+K'\cdot C)=0$. In particular $(K'\cdot C)<0$. Replacing φ' by its Stein factorization, we may apply [Mo] Cor.(3.6) and Th.(3.3) to deduce that there is some integral divisor E on X' contracted by φ'. Suppose that $\varphi'(E)$ is a curve. By [Mo] Th.(3.3) it follows that a fibre F of φ' contained in E would be isomorphic to $\mathbb{P}^1$, with normal bundle of degree -1. But the relation $(2H'+K'\cdot F)=0$ implies that the degree of the normal bundle of F must be even. This contradiction shows that E is contracted to a point. Let $\tilde E$ be the proper transform of E via σ. The restriction of φ to H, which is essentially the adjunction mapping of H, contracts $\tilde E\cap H$ to a point. Therefore, it follows by (0.5) and (0.6) that $\tilde E$ is an exceptional divisor. This is absurd. Thus φ' is finite and (1.2) is proved.

(1.3) <u>Remark</u>. The above lemma should be compared with $[S_3]$, Th.(1.4 Under hypotheses similar to that of (1.2), Sommese proved that <u>the restriction</u> of the adjoint system of X' to a generic member of itself is ample. Thanks to the powerful results due to Mori, we were able to deduce the ampleness of the adjoint system on X'. This will allow us to use Bădescu's theorem quoted in (0.8).

(1.4) Lemma. Let (X^3,H) be as in §0. Assume the map $\varphi=\varphi_{|K+2H|}$ is finite. If $D\in|K+2H|$, we have $q(D)=q(X)$. Moreover, if D is smooth, it follows:

a) $(\omega_D^2)=8\chi(\mathcal{O}_D)$ if $p_g(H)=0$

b) $p_g(D)=0$ if $p_2(H)=0$.

Proof. Since D is ample on X, we have $q(D)=q(X)$ by Lefschetz's theorem. Assume that D is smooth (hence irreducible) and $p_g(H)=0$. It follows by adjunction that we have:

(1) $|K_X+H|=\phi$.

By duality, $H^3(\mathcal{O}_X(-H))=H^0(\mathcal{O}_X(K+H))=0$. Together with Kodaira vanishing this gives:

(2) $\chi(\mathcal{O}_X(-H))=\chi(\mathcal{O}_X(K+H))=0$.

The exact sequence:

(3) $0\to\mathcal{O}_X(-H)\to\mathcal{O}_X\to\mathcal{O}_H\to 0$

and (2) imply:

(4) $\chi(\mathcal{O}_X)=\chi(\mathcal{O}_H)$.

Now, use the exact sequence:

$$0\to\mathcal{O}_X\to\mathcal{O}_X(D)\to\mathcal{O}_X(D)\otimes\mathcal{O}_D\to 0.$$

It follows:

(5) $\chi(\mathcal{O}_X(D)\otimes\mathcal{O}_D)=\chi(\mathcal{O}_X(D))-\chi(\mathcal{O}_X)$.

By Riemann-Roch theorem on D, we get:

$$(6)\ \chi(\mathcal{O}_X(D)\otimes\mathcal{O}_D)=\tfrac{1}{2}(\mathcal{O}_X(D)\otimes\mathcal{O}_D\cdot\mathcal{O}_X(D)\otimes\mathcal{O}_D\otimes\omega_D^{-1})+\chi(\mathcal{O}_D).$$

Next, the exact sequence:

$$0\to\mathcal{O}_X(K+H)\to\mathcal{O}_X(D)\to\mathcal{O}_X(H)\otimes\mathcal{O}_H\otimes\omega_H\to 0,$$

together with (2) imply:

$$(7)\ \chi(\mathcal{O}_X(D))=\chi(\mathcal{O}_X(H)\otimes\mathcal{O}_H\otimes\omega_H).$$

Riemann-Roch theorem on H gives:

$$(8)\ \chi(\mathcal{O}_X(H)\otimes\mathcal{O}_H\otimes\omega_H)=\tfrac{1}{2}(\mathcal{O}_X(H)\otimes\mathcal{O}_H\otimes\omega_H\cdot\mathcal{O}_X(H)\otimes\mathcal{O}_H)+\chi(\mathcal{O}_H).$$

Putting together (5), (7), (8) and (4) it follows:

$$(9)\ \chi(\mathcal{O}_X(D)\otimes\mathcal{O}_D)=\tfrac{1}{2}(\mathcal{O}_X(H)\otimes\mathcal{O}_H\otimes\omega_H\cdot\mathcal{O}_X(H)\otimes\mathcal{O}_H).$$

This equality cupled with (6) gives:

$$(10)\ (\mathcal{O}_X(H)\otimes\mathcal{O}_H\otimes\omega_H\cdot\mathcal{O}_X(H)\otimes\mathcal{O}_H)+(\mathcal{O}_X(D)\otimes\mathcal{O}_D\cdot\mathcal{O}_X(-D)\otimes\mathcal{O}_D\otimes\omega_D)=2\chi(\mathcal{O}_D).$$

Since we have by adjunction $\omega_H=\mathcal{O}_X(H+K)\otimes\mathcal{O}_H$ and $\omega_D=\mathcal{O}_X(2H+2K)\otimes\mathcal{O}_D$, the relation (10) turns out to be equivalent to $(\omega_D)^2=8\chi(\mathcal{O}_D)$, which is a).

Assume now that $p_2(H)=0$. It follows that $|2K_X+2H|=\emptyset$, so we get by duality:

$$(11)\ H^3(\mathcal{O}_X(-D)=H^0(\mathcal{O}_X(2K+2H))=0.$$

Moreover, we have: $0=p_g(H)=H^2(\mathcal{O}_H)$. This equality, the exact sequence (3) and Kodaira vanishing give:

(12) $H^2(\mathcal{O}_X)=0$.

Finally, we use the exact sequence:

$$0\to\mathcal{O}_X(-D)\longrightarrow\mathcal{O}_X\longrightarrow\mathcal{O}_D\longrightarrow 0;$$

it follows by (11) and (12) that we have: $0=H^2(\mathcal{O}_D)=p_g(D)$. This proves b) and we are done.

Proof of Theorem II

It is easy to see that, in each of the situations described in cases a) to e) of the theorem, H is a ruled surface. Conversely, according to (0.3) and (0.4), we only have to prove that if the map $\varphi=\varphi_{|K+2H|}$ is generically finite, the pair (X,H) must be as in case e). We divide the rest of the proof into five steps.

Step I. Assume that $\varphi=\varphi_{|K+2H|}$ is generically finite and replace (X,H) by its reduction (X',H'); it follows by (1.2) that the adjoint system $|K+2H|$ is ample. We want to prove that a general member $D\in|K+2H|$ is a geometrically ruled surface.

Step II. Since H is ruled, we have $p_g(H)=p_2(H)=0$ and it follows by (1.4) that $(\omega_D)^2=8(1-q)$, where we let $q:=q(D)=q(X)$. Therefore, to prove step I, it is enough to prove that D is birationally ruled. We treat separately the regular and the irregular case.

Step III. Assuming that q=0, we shall see that $p_2(D)=0$. Therefore D is rational by Castelnuovo's criterion. We use first the exact sequence:

$$0 \to \mathcal{O}_X(3K+2H) \to \mathcal{O}_X(4(K+H)) \to \omega_D^2 \to 0$$

Since $p_4(H)=0$, it follows $|4(K_X+H)|=\emptyset$, so, to prove that $p_2(D)=0$, it would be enough to know that $H^1(\mathcal{O}_X(3K+2H))=0$, or equivalently:

(13) $H^2(\mathcal{O}_X(-2K-2H))=0$.

Next, look at the exact sequence:

$$0 \to \mathcal{O}_X(-2K-3H) \to \mathcal{O}_X(-2K-2H) \to \omega_H^{-2} \to 0.$$

We have by duality $H^2(\omega_H^{-2}) = p_3(H)=0$, so, to prove (13) it would be sufficient to prove:

(14) $H^2(\mathcal{O}_X(-2K-3H))=0$.

We also have the exact sequence:

(15) $0 \to \mathcal{O}_X(K+H) \to \mathcal{O}_X(2K+3H) \to \mathcal{O}_X(H)\otimes\mathcal{O}_D\otimes\omega_D \to 0$.

By (0.3) applied to the pair $(D, \mathcal{O}_X(H)\otimes\mathcal{O}_D)$, we may assume that the linear system $|\mathcal{O}_X(H)\otimes\mathcal{O}_D\otimes\omega_D|$ has no base-points (otherwise D is rational!). Since $H^1(\mathcal{O}_X(K+H))=0$ by Kodaira vanishing and D is ample, the sequence (15) shows that $|2K+3H|$ has only finitely many base-points. We shall see below that $(2K+3H)^3>0$. Therefore, the relation (14) is a consequence of [R], Th.3. Suppose now that $(2K+3H)^3=0$. It follows that $|2K+3H|$ is base-points free. By (0.4) we may assume that $(\mathcal{O}_X(H)\otimes\mathcal{O}_D\otimes\omega_D)^2>0$ (otherwise D is rational!).

Therefore, $\dim \varphi_{|2K+3H|}(X)=2$. If F denotes a connected component of a general fibre of $\varphi_{|2K+3H|}$, we have:

$$\mathcal{O}_F \simeq \mathcal{O}_X(2K+3H)\otimes \mathcal{O}_F \simeq \omega_F^2 \otimes \mathcal{O}_X(3H)\otimes \mathcal{O}_F .$$

Therefore F is a rational curve, so $\deg \omega_F = -2$ and $4=3\cdot \deg \mathcal{O}_X(H)\otimes \mathcal{O}_F$, which is absurd. Thus D is rational if $q=0$.

<u>Step IV.</u> Assume $q>0$. We shall prove that D is ruled showing that the (non-empty) fibres of the Albanese mapping of D are rational curves. Consider the following standard commutative diagram:

$$\begin{array}{ccccc}
H & \xrightarrow{\ i\ } & X & \xleftarrow{\ j\ } & D \\
\downarrow{\scriptstyle \alpha_H} & & \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_D} \\
\operatorname{Im}\alpha_H & \xrightarrow[\sim]{\ u\ } & \operatorname{Im}\alpha_X & \xleftarrow[\sim]{\ v\ } & \operatorname{Im}\alpha_D \\
\cap & & \cap & & \cap \\
\operatorname{Alb}(H) & \xrightarrow[\sim]{\operatorname{Alb}(i)} & \operatorname{Alb}(X) & \xleftarrow[\sim]{\operatorname{Alb}(j)} & \operatorname{Alb}(D)
\end{array}$$

By Lefschetz's theorem Alb(i) and Alb(j) are isomorphisms. Since H is ruled, $\dim \operatorname{Im}\alpha_H=1$. By (0.5) u and v are also isomorphisms. As it is well-known, the fibres of α_H and α_D (hence also of α_X) are connected. Let F be a general (hence smooth, irreducible) closed fibre of α_X. Since H is ruled, $g(H \cap F)=0$. Then, by a classical result (see for instance [N]), F must be a rational surface satisfying:

(16) $(\omega_F)^2=9$ or 8.

By adjunction formula we get:

(17) $-2=(\mathcal{O}_X(H)\otimes \mathcal{O}_F \cdot \mathcal{O}_X(H)\otimes\mathcal{O}_F\otimes\omega_F)$.

Since α_D has connected fibres, it follows:

$$0\leq g(D\cap F)=\frac{1}{2}(\mathcal{O}_X(D)\otimes\mathcal{O}_F\cdot\mathcal{O}_X(D)\otimes\mathcal{O}_F\otimes\omega_F)+1=(\mathcal{O}_X(2H)\otimes\mathcal{O}_F\otimes\omega_F\cdot\mathcal{O}_X(H)\otimes\mathcal{O}_F\otimes\omega_F)+1$$

Together with (17) this gives:

(18) $0\leq g(D\cap F)=(\omega_F)^2-(\mathcal{O}_X(H)\otimes\mathcal{O}_F)^2-5.$

We want to prove that:

(19) $g(D\cap F)=0.$

Assume this would be false. It follows by (18) that $(\mathcal{O}_X(H)\otimes\mathcal{O}_F)^2<$ $<(\omega_F)^2-5$; this inequality cupled with (16) implies that either:

$(\omega_F)^2=9$, so $F\simeq\mathbb{P}^2$ and $\mathcal{O}_X(H)\otimes\mathcal{O}_F\simeq\mathcal{O}(1)$, or

$(\omega_F)^2=8$, $(\mathcal{O}_X(H)\otimes\mathcal{O}_F)^2=2$, so $F\simeq\mathbb{P}^1\times\mathbb{P}^1$ and $\mathcal{O}_X(H)\otimes\mathcal{O}_F\simeq\mathcal{O}(1,1)$.

In the first case we get $\mathcal{O}_X(K+2H)\otimes\mathcal{O}_F\simeq\mathcal{O}(-1)$ and in the second case $\mathcal{O}_X(K+2H)\otimes\mathcal{O}_F\simeq\mathcal{O}_F$. Both possibilities are absurd since $\mathcal{O}_X(K+2H)$ is ample. Therefore the relation (19) is true and D is ruled.

<u>Step V.</u> By the first step we have a geometrically ruled surface $D'\in|K'+2H'|$ which is an ample divisor on X'. We may apply (0.8) to deduce that (X',H') is as stated in case e). The proof of Theorem II is complete.

(1.5) <u>Remark.</u> Sommese (see [S_3], Th.(2.4)) proved a result related to Theorem II, assuming only the ampleness of H. However, his conclusion, at least in case of $\mathbb{P}^2$-bundles over a curve and hyperquadric fibrations, is (according to the point of view in [S_3]) only of birational type. Our interest in proving the above form of Theorem II was motivated by its role in the proof of Theorem I. On the other side, since Theorem (2.4) in [S_3] was not the main purpose of that paper and its proof is rather involved, we took the opportunity to give what we think is a natural argument, in the version formulated above.

Proof of Theorem I

It is known (see for instance [Bu]) that a (nondegenerate, smooth, connected) surface $X^2_d \subset \mathbb{P}^{s+2}_{\mathbb{C}}$ with $d \leq 2s+1$ is ruled and if $d=2s+2$ it is either ruled or K3. We offer below a simple argument for the sake of completeness. Let $C \subset \mathbb{P}^{s+1}$ be a (smooth, connected, nondegenerate) curve of degree d and genus g such that $d \leq 2s+2$. Combining Riemann-Roch and Clifford's theorem it follows that either $d \geq 2g$ or C is a canonical curve - see [G-H], p.253. Therefore, if $X^2_d \subset \mathbb{P}^{s+2}_{\mathbb{C}}$ is a surface as above with $d \leq 2s+1$ and H denotes its general hyperplane section, we have $d \geq 2g(H)$. Adjunction formula yields $(H \cdot K) < 0$. Thus $p_m(X)=0$ for any $m \geq 1$ and X is ruled by Enriques' criterion. If $d=2s+2$ and H is embedded as a canonical curve, it follows via Noether's theorem (see [G-H], p.253) that $q(X)=0$, $H^1(\mathcal{O}_X(H))=0$ and X is linearly normal. By adjunction $(H \cdot K)=0$ and $H^2(\mathcal{O}_X(H))=H^0(\mathcal{O}_X(K-H))=0$ since $(H \cdot K-H) < 0$. We get by Riemann-Roch theorem:

$$s+3=h^0(\mathcal{O}_X(H))=\chi(\mathcal{O}_X(H))=s+2+p_g(X), \text{ so } p_g(X)=1;$$

since $(H \cdot K)=0$, we must have $\omega_X \simeq \mathcal{O}_X$. Thus X is a K3 surface. Case h) of Theorem I follows by induction on r, using adjunction formula and Lefschetz's theorem. Using (0.3) and (0.4) we get the cases b) to e). To finish the proof, by (0.4), (0.5) and Theorem II, it is enough to check the following:

If (X^4,H) is a pair such that the reduction of H is one of those listed in case e) of Theorem II, the reduction of (X^4,H) is $X'\simeq\mathbb{P}^4$, $H'\in|\mathcal{O}(2)|$; moreover, there is no (smooth, projective) five-fold whose hyperplane section is X, with normal bundle $\mathcal{O}_X(H)$.

So, let (X^4,H) be such that H has a reduction which is one of those listed in Theorem II, case e). In particular the adjunction mapping (both for X and for H) is generically finite. By (0.7) and $[F_1]$, Lemma (5.7), it follows that $\mathbb{P}^3$ with normal bundle $\mathcal{O}(3)$, or Q^3 with normal bundle $\mathcal{O}(2)$, or a $\mathbb{P}^2$-bundle over a smooth curve with normal bundle inducing $\mathcal{O}(2)$ on each fibre, is an ample divisor on the reduction of X. Using Lefschetz's theorem one can prove (see for instance $[B_1]$ Th.1 and Th.4, $[B_2]$ Th.2 and p.20, or $[F_1]$) that this is possible only for Q^3, and the reduction (X',H') of (X,H) must be $X'\simeq\mathbb{P}^4$, $H'\in|\mathcal{O}(2)|$. It also follows that, for $r\geqslant 5$, there are no other possibilities and the proof of Theorem I is complete.

(1.6) Remarks. a) Concerning the pairs (X,H) with $d\leqslant 2s+1$ listed in Theorem I, case f) we remark that if $X'\simeq\mathbb{P}^3$ (respectively $X'\simeq Q^3$ or $X'\simeq\mathbb{P}^4$), X is obtained by blowing-up at most 6 (respectively 5) points and this cases really occur.

b) Buium (see [Bu]) investigated the abstract structure of surfaces $X^2_d\subset\mathbb{P}^{s+2}_{\mathbb{C}}$ with $d\leqslant 2s+5$.

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