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ALMOST INDUCTIVE LIMIT AUTOMORPHISMS AND EMBEDDINGS INTO AF - ALGEBRAS (PRELIMINARY VERSION)

by

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April 1984

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ALMOST INDUCTIVE LIMIT AUTOMORPHISMS AND EMBEDDINGS INTO AF - ALGEBRAS

by Dan Voiculescu

In this paper we obtain results concerning the following question: when is the crossed product of an AF-algebra A by an action of $\mathbb{Z}$, isomorphic to a C^*-subalgebra of an AF-algebra? This is a natural question in view of the work concerning the corresponding question for crossed products of commutative C^*-algebras ([12], [13], [16], [17]). Our approach uses the fact that such an embedding of the crossed product can be constructed in case the automorphism α generating the action of $\mathbb{Z}$ has an increasing sequence of almost invariant finite-dimensional C^*-subalgebras converging to A (such automorphisms will be called almost inductive limit). We prove the existence of an embedding of $A \times_\alpha \mathbb{Z}$ into an AF-algebra in case α^n is approximately inner for some $n \geq 1$ and $K_0(A) \otimes \mathbb{Q}$ is ultrasimplicial in the sense of [6]. It is not known whether the ultrasimpliciality assumption is actually a restriction ([8]), in any case this assumption is satisfied if $K_0(A)$ has finite rank ([8]). In particular if A is an UHF- algebra, then any crossed product $A \times_\alpha \mathbb{Z}$ can be embedded into an AF-algebra.

The paper has three sections.

§ 1 deals with preliminaries.

§ 2 considers almost inductive limit automorphisms. There is an obvious analogy with Halmos' quasitriangularity [7]. We show that almost inductive limit automorphisms are approximable by inductive limit automorphisms and use this to prove the embedding result for crossed products by almost inductive limit automorphisms.

§ 3 contains our results on the embeddability of $A \times_{\alpha} \mathbb{Z}$ when α is approximately inner and $K_0(A) \otimes \mathbb{Q}$ ultrasimplicial. An essential part in the proof is a lemma (Lemma 3.1) about periodic pseudoorbits of finite-dimensional C^*-algebras for the automorphism α.

§ 1

Throughout this paper A will denote an AF-algebra with unit. By a nest of finite-dimensional C^*-subalgebras of A, we shall mean an increasing sequence

$$\mathbb{C}1 = A_0 \subset A_1 \subset A_2 \subset \ldots$$

of finite-dimensional C^*-subalgebras such that $A = \bigcup_{n \geq 0} A_n$

All C^*-algebras considered will be separable and unital, and C^*-subalgebras will be always assumed to contain the unit of the bigger algebra. The set of all finite-dimensional C^*-subalgebras of A will be denote by $\mathcal{F}(A)$.

The $*$-homomorphisms will be assumed to be unit-preserving and for a $*$-homomorphism $\psi : B \longrightarrow C$ we shall denote by $\psi_* : K_0(B) \longrightarrow K_0(C)$ the corresponding homomorphism between the K_0-groups. For a unitary u in a C^*-algebra, $\mathrm{Ad}\, u$ will denote the corresponding inner automorphism.

If C_1, C_2 are C -subalgebras of C and $\varepsilon > 0$ we shall write

$$C_1 \overset{\varepsilon}{\subset} C_2$$

if

$$\sup_{\substack{x \in C_1 \\ \|x\| \leq 1}} \Big(\inf_{\substack{y \in C_2 \\ y \leq 1}} \|x-y\| \Big) \leq \varepsilon$$

and $d(C_1,C_2)$ is defined by the formula

$$d(C_1,C_2)=\inf\{\varepsilon>0 \mid C_1 \overset{\varepsilon}{\subset} C_2 \text{ and } C_2 \overset{\varepsilon}{\supset} C_1\}$$

We shall frequently use the following standard approximation result:

if C_1, C_2 are C^*-subalgebras of C, C_1 is finite-dimensional and $\varepsilon>0$ then there is $\delta>0$ depending only on and the dimension of C_1 such that

$$C_1 \overset{\delta}{\subset} C_2 \Rightarrow \exists\, u \in C \text{ unitary such that}$$

$$\operatorname{Ad} u\,(C_1) \subset C_2 \text{ and } \|u-1\|<\varepsilon$$

There is also a much stronger approximation result due to E.Christensen (Thm.6.4 in [1], for the history of such results see [11]) which we shall also need:

if C_1, C_2 are C^*-subalgebras of C, C_1 is finite-dimensional, $0 \le \gamma < 10^{-4}$ and $C_1 \overset{\gamma}{\subset} C_2$, then there is a unitary $u \in C$ such that $\operatorname{Ad} u(C_1) \subset C_2$ and $\|u-1\| < 64\gamma^{1/2}$.

Another fact we will need is that *if B is a finite-dimensional C^*-algebra and $\varphi_1, \varphi_2 : B \longrightarrow A$ are $*$-homomorphism (A is AF), then $\varphi_1 = \operatorname{Ad} u \circ \varphi_2$ if and only if we have $\varphi_{1*} = \varphi_{2*}$* (see Lemma 7.7 in [4]).

We also recall that *an automorphism α of A is called approximately inner if it is a point-norm limit of inner automorphisms $\operatorname{Ad} u$ and (A being AF) this is equivalent to the requirement $\alpha_* = \mathrm{id}_*$* (see [5]).

§ 2.

2.1. *Definition.* *An automorphism α of A is called an almost inductive limit automorphism if there exists a nest of finite-dimensional C^*-subalgebras $(A_n)_{n\geq 0}$ of A, such* that

$$\lim_{n\to\infty} d(\alpha(A_n),A_n)=0$$

2.2.Definition.An automorphism α of A is called an inductive limit automorphism if there exists a nest of finite-dimensional C^*-subalgebras $(A_n)_{n\geqslant 0}$ of A,such that

$$\alpha(A_n)=A_n \ .$$

It is easily seen that the following is an alternative definition of almost-inductive-limit automorphisms:

α is an almost-inductive-limit automorphism if for every $B \in \mathcal{F}(A)$ and $\varepsilon>0$,we can find $C \in \mathcal{F}(A)$ such that $B \overset{\varepsilon}{\subset} C$ and $d(\alpha(C),C)<\varepsilon$

It is also easy to see that if $(A_n)_{n\geqslant 0}$ is a nest of finite-dimensional C^*-subalgebras then $d(\mathrm{Ad}\,u(A_n),A_n)\longrightarrow 0$ and hence:

α is an almost-indctive-limit automorphism if and only if $\mathrm{Ad}\,u\circ\alpha$ is an almost-inductive limit automorphism.

2.3.Proposition.If α is an almost inductive limit automorphism of A and $\varepsilon>0$,then there is a unitary $u\in A$ such that $\mathrm{Ad}\,u\circ\alpha$ is an inductive limit automorphism and $\|u-1\|<\varepsilon$.

Proof.Let $(B_n)_{n\geqslant 0}$ be a fixed nest of finite-dimensional C^*-subalgebras of A.We shall construct recurrently C^*-subalgebras A_n,automorphisms α_n and unitaries u_n for $n\geqslant 0$,so that:

$A_0=\mathbb{C}1, u_0=1, \alpha_0=\alpha$

$A_n\subset A_{n+1}, \mathrm{Ad}\,u_{n+1}\circ\alpha_n=\alpha_{n+1}, \|u_{n+1}-1\|<\varepsilon\cdot 2^{-n-1}$

$B_j \overset{(n+1)^{-1}}{\subset} A_{n+1}$ and $\alpha_{n+1}(A_j)=A_j$ for $0\leq j\leq n+1$.

Clearly, having constructed A_n, u_n, α_n the proof of the theorem will be concluded since $u = \lim_{n\to\infty} u_n \ldots u_o$ is then well defined, $\| u-1 \| < \varepsilon$ and the nest $(A_n)_{n \geq 0}$ is invariant for $\mathrm{Ad}\, u \circ \alpha$.

Thus assume we have found A_j, α_j, u_j with the desired properties for $0 \leq j \leq n$ and let us show that we can find $A_{n+1}, \alpha_{n+1}, u_{n+1}$.

$$\text{Let } \gamma = \left(\frac{\varepsilon}{(1000\,(n+1))^{n+2}} \right)^{2^{n+1}}$$

and assume $\varepsilon < 10^{-4}$ which is no loss of generality. Since α_n is an almost inductive limit automorphism, we can find $A_{n+1} \in \mathcal{F}(A)$ such that $A_{n+1} \supset A_n, A_{n+1} \overset{(n+1)^{-1}}{\supset} B_j$ for $0 \leq j \leq n+1$ and $d(\alpha_n(A_{n+1}), A_{n+1}) < \gamma_0$ .By Christensen's theorem there is a unitary $v_o \in A$ so that

$$(\mathrm{Ad}\, v_o \circ \alpha_n)(A_{n+1}) = A_{n+1} \text{ and}$$

$$\| v_o - 1 \| < \gamma_1 = 2(n+1)10^2\, \gamma_0^{\,1/2} .$$

Putting $\gamma_j = (2(n+1)10^2)^j\, \gamma_0^{\,1/2^j}$, we shall find recurrently unitaries $v_j \in A_{n+2-j}$ $(1 \leq j \leq n)$ such that

$$(\mathrm{Ad}\, v_j \circ \ldots \circ \mathrm{Ad} v_o \circ \alpha_n)(A_{n+1-j}) = A_{n+1-j} (1 \leq j \leq n)$$

and $\| v_j - 1 \| < \gamma_{j+1}$.

Indeed, assume we have found v_j for $j \leq k$, then

$$d((\mathrm{Ad}\, v_k \circ \ldots \circ \mathrm{Ad}\, v_o \circ \alpha_n)(A_{n-k}), A_{n-k}) \leq$$

$$\leq 2\, \| v_k \cdots v_o - 1 \| \leq 2\, (\gamma_1 + \ldots + \gamma_{k+1}) \leq$$

$$\leq 2(n+1)\, \gamma_{k+1}$$

$(\mathrm{Ad}\,v_k \circ \ldots \circ \mathrm{Ad}\,v_0 \circ \alpha_n)(A_{n-k}) \subset \ldots$

$\subset (\mathrm{Ad}\,v_k \circ \ldots \circ \mathrm{Ad}\,v_0 \circ \alpha_n)(A_{n+1-k}) = A_{n+1-k}$

and hence by Christensen's theorem there is $v_{k+1} \in A_{n+1-k}$ such that

$(\mathrm{Ad}\,v_{k+1} \circ \ldots \circ \mathrm{Ad}\,v_0 \circ \alpha_n)(A_{n-k}) = A_{n-k}$

and

$\| v_{k+1} - 1 \| < 10^2 (2(n+1) \gamma_{k+1})^{1/2} \leq \gamma_{k+2}$

which concludes the proof of the existence of the v_j's $(1 \leq j \leq n)$. Defining $u_{n+1} = v_n \ldots v_0$, we have

$\| u_{n+1} - 1 \| \leq \gamma_1 + \ldots + \gamma_{n+1} \leq (n+1) \gamma_{n+1} < \varepsilon . 2^{-n}$

and for $0 \leq j \leq n$

$(\mathrm{Ad}\,u_n \circ \alpha_n)(A_{n+1-j}) = (\mathrm{Ad}\,v_n \circ \ldots \circ \mathrm{Ad}\,v_{j+1})(A_{n+1-j}) = A_{n+1-j}.$

Q.E.D.

2.4. Definition. An automorphism α of A is called limit periodic if there is a nest $(A_n)_{n \geq 0}$ of finite-dimensional C^*-subalgebras and a sequence of positive integers $(d_n)_{n \geq 0}$, $d_0 = 1$ such that $\alpha(A_n) = A_n$ $(n \geq 0)$ and $(\alpha | A_n)^{d_n} = \mathrm{id}\, A_n$ $(n \geq 0)$. A sequence $(d_n)_{n \geq 0}$ with the above properties is called a sequence of periods for α.

2.5. Proposition. If α is an inductive limit automorphism of A with respect to a given nest $(A_n)_{n \geq 0}$ of finite dimensional C^*-subalgebras and $\varepsilon > 0$, then there is a unitary u such that $\mathrm{Ad}\,u \circ \alpha$ is limit periodic with respect to the nest $(A_n)_{n \geq 0}$ and $\| u - 1 \| < \varepsilon$.

Proof. Let m_n be the least positive integer such that $(\alpha | A_n)^{m_n}$ is an inner automorphism of A_n. Then we can find a sequence of integers $(d_n)_{n \geq 0}$ such that $d_0=1$,

$d_n \mid d_{n+1}, m_n^2 \mid d_n$ and $\sum_{n \geq 1} d_n^{-1} < \varepsilon/10$. We shall construct recurrently unitaries $u_n \in A_n$ and automorphims α_n of A so that

$$\alpha_0 = \alpha \quad ,u_0=1$$

$$\alpha_{n+1} = \mathrm{Ad}\, u_{n+1} \circ \alpha_n, u_{n+1} \in A'_n \cap A_{n+1}$$

$$\|u_{n+1}-1\| < 2\pi \, d_{n+1}^{-1}$$

$$\alpha_{n+1}(A_j)=A_j, (\alpha_{n+1}|A_{n+1})^{d_{n+1}} = \mathrm{id}_{A_{n+1}} \quad (n \geq 0, j \geq 0).$$

Assume u_j, α_j have been constructed for $0 \leq j \leq n$ and let us show how one constructs u_{n+1}, α_{n+1}. Since $\alpha_n | A_{n+1}$ and $\alpha | A_{n+1}$ differ by an inner automorphism of A_{n+1}, we have that $(\alpha_n | A_{n+1})^{m_{n+1}} = \mathrm{Ad}\, v$ for some $v \in A_{n+1}$. Then we have $\alpha_n(v) = \gamma v$ where $\gamma \in Z(A_{n+1})$ the center of A_{n+1} and

$$\gamma \alpha_n(\gamma) \ldots \alpha_n^{m_{n+1}-1}(\gamma) = 1 \text{ since } \alpha_n^{m_{n+1}}(v) = \mathrm{Ad} v(v) = v.$$

It is easily seen that we can find then a unitary $\delta \in Z(A_{n+1})$ such that $(\gamma \delta^{-1} \alpha_n(\delta))^{m_{n+1}} = 1$. But then for $\tilde{v} = \delta v$ and $\tilde{\gamma} = \gamma(\delta^{-1}\alpha_n(\delta)$ we have

$(\alpha_n | A_{n+1})^{m_{n+1}} = \mathrm{Ad}\, \tilde{v}$ and $\alpha_n(\tilde{v}) = \tilde{\gamma} \tilde{v}$ where

$\tilde{\gamma} \in Z(A_{n+1})$, $\tilde{\gamma}^{m_{n+1}} = 1$. Hence for $w = \tilde{v}^{d_{n+1}/m_{n+1}}$ we have

$$(\alpha_n | A_{n+1})^{d_{n+1}} = \mathrm{Ad}\, w \text{ and}$$

$$\alpha_n(w) = \tilde{\gamma}^{d_{n+1}/m_{n+1}} w = w$$

since $m_{n+1}^2 \mid d_{n+1}$.

Since $d_n \mid d_{n+1}$ we have

$(Ad\ w \mid A_n) = (\alpha_n^{d_{n+1}} \mid A_n) = id_{A_n}$ and hence $w \in A'_n \cap A_{n+1}$.

Let g be a Borel function such that defining $u_{n+1} = g(w)$ we have

$u_{n+1}^{d_{n+1}} = w^{-1}$ and $\| u_{n+1} - 1 \| < 2\pi\, d_{n+1}^{-1}$.

Then $u_{n+1} \in A'_n \cap A_{n+1}$, $\alpha_n(u_{n+1}) = u_{n+1}$ and hence

$\alpha_n \circ Ad\ u_{n+1} = Ad\ u_{n+1} \circ \alpha_n$.

For $\alpha_{n+1} = Ad\ u_{n+1} \circ \alpha_n$ we have

$\alpha_{n+1}(A_j) = (Ad\ u_{n+1} \circ \alpha_n)(A_j) =$

$= Ad\ u_{n+1}(A_j) = A_j$

$(\alpha_{n+1} \mid A_{n+1})^{d_{n+1}} =$

$= (Ad\ u_{n+1} \mid A_{n+1})^{d_{n+1}} \circ (\alpha_n \mid A_{n+1})^{d_{n+1}} =$

$= (Ad\ w^{-1} \mid A_{n+1}) \circ (Ad\ w \mid A_{n+1}) = id_{A_{n+1}}$

This concludes the proof of the existence of the (u_n, α_n) for all $n \in \mathbb{N}$.

Now $u = \lim u_n \ldots u_1$ is well-defined and has the desired properties.

Q.E.D.

2.6.Corollary. If α is an almost inductive limit automorphism and $\varepsilon > 0$ then there is a unitary u such that Ad u $\circ \alpha$ is limit periodic and $\|u-1\| < \varepsilon$.

We pass now to embedding the crossed product of A by the action of $\mathbb{Z}$ generated by α into an AF-algebra. We shall denote this crossed product by $A \times_\alpha \mathbb{Z}$.

2.7.Theorem. If α is an almost inductive limit automorphism of A, then $A \times_\alpha \mathbb{Z}$ can be embedded into $A \otimes B$, where B is an UHF-algebra.

The theorem follows in view of Corollary 2.6 from the following lemma.

2.8.Lemma. Let α be a limit periodic automorphism of A and $(d_n)_{n \geq 0}$ a sequence of periods for α such that $\lim_{n \to \infty} d_n = \infty$. Then $A \times_\alpha \mathbb{Z}$ can be embedded into $A \otimes B$ where B is the UHF-algebra $\varinjlim \mathcal{L}(\mathbb{C}^{d_n})$.

Proof. Replacing the nest $(A_n)_{n \geq 0}$ with respect to which α is limit periodic by a subnest we may assume that $\sum d_n^{-1} < \infty$.

Consider in $B_n = \mathcal{L}(\mathbb{C}^{d_n})$ the matrix units $(e(n;i,j))_{1 \leq i,j \leq d_n}$ corresponding to the canonical basis $(e_k^{(n)})_{1 \leq k \leq d_n}$ of $\mathbb{C}^{d_n}$ and consider the unitary

$$s_n = e(n;1,2) + \ldots + e(n;d_n-1,d_n) + e(n;d_n,1).$$

We define injective $*$-homomorphisms

$$\rho_n : A_n \longrightarrow A_n \otimes B_n$$

by

$$\rho_n(x) = \sum_{1 \leq j \leq d_n} \alpha^j(x) \otimes e(n;j,j)$$

so that

$$\rho_n(\alpha(x))=(\mathrm{Ad}(1\otimes s_n))(\rho_n(x))$$

Next, we construct some special embeddings

$j_n: A_n \otimes B_n \longrightarrow A_{n+1} \otimes B_{n+1}$, $k_n: B_n \longrightarrow B_{n+1}$ so that

$$j_n(x\otimes b)=x\otimes k_n(b)$$

and $k_n(b)=\sum_{1\leq j\leq m_n} W_{n,j}bW_{n,j}^*$

where $m_n=d_{n+1}/d_n$ and the isometries $W_{n,j}: \mathbb{C}^{d_n} \longrightarrow \mathbb{C}^{d_{n+1}}$

are given by the formulae

$$W_{n,j}e_s^{(n)}=\frac{1}{\sqrt{m_n}}\exp\left(\frac{2\pi sji}{d_{n+1}}\right)\sum_{0\leq k\leq m_n-1}\exp\left(\frac{2\pi jki}{m_n}\right)e_{s+kd_n}^{(n+1)}$$

It is easily seen that the $W_{n,j}$'s $(1\leq j\leq m_n)$ have pairwise orthogonal ranges and it is easy to check that

$$W_{n,j}s_n=\exp\left(\frac{2\pi ji}{d_{n+1}}\right)s_{n+1}W_{n,j}$$

so that

$$k_n(s_n)=s_{n+1}\sum_{1\leq j\leq m_n}\exp\left(\frac{2\pi ji}{d_{n+1}}\right)W_{n,j}W_{n,j}^*$$

This implies

$$\|k_n(s_n)-s_{n+1}\|\leq\frac{2\pi}{d_n}$$

and hence

$$\|j_n(1\otimes s_n)-1\otimes s_{n+1}\|\leq\frac{2\pi}{d_n}$$

Since $\alpha|A_n$ has order d_n it is easy to check that the diagram

$$\begin{array}{ccc} A_n\otimes B_n & \xrightarrow{j_n} & A_{n+1}\otimes B_{n+1} \\ \uparrow \rho_n & & \uparrow \rho_{n+1} \\ A_n & \hookrightarrow & A_{n+1} \end{array}$$

is commutative.

Consider D the inductive limit of the $(A_n \otimes B_n, j_n)$ and $\varphi_n : A_n \otimes B_n \longrightarrow D$, $\rho : A \longrightarrow D$ the corresponding injective $*$-homomorphisms so that the diagram

$$\begin{array}{ccc} A_n \otimes B_n & \xrightarrow{\varphi_n} & D \\ \uparrow \rho_n & & \uparrow \rho \\ A_n & \hookrightarrow & A \end{array}$$

be commutative. Then $u = \lim\limits_{n \to \infty} \varphi_n(1 \otimes s_n)$ is well-defined and

$\rho(\alpha(a)) = u\, \rho(a) u^*$ for $a \in A$.

Since the C^*-algebra generated by $\rho_n(A_n)$ and $1 \otimes s_n$ is the crossed product $A_n \times_{\alpha_n} \mathbb{Z}/d_n\mathbb{Z}$ there is an automorphism $\beta_n(\zeta)$ of this algebra, if $\zeta^{d_n} = 1$, such that $\beta_n(\zeta) \circ \rho_n = \rho_n$ and $\beta_n(\zeta)(1 \otimes s_n) = \zeta(1 \otimes s_n)$. Hence if $\zeta^{d_n} = 1$ for some n, then there is an automorphism $\beta(\zeta)$ of the C^*-algebra generated by $\rho(A)$ and u such that $\beta(\zeta) \circ \rho = \rho$ and $\beta(\zeta)(u) = \zeta u$. Since $d_n \longrightarrow \infty$ passing to the limit we see that $\beta(\zeta)$ exists for arbitrary ζ with $|\zeta| = 1$ and hence that the covariant representation $(\rho(A), u)$ gives an embedding of $A \times_\alpha \mathbb{Z}$ into D.

It is also obvious that D is isomorphic to $A \otimes B$.

Q.E.D.

2.9. __Remark.__ The construction of the approximate intertwinings $W_{n,j}$ in the proof of the preceding lemma is related to a construction in [12]. We could have also used the construction in [13], but the corresponding formulae would have been cumbersome.

§ 3.

We shall prove in this section that certain automorphisms of AF-algebras are almost inductive limits. In view of Theorem 2.8 this will yield results about embeddings of crossed products into AF-algebras.

For the first lemma in this section we shall have to make an additional assumption concerning A. We shall need to assume that there exists a nest $(A_n)_{n \geq 0}$ of finite-dimensional C^*-subalgebras in A so that the maps $K_o(A_n) \longrightarrow K_o(A)$ be injective for all $n \geq 0$. Recall that $K_o(A)$ with its natural ordering is called the dimension group of A (see [4]) and in terms of dimension groups the condition we are interested in, is equivalent to the requirement that the dimension group $K_o(A)$ be ultrasimplicial in the sense of ([6]). This means $K_o(A)$ is isomoprhic as an ordered group to an inductive limit of ordered groups

$$\mathbb{Z}^{n(1)} \xrightarrow{\varphi_1} \mathbb{Z}^{n(2)} \xrightarrow{\varphi_2} \mathbb{Z}^{n(3)} \longrightarrow \cdots$$

where the φ_j's in addition to being positive are also injective, the groups $\mathbb{Z}^{n(k)}$ being endowed with the simplicial ordering $(a_1,\dots,a_{n(k)}) \geq 0 \Longleftrightarrow a_j \geq 0$ for $1 \leq j \leq n(k)$.

The class of ultrasimplicial dimension groups has been studied by G.A.Elliot ([6]), N.Riedel ([14],[15]) and D.Handelman ([8]).

3.1. Lemma. Let α be an automorphism of A and n a positive integer so that α^n is approximately inner. Assume moreover that the dimension group $K_o(A)$ is ultrasimplicial. Then, given $D \in \mathcal{F}(A)$ and $m \in \mathbb{N}$ such that $n \mid m$, there are $B_j \in \mathcal{F}(A)$ $(1 \leq j \leq m)$ such that

$$d(\alpha(B_j), B_{j+1}) < \frac{5\pi}{m} \text{ for } 1 \leq j \leq m-1,$$

$$d(\alpha(B_m), B_1) < \frac{5\pi}{m}$$

and $B_j \supset D$ for $1 \leq j \leq m$.

Proof. Given $\varepsilon > 0$, we can find $B, C \in \mathcal{F}(A)$ and unitaries $u_0 = 1, u_1, \ldots, u_n, u \in A$ so that

$$(\mathrm{Ad}\, u_j \circ \alpha^j)(C) \supset D$$

$$\|u_j - 1\| < \varepsilon \qquad \text{for } 0 \leq j \leq m$$

and

$$B \supset C$$

$$(\mathrm{Ad}\, u \circ \alpha^{-m})(B) \supset C$$

$$\|u - 1\| < \varepsilon .$$

Moreover $K_0(A)$ being ultrasimplicial we may assume B is such that $i_* : K_0(B) \longrightarrow K_0(A)$ is injective, where $i: B \longrightarrow A$ is the inclusion. Since α^m is approximately inner, there is a unitary w_1 such that

$$(\mathrm{Ad}\, w_1^* \circ \mathrm{Ad}\, u \circ \alpha^{-m})(B) = B$$

Let $k: C \longrightarrow B$ be given by the inclusion and let $\varphi: C \longrightarrow B$ coincide with the restriction of $\mathrm{Ad}\, w_1^*$ to C. We have

$$i_* \circ k_* = i_* \circ \varphi_*$$

and i_* being injective, this implies

$$k_* = \varphi_*$$

Thus there exists a unitary $w_2 \in B$ such that

$$w_2 x w_2^* = w_1^* x w_1 \text{ for } x \in C.$$

Then for $w = w_1 w_2$ we have

$$\mathrm{Ad}\, w(B) = \mathrm{Ad}\, w_1(B) = (\mathrm{Ad}\, u \circ \alpha^{-m})(B)$$

and

$$\mathrm{Ad}\, w \mid C = \mathrm{id}_C$$

or equivalently $w \in C' \cap A$.

Choosing a unitary $v \in C' \cap A$ such that

$\| v^m - w \| < \varepsilon$ and $\| v - 1 \| < 2\pi / m$

we define

$$B_j = (\mathrm{Ad}\, u_j \circ \alpha^j \circ \mathrm{Ad}\, v^j)(B), \text{ for } 0 \leq j \leq m.$$

We have

$$d(\alpha(B_j), B_{j+1}) =$$

$$= d((\mathrm{Ad}\, \alpha(u_j) \circ \alpha^{j+1} \circ \mathrm{Ad}\, v^j)(B), (\mathrm{Ad}\, u_{j+1} \circ \alpha^{j+1} \circ \mathrm{Ad}\, v^{j+1})(B)) <$$

$$< 4\varepsilon + d((\alpha^{j+1} \circ \mathrm{Ad}\, v^j)(B), (\alpha^{j+1} \circ \mathrm{Ad}\, v^{j+1})(B)) =$$

$$= 4\varepsilon + d(B, \mathrm{Ad}\, v(B)) \leq 4\varepsilon + 4\pi / m$$

where $0 \leq j \leq m-1$.

Moreover

$$d(B_m, B_o) =$$

$$= d((\mathrm{Ad}\, u_m \circ \alpha^m \circ \mathrm{Ad}\, v^m)(B), B) \leq$$

$$\leq 2\varepsilon + d((\alpha^m \circ \mathrm{Ad}\, v^m)(B), B) \leq$$

$$\leq 4\varepsilon + d((\alpha^m \circ \mathrm{Ad}\, w)(B), B) =$$

$$= 4\varepsilon + d((\alpha^m \circ \mathrm{Ad}\, u \circ \alpha^{-m})(B), B)$$

so that

$$d(\alpha(B_m), B_1) \leq$$

$$\leq 6\varepsilon + d(\alpha(B_o), B_1) \leq 10\varepsilon + \frac{4\pi}{m}.$$

We also have

$$B_j \supset (\mathrm{Ad}\, u_j \circ \alpha^j \circ \mathrm{Ad}\, v^j)(C) =$$

$$= (\mathrm{Ad}\, u_j \circ \alpha^j)(C) \supset D$$

Thus, taking $\varepsilon < \dfrac{\pi}{10\, m}$, $B_1, \ldots, B_m$ will have the desired properties.

Q.E.D.

For the next lemma it will be convenient to consider for an automorphism the condition ΨO (ΨO =pseudoorbits).

ΨO : given $\varepsilon > 0$ and $D \in \mathcal{F}(A)$ there is $m \in \mathbb{N}$ and there are $B_j, j \in \mathbb{Z}/m\mathbb{Z}$ such that $B_j \supset D$ and $d(\alpha(B_j), B_{j+1}) < \varepsilon$ for all $j \in \mathbb{Z}/m\mathbb{Z}$.

Consider also $\mathcal{U}$ the UHF-algebra

$$\mathcal{U} = \bigotimes_{k \geq 1} \mathcal{L}(\mathbb{C}^k)$$

and the automorphism $\sigma = \bigotimes_{k \geq 1} \operatorname{Ad} s_k$ where s_k is the cyclic permutation of the canonical basis of $\mathbb{C}^k$.

3.2. Lemma. If α satisfies ΨO then $\alpha \otimes \sigma$ is an almost inductive limit automorphism of $A \otimes \mathcal{U}$.

Proof. Let $(e(k;i,j))_{1 \leq i,j \leq k}$ be the canonical matrix units in $\mathcal{L}(\mathbb{C}^k)$, so that

$$s_k = e(k;1,2) + \ldots + e(k;k-1,k) + e(k;k,1).$$

Consider also

$$E(k;i,j) = I_{\mathbb{C}^1} \otimes \ldots \otimes I_{\mathbb{C}^{k-1}} \otimes e(k;i,j) \otimes I_{\mathbb{C}^k} \otimes \ldots \in \mathcal{U}.$$

Let further $\mathcal{U}_k$ be the C^*-subalgebra

$$\Big(\bigotimes_{1 \leq j \leq k} \mathcal{L}(\mathbb{C}^j)\Big) \otimes I_{\mathbb{C}^{k+1}} \otimes \ldots \text{ of } \mathcal{U}.$$

Note also that an ε-pseudoorbit of length m, yields ε-pseudoorbits of lengths md for every $d \geq 1$, so that there is no loss of generality in condition ΨO to require that the length m of the pseudoorbit be greater than a given number.

Thus there is a nest of C^*-subalgebras $(D_j)_{j \geq 1}$ of A and there are integers $1 < m_1 < m_2 < \ldots$ such that there are $1/k$-pseudoorbits $(B_j^{(k)})_{1 \leq j \leq m_k}$ with $B_j^{(k)} \supset D_k$. Let then

$$X_k = \sum_{1\le j\le m_k} B_j^{(k)} \otimes E(m_k; j,j) \in \mathcal{F}(A \otimes \mathcal{U})$$

so that $X_k \supset D_k \otimes 1_{\mathcal{U}}$

If $x \in X_k$, $\|x\| \le 1$, then $x = \sum_{1\le j\le m_k} x_j \otimes E(m_k; j,j)$ and there are $y_j \in B_j^{(k)}$ such that

$$\|y_1 - \alpha(x_2)\| < 1/k, \ldots, \|y_{m_k - 1} - \alpha(x_{m_k})\| < 1/k$$

$$\|y_{m_k} - \alpha(x_1)\| < 1/k$$

Thus for $y = \sum_{1\le j\le m_k} y_j \otimes E(m_k; j,j)$ we have

$$\|y - (\alpha \otimes \sigma)x\| < 1/k.$$

This shows that $(\alpha \otimes \sigma)(X_k) \overset{1/k}{\subset} X_k$. Note that $\mathcal{U}_n$ for $n < m_k$ is in the commutant of X_k. Thus for a sequence $n_1 \le n_2 \le \ldots$ such that $\lim_{j\to\infty} n_j = \infty$, $n_k < m_k$, $\dim \mathcal{U}_{n_k} < k^{1/2}$ we have that

$$(\alpha \otimes \sigma)(X_k \mathcal{U}_{n_k}) \overset{k^{-1/2}}{\subset} X_k \mathcal{U}_{n_k}$$

$$X_k \mathcal{U}_{n_k} \supset D_k \otimes \mathcal{U}_{n_k}$$

Since $X_k \mathcal{U}_{n_k} \in \mathcal{F}(A \otimes \mathcal{U})$ this implies $d((\alpha\otimes\sigma)(X_k \mathcal{U}_{n_k}), X_k \mathcal{U}_{n_k}) \longrightarrow 0$ and the fact that $\alpha \otimes \sigma$ is almost inductive limit.

Q.E.D.

3.3. Remark. It is easily seen that in the preceding lemma, if instead of passing from α to $\alpha \otimes \sigma$, we would have required that α satisfy some Rohlin-type condition, like the one in [10], then by a slight adaptation the same kind of proof would have shown that α is an almost inductive limit automorphism. This applies also to Proposition 3.4.

For the next proposition it will be necessary to keep in mind that $K_0(A\otimes\mathcal{U})$ is isomorphic as a dimension group to $K_0(A)\otimes\mathbb{Q}$ (see [6]).

3.4. Proposition. If α^n is approximately inner for some $n\geq 1$, and $K_0(A)\otimes\mathbb{Q}$ is ultrasimplicial, then $\alpha\otimes\sigma$ is an almost inductive limit automorphism.

Proof. Remarking that $(\mathcal{U},\sigma)$ is isomorphic to $(\mathcal{U}\otimes\mathcal{U},\sigma\otimes\sigma)$, the proposition follows from Lemma 3.1 and Lemma 3.2 applied to $(A\otimes\mathcal{U},\alpha\otimes\sigma)$.

Q.E.D.

3.5. Theorem. If α^n is approximately inner for some $n\geq 1$ and if $K_0(A)\otimes\mathbb{Q}$ is ultrasimplicial then $A\times_\alpha\mathbb{Z}$ can be embedded into $A\otimes\mathcal{U}$.

Proof. $A\times_\alpha\mathbb{Z}$ embeds into $(A\otimes\mathcal{U})\times_{\alpha\otimes\sigma}\mathbb{Z}$ which in turn because of Proposition 3.4 and of Theorem 2.7 can be embedded into $A\otimes\mathcal{U}$.

Q.E.D.

It is known that tnere exist AF-algebra for ehich $K_0(A)$ is not ultrasimplicial ([6], [14]) but as pointed out in [8] it is on open question whether all divisible dimension groups are ultrasimplicial. In support of an affirmative answer to this question is the result contained in [8], that finite rank divisible dimension groups are ultrasimplicial. This has as a consequence the following corollary.

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3.6. Corollary. If α^n is approximately inner for some $n \geq 1$ and if $K_0(A)$ has finite rank then $A \times_\alpha \mathbb{Z}$ can be embedded into $A \otimes \mathcal{U}$.

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