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EXTENSIONS AND RANK 3 VECTOR BUNDLES WITH GIVEN CHERN CLASSES ON HOMOGENOUS RATIONAL 3 - FOLDS

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Constantin BANICA and Justin COANDA

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EXTENSIONS AND RANK 3 VECTOR BUNDLES WITH GIVEN CHERN CLASSES ON HOMOGENOUS RATIONAL 3 - FOLDS

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Constantin BANICA *) and Justin COANDA *)

*) The National Institute for Scientific and Technical Creation
Department of Mathematics, Bd. Pacii 220, 79622 Bucharest, Romania

Introduction

It is known ([1],[6],[14]) that any topological vector bundle on the projective space $\mathbb{P}_3$ has an algebraic structure.On the other hand, in [11] it is shown that any topological vector bundle on a rational surface (complex,projective,nonsingular) has an algebraic structure.

In [2] one classifies the topological vector bundles on complex rational 3-folds,using ideas from [1],and one proves that the rank 2 topological vector bundles on homogeneous rational 3-folds ($\mathbb{P}_3$, the quadric Q_3 ,the flag manifold $\mathbb{F}_{(1,2)}$, $\mathbb{P}_2 \times \mathbb{P}_1$ and $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$) have algebraic structures.

In this paper we prove the following:

Theorem Any rank 3 topological vector bundle on a homogeneous rational 3-fold X has an algebraic structure.

Consequently,any topological vector bundle on a homogeneous rational 3-fold has an algebraic structure.

In [14] Vogelaar proves his theorem on the existence of algebraic structures on rank 3 topological vector bundles on $\mathbb{P}_3$ by constructing algebraic vector bundles as extensions:

$$0 \longrightarrow 2\mathcal{O}_{\mathbb{P}_3} \longrightarrow E(t) \longrightarrow \mathcal{J}_Y(k) \longrightarrow 0$$

where Y is a union of disjoint global complete intersection curves in $\mathbb{P}_3$ and the extension is determined by two global sections of det $N_{Y|\mathbb{P}_3} \otimes \mathcal{O}_{\mathbb{P}_3}(-k)$ which generates this bundles

Then,by number theoretical arguments,he shows that one can choose Y,t and k such that the Chern classes of E are equal to given Chern classes (c_1,c_2,c_3) satisfying the topological obstruction $c_1c_2 \equiv c_3 \pmod 2$.

In order to prove our theorem we use Vogelaar's extensions but we do it in their general form: extensions of ideals of curves by arbitrary rank 2 vector bundles. In particular, we obtain a short proof of Vogelaar's theorem (see 2.1.).

As the method of extensions is the main tool in proving the existence of algebraic structures on topological vector bundles, we take this opportunity to make some remarks about it. Some of them are given only for the coherence of the discussion and are not intended to be published.

1. Preliminaries

1.1. Let X be a connected, nonsingular, projective 3-fold over $\mathbb{C}$ and $Y \subset X$ a locally complete intersection (=l.c.i.) subscheme of codimension 2. Let L be an invertible $\mathcal{O}_X$-module.

A theorem due essentially to Serre ([12] and [10], Ch.I, §5) asserts that if $\det N_{Y|X} \simeq L|Y$ and $H^2(X, L^{-1})=0$, then there exists an algebraic rank 2 vector bundle F on X appearing as an extension:

(1) $$0 \longrightarrow \mathcal{O}_X \longrightarrow F \longrightarrow \mathcal{J}_Y \otimes L \longrightarrow 0$$

The Chern classes of F are given by:

(2) $c_1(F)=c_1(L)$, $c_2(F)=[Y]$

Serre's construction has been extended by Vogelaar to rank 3 vector bundles. As stated in [10], Vogelaar's construction can be described as it follows. Suppose $\det N_{Y|X} \otimes L^{-1}$ is generated by 2 global sections t_1, t_2 and that $H^2(X, L^{-1})=0$. Then there exists an algebraic rank 3 vector bundle E on X which appears as an extension :

$$0 \longrightarrow 2\,\mathcal{O}_X \longrightarrow E \longrightarrow \mathcal{J}_Y \otimes L \longrightarrow 0$$

We shall use this construction in its general form. We start with X, Y, L as above and with a rank 2 vector bundle F on X. Suppose $H^2(X,F)=0$ and that $\det N_{Y|X} \otimes F$, considered as a rank 2 vector bundle on Y, has a global section vanishing nowhere. Then there exists an algebraic rank 3 vector bundle E on X given by an extension:

(3) $$0 \longrightarrow F \longrightarrow E \longrightarrow \mathcal{J}_Y \longrightarrow 0$$

Indeed, as $\det N_{Y|X} \otimes F \simeq \mathcal{E}xt^1(\mathcal{J}_Y, F)$ a global section t of $\det N_{Y|X} \otimes F$ determines an element of $H^0(X, \mathcal{E}xt^1(\mathcal{J}_Y, F))$ and this one lifts to an element e of $\mathrm{Ext}^1(\mathcal{J}_Y, F)$, because $H^2(X,F)=0$. If (3) is the extension determined by e then: E is locally free if and only if t vanishes nowhere (the problem is local, hence one can apply [10].,Ch.I,Lemma 6.4.3.).

The Chern classes of E are given by:

$$(4)\quad \begin{cases} c_1(E)=c_1(F) \\ c_2(E)=c_2(F)+[Y] \\ c_3(E)=(c_1(F)+c_1(X)).[Y]-2\chi(\mathcal{O}_Y) \end{cases}$$

where $c_1(X)=c_1(T_X)$.

The first relation in (4) is a consequence of the fact that $c_1=0$ for a coherent sheaf of dimension ≤ 1 (or,alternatively, restrict the extension (3) to $X\setminus Y$ and observe that the map $H^2(X,\mathbb{Z})\longrightarrow H^2(X\setminus Y,\mathbb{Z})$ is injective).The second relation results from Porteous' formula [8] and the third one using the Theorem of Riemann-Roch. If $X=\mathbb{P}_3$,one can derive the formulae (4) using the Hilbert polynomials and identifying coefficients.

1.2.Let E be an algebraic rank 3 vector bundle on X.It results from the Theorem of Riemann-Roch that the Chern classes of E must satisfy the relation

$$(5)\quad (c_1+c_1(X)).c_2\equiv c_3 \pmod 2$$

(one identifies $H^6(X,\mathbb{Z})$ with $\mathbb{Z}$ in the usual manner).

On the other hand,in [2] it is shown that the systems (c_1,c_2,c_3) which verify the congruence (5) parametrise the isomorphism classes of topological rank 3 vector bundles on X (X being here a connected,complex manifold of dimension 3).

If E is an algebraic vector bundle of rank r on X and L an invertible $\mathcal{O}_X$-module,then:

$$(6)\quad \begin{cases} c_1(E\otimes L)=c_1(E)+rc_1(L) \\ c_2(E\otimes L)=c_2(E)+(r-1)c_1(E).c_1(L)+\binom{r}{2}c_1(L)^2 \\ c_3(E\otimes L)=c_3(E)+(r-2)c_2(E).c_1(L)+\binom{r-1}{2}c_1(E).c_1(L)^2+ \\ \qquad +\binom{r}{3}c_1(L)^3 \end{cases}$$

1.3. We recall now the "doubling" method of Ferrand [3].

Let X,Y be as in 1.1. and L an invertible $\mathcal{O}_Y$-module. Suppose we have an epimorphism $\mathcal{I}_Y/\mathcal{I}_Y^2 \xrightarrow{u} L \longrightarrow 0$. Let Z be the closed subscheme of X determined by the ideal:

$$\mathcal{I}_Z = \ker(\mathcal{I}_Y \longrightarrow \mathcal{I}_Y/\mathcal{I}_Y^2 \xrightarrow{u} L).$$

Then Z is a l.c.i. subscheme of X of codimension 2 having the properties:

(i) $|Z| = |Y|$

(ii) $\det N_{Z|X}|Y \simeq \det N_{Y|X} \otimes L^{-1}$.

As $\mathcal{I}_Y/\mathcal{I}_Z \simeq L$ we have an exact sequence:

(7) $$0 \rightarrow L \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

from which we deduce that

(iii) $\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_Y) + \chi(L)$.

We also have:

(iv) $[Z] = 2[Y]$.

Let F be a rank 2 vector bundle on X. If $\det N_{Y|X} \otimes L^{-1} \otimes F$ has a global section vanishing at no point and if $H^1(Y, \det N_{Y|X} \otimes F) = 0$ then $\det N_{Z|X} \otimes F$ has a global section vanishing nowhere (one tensors (7) with $\det N_{Z|X} \otimes F$,one uses (ii) and so on).

1.4. One knows that the only homogeneous rational 3-folds are: $\mathbb{P}_3, Q_3, \mathbb{F}_{(1,2)}, \mathbb{P}_2 \times \mathbb{P}_1$ and $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$. We prove the theorem inspecting each case. Here we shall make some general remarks concerning the proof.

Let X be a homogeneous rational 3-fold. We identify $H^6(X,\mathbb{Z})$ with $\mathbb{Z}$, using the orientation induced by the complex structure, and $H^2(X,\mathbb{Z})$ with Pic X by taking the Chern class c_1.

In each case the cup-product:

$$H^2(X,\mathbb{Z}) \times H^4(X,\mathbb{Z}) \longrightarrow H^6(X,\mathbb{Z}) \simeq \mathbb{Z}$$

is a perfect pairing, that is, identifies $H^4(X,\mathbb{Z})$ with

$Hom_{\mathbb{Z}}(H^2(X,\mathbb{Z}),\mathbb{Z})$.

One can choose a $\mathbb{Z}$-basis $e_1,\ldots,e_s$ ($s \leq 3$) of $H^4(X,\mathbb{Z})$ such that there are disjoint nonsingular rational curves $C^{(i)} \subset X$, $i=1,\ldots,s$ with

$$[C^{(i)}] = e_i,\ i=1,\ldots,s.$$

Moreover, $e_1,\ldots,e_s$ can be choosen such that if $n \in \mathbb{N}$ there exists disjoint nonsingular rational curves $C_j^{(i)}$, $i=1,\ldots,s$, $j=1,\ldots,n$ with

$$[C_j^{(i)}] = e_i,\ i=1,\ldots,s, j=1,\ldots,n.$$

Let $L_1,\ldots,L_s$ be invertible $\mathcal{O}_X$-modules such that $f_1=c_1(L_1)$, $\ldots,f_s=c_1(L_s)$ is the dual basis of $e_1,\ldots,e_s$. (In fact, $f_1,\ldots,f_s$ can be realized as fundamental classes of nonsingular rational surfaces).

If $C \subset X$ is a nonsingular curve and L an invertible $\mathcal{O}_X$-module then:

(8) $\chi(L|C)=c_1(L).[C]+\chi(\mathcal{O}_C)$.

It follows from (8) (or from the geometric realization of the dual basis) that if $C \subset X$ is a nonsingular rational curve with $[C]=e_i$ then, identifying C with $\mathbb{P}_1$, we have:

$$L_j|C \simeq \mathcal{O}_C(\delta_{ij}),\ j=1,\ldots,s.$$

We also have:

$$\det N_{C|X} \simeq (\det T_X|C) \otimes \mathcal{O}_C(-2).$$

If $t_1,\ldots t_s$ are integers ≥ 1 then $L_1^{t_1} \otimes \ldots \otimes L_s^{t_s}$ is ample.

Firstly, we construct, using Serre's method, an algebraic rank 2 vector bundle F on X as an extension:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F \longrightarrow \mathcal{J}_Y \otimes L \longrightarrow 0$$

where Y is a union of disjoint nonsingular rational curves $C_j^{(i)}$ $i=1,\ldots,s, j=1,\ldots,n_i$ with $[C_j^{(i)}] = e_i$, and $L=\det T_X \otimes L_1^{-2} \otimes \ldots \otimes L_s^{-2}$.

We have:

$$\begin{cases} c_1(F)=c_1(X)-2f_1-\ldots-2f_s \\ c_2(F)=n_1e_1+\ldots+n_se_s \end{cases}$$

We construct then an algebraic rank 3 vector bundle E on X as a Vogelaar extension:

$$0 \longrightarrow F\otimes K \longrightarrow E\otimes T \longrightarrow \mathcal{J}_Z \longrightarrow 0$$

with Z union of disjoint nonsingular rational curves, some of them doubled à la Ferrand and such that $Z\cap Y=\emptyset$, with $T=L_1^{t_1}\otimes\cdots\otimes L_s^{t_s}$, $t_i \gg 0$ and $K=L_1^{k_1}\otimes\ldots\otimes L_s^{k_s}$, $k_i\gg 0$.

In each case we have to verify the conditions:

(9) $H^2(X,F\otimes K)=0$

(10) det $N_{Z|X}\otimes F\otimes K$ has a section vanishing nowhere.

In order to verify (9), firstly we deduce the isomorphisms:

$$H^2(X,F\otimes K)\simeq H^2(X,\mathcal{J}_Y\otimes L\otimes K)\simeq H^1(Y,L\otimes K|Y)$$

from vanishing theorems for ample invertible sheaves. If $C\subset X$ is one of the rational curves contained in Y the choice of $k_1,\ldots,k_s$ will imply that $L\otimes K|C\simeq \mathcal{O}_C(k)$ with $k\geq 0$ and thus $H^1(C,L\otimes K|C)=0$. Consequently, $H^2(X,F\otimes K)=0$.

In order to verify (10) we notice that if $C\subset X$ is a nonsingular rational curve such that $C\cap Y=\emptyset$ then we have an exact sequence:

$$0\longrightarrow \mathcal{O}_C \longrightarrow F|C \longrightarrow L|C \longrightarrow 0.$$

Suppose $L|C\simeq\mathcal{O}_C(a)$. If $a\leq 1$ then $F|C\simeq\mathcal{O}_C\oplus\mathcal{O}_C(a)$. If $a\geq 2$ then $F|C\simeq\mathcal{O}_C(u)\oplus\mathcal{O}_C(v)$ with $u\geq 0, v\geq 0$. The choice of $k_1,\ldots k_s$ will imply that det $N_{C|X}\otimes F\otimes K$ has a section vanishing nowhere.

If C is doubled we apply the last remark in 1.3.

2. Proof of the theorem

2.1. X=the projective space $\mathbb{P}_3$.

$H^2(X,\mathbb{Z})$ has a basis consisting of the element $h=c_1(\mathcal{O}_X(1))$, the dual basis of $H^4(X,\mathbb{Z})$ is h^2 and h^3 is the canonical generator of $H^6(X,\mathbb{Z})$. If $C\subset X$ is a curve then $[C]=(\deg C).h^2$.

We consider the Chern classes as integers. Recall that $c_1(X)=4$ and that the normal bundle $N_{C|X}$ of a nonsingular curve $C\subset X$ is generated by global sections.

We have to show that if $c_1,c_2,c_3\in\mathbb{Z}$ satisfy $c_1c_2\equiv c_3 \pmod 2$ then there exists an algebraic rank 3 vector bundle E on X with $c_i(E)=c_i$, $i=1,2,3$.

Firstly, we construct, by Serre's method, a rank 2 vector bundle F on X as an extension:

$$0\longrightarrow \mathcal{O}_X \longrightarrow F \longrightarrow \mathcal{J}_Y(2) \longrightarrow 0$$

where Y is a union of d disjoint lines. We have:

$$c_1(F)=2 \quad ,c_2(F)=d.$$

Then we construct, by Vogelaar's method, E as an extension:

$$0\longrightarrow F(k)\longrightarrow E(t)\longrightarrow \mathcal{J}_Z \longrightarrow 0$$

where Z is a union of disjoint nonsingular rational curves such that $Z\cap Y=\emptyset$. The construction is possible for $k\geq 0$. We have:

$$\begin{cases} c_1(E(t))=c_1(F(k)) \\ c_2(E(t))=c_2(F(k))+\deg Z \\ c_3(E(t))=(c_1(F(k))+4)\deg Z -2\chi(\mathcal{O}_Z). \end{cases}$$

We have to choose $d\geq 1$, Z, t and $k\geq 0$ such that:

$$\begin{cases} c_1(E(t))=c_1+3t \\ c_2(E(t))=c_2+2c_1t+3t^2 \\ c_3(E(t))=c_3+c_2t+c_1t^2+t^3 \end{cases}$$

In other words, we have to solve the system:

$$\begin{cases} c_1(E(t))=2k+2 \\ c_2(E(t))=d+2k+k^2+\deg Z \\ c_3(E(t))=(2k+6)\deg Z-2\chi(\mathcal{O}_Z) \end{cases}$$

where we denote by $c_1(E(t))$ the expression c_1+3t etc.

We choose the class of t (mod 2) such that $t \equiv c_1 \pmod 2$. Then we define k by the formula:

$$k=\frac{1}{2}c_1(E(t))-1=\frac{1}{2}(c_1+3t)-1$$

If $t \gg 0$, then $k \gg 0$. The relation $c_1c_2 \equiv c_3 \pmod 2$ implies that $c_1(E(t)).c_2(E(t)) \equiv c_3(E(t)) \pmod 2$ (recall that $c_1(E(t))$ means c_1+3t, etc.)

As $c_1(E(t))= 2k+2$ we see that $c_3(E(t)) \equiv 0 \pmod 2$. The last equation of the system can be written:

$$\frac{1}{2}c_3(E(t))=(k+3)\deg Z-\chi(\mathcal{O}_Z)$$

From this moment we can solve the system by several methods.

<u>The first method.</u> We divide $\frac{1}{2}c_3(E(t))$ by $k+3$:

$$\frac{1}{2}c_3(E(t))=(k+3)q-r, \text{ with } 1 \leq r \leq k+3.$$

If $t \gg 0$, $\frac{1}{2}c_3(E(t)) \sim \frac{1}{2}t^3$, $k+3 \sim \frac{3}{2}t$ hence $q \sim \frac{1}{3}t^2$. It results that for $t \gg 0$ we have $r \leq q$. Now let Z be a disjoint union of $r-1$ lines and a nonsingular rational curve of degree $q-r+1$. Then:

$$\deg Z=q, \quad \chi(\mathcal{O}_Z)=r.$$

If $t \gg 0$, then $c_2(E(t))-(2k+k^2+q) \sim 3t^2-(\frac{9}{4}t^2+\frac{1}{3}t^2)=\frac{5}{12}t^2$.

It results that if we define d by the formula:

$$d=c_2(E(t))-(2k+k^2+q)$$

then $d \geq 1$ for $t \gg 0$.

<u>The second method.</u> We use two elementary lemmas:

<u>Lemma 1.</u> Let a,b be coprime natural numbers. If $n \geq ab$ then there exist natural numbers x,y such that $n=xa+yb$.

<u>Proof.</u> a and b being coprime the numbers $a, 2a, \ldots, ba$ have distinct classes (mod b). It follows that there exists $x \in \{1,\ldots,b\}$ such that $n \equiv xa \pmod b$. Then $n-xa=yb$ and the condition $n \geq ab$ implies that $y \geq 0$.

Lemma 2. Let M and $N\geq 2$ be natural numbers such that $M\geq 2N^2$. Then there exist natural numbers x, y such that:

$M=N(2x+y)-(x+y)$.

Moreover, we have $\frac{M}{N}\leq 2x+y\leq\frac{M}{N-1}$.

Proof. The asserted equality can be written as:

$$M=(2N-1)x+(N-1)y.$$

Now, the existence of x and y follows from Lemma 1 since 2N-1 and N-1 are coprime and $M\geq 2N^2>(2N-1)(N-1)$. The inequalities are the consequence of the fact that $0<x+y\leq 2x+y$.

If $t\gg 0$ then $\frac{1}{2}c_3(E(t))\geq 2(k+3)^2$, hence there exist $x,y\in\mathbb{N}$ such that:

$$\frac{1}{2}c_3(E(t))=(k+3)(2x+y)-(x+y).$$

Let Z be a union of x conics and y lines. We have :

$$\deg Z=2x+y\ ,\ \chi(\mathcal{O}_Z)=x+y.$$

If $t\gg 0$ then $\deg Z=2x+y\sim\frac{1}{3}t^2$ and one continues as in the first method.

Alternatively, we can use lines and conics degenerated to double lines. This suggests a method which works in all cases: one uses only curves $C_j^{(i)}$ and Ferrand double structures on them.

2.2. X= the quadric hypersurface $Q_3\subset\mathbb{P}_4$.

The restriction morphism $\mathrm{Pic}\,\mathbb{P}_4\longrightarrow\mathrm{Pic}\,X$ is an isomorphism, hence $H^2(X,\mathbb{Z})$ is generated by $f=c_1(\mathcal{O}_X(1))$.

If $L\subset X$ is a line then $[L]$ is the dual generator of $H^4(X,\mathbb{Z})$. We denote this generator by e. We have $f^2=2e$ and fe is the canonical generator of $H^6(X,\mathbb{Z})$. If $C\subset X$ is a curve then $[C]=(\deg C).e$ (where deg C is computed in $\mathbb{P}_4$). We consider the Chern classes as integers. We have $\det T_X\simeq\mathcal{O}_X(3)$ therefore $c_1(X)=3$.

We have to show that if $c_1,c_2,c_3\in\mathbb{Z}$ satisfy $(c_1+1)c_2\equiv c_3 \pmod 2$ then there exists an algebraic rank 3 vector bundles E on X with $c_i(E)=c_i$, i=1,2,3.

Firstly we make some remarks. If $L\subset X$ is a line then $N_{L|X}\simeq \mathcal{O}_L\oplus\mathcal{O}_L(1)$. We double L starting with an epimorphism $\mathcal{I}_L/\mathcal{I}_L^2=\mathcal{O}_L\oplus\mathcal{O}_L(-1)\longrightarrow\mathcal{O}_L(-1)$. Let D be the curve obtained. Then deg D =2 and $\chi(\mathcal{O}_D)=1$.

If $C\subset X$ is a nonsingular rational curve then det $N_{C|X}\simeq$ $\simeq(\det T_X|C)\otimes\omega_C$ is very ample on C. We also notice that there exist such curves on X of any degree (it is enough to consider nonsingular curves of type $(1,a)$, $a\geqslant 0$, on nonsingular hyperplane sections $Q_2=H\cap X$).

We shall construct a rank 2 vector bundle F on X as an extension:

$$0\longrightarrow\mathcal{O}_X\longrightarrow F\longrightarrow\mathcal{I}_Y(1)\longrightarrow 0$$

where Y is a union of d disjoint lines. We have

$$c_1(F)=1,c_2(F)=d.$$

Then we construct E as an extension:

$$0\longrightarrow F(k)\longrightarrow E(t)\longrightarrow\mathcal{I}_Z\longrightarrow 0$$

where Z is a union of disjoint nonsingular rational curves, some of them doubled, and such that $Z\cap Y=\phi$. The construction is possible for $k\geqslant 0$. We have to solve the system:

$$\begin{cases}c_1(E(t))=2k+1\\ c_2(E(t))=d+2k+2k^2+\deg Z\\ c_3(E(t))=(2k+4)\deg Z-2\chi(\mathcal{O}_Z)\end{cases}$$

where:

$$\begin{cases}c_1(E(t))=c_1+3t\\ c_2(E(t))=c_2+4c_1t+6t^2\\ c_3(E(t))=c_3+c_2t+2c_1t^2+2t^3\end{cases}$$

We can solve the system by any of the methods used in the case $X=\mathbb{P}_3$.

2.3. X=the flag manifold $\mathbb{F}_{(1,2)}$.

We consider X as the incidency manifold in $\mathbb{P}_2\times\mathbb{P}_2^*$. Let

$\pi : X \longrightarrow \mathbb{P}_2$, $\rho : X \longrightarrow \mathbb{P}_2^*$ be the projection maps. A basis of $H^2(X,\mathbb{Z})$ consists of the elements $f = c_1(\pi^* \mathcal{O}_{\mathbb{P}_2}(1))$ and $g = c_1(\rho^* \mathcal{O}_{\mathbb{P}_2^*}(1))$. The dual basis of $H^4(X,\mathbb{Z})$ is g^2, f^2. We have the relations:

$$fg = f^2 + g^2 \quad , f^3 = g^3 = 0$$

and $f^2 g = f g^2$ is the canonical generator of $H^6(X,\mathbb{Z})$.

If $z \in \mathbb{P}_2$ then $[\pi^{-1}(z)] = f^2$ and if $\ell \in \mathbb{P}_2^*$ then $[\rho^{-1}(\ell)] = g^2$. We have $c_1(X) = 2f + 2g$. Let $C \subset X$ be a curve of the form $\pi^{-1}(z)$ or $\rho^{-1}(\ell)$. Then $N_{C|X} \simeq 2\mathcal{O}_C$. We double C starting with an epimorphism $\mathcal{I}_C/\mathcal{I}_C^2 = 2\mathcal{O}_C \longrightarrow \mathcal{O}_C(1)$. If D is the curve obtained then $[D] = 2[C]$ and $\chi(\mathcal{O}_D) = 3$.

If $K = \pi^* \mathcal{O}_{\mathbb{P}_2}(k) \otimes \rho^* \mathcal{O}_{\mathbb{P}_2^*}(\ell)$ then $H^i(X,K) = 0$ for $i \geq 1, k \geq 0, \ell \geq 0$. We have to show that if $c_i \in H^i(X,\mathbb{Z}), i = 1,2,3$ satisfy $c_1 c_2 \equiv c_3 \pmod 2$ then there exists an algebraic rank 3 vector bundle E on X with $c_i(E) = c_i, i = 1,2,3$.

Firstly, we construct a rank 2 vector bundle F on X as an extension:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F \longrightarrow \mathcal{I}_Y \longrightarrow 0$$

with Y a disjoint union of p curves of type $\pi^{-1}(z)$, $z \in \mathbb{P}_2$ and q curves of type $\rho^{-1}(\ell)$, $\ell \in \mathbb{P}_2^*$. We get:

$$c_1(F) = 0, \; c_2(F) = pf^2 + qg^2.$$

We construct E as an extension:

$$0 \longrightarrow F \otimes K \longrightarrow E \otimes T \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

where $K = \pi^* \mathcal{O}_{\mathbb{P}_2}(k) \otimes \rho^* \mathcal{O}_{\mathbb{P}_2^*}(m)$, $T = \pi^* \mathcal{O}_{\mathbb{P}_2}(t) \otimes \rho^* \mathcal{O}_{\mathbb{P}_2^*}(u)$, and Z is a union of disjoint curves of type $\pi^{-1}(z)$ or $\rho^{-1}(\ell)$, some of them doubled (as above) and such that $Z \cap Y = \emptyset$. The construction is possible for $k \geq 1$ and $m \geq 1$.

We have to solve the system:

$$\begin{cases} c_1(E\otimes T)=2kf+2mg \\ c_2(E\otimes T)=(p+k^2+2km)f^2+(q+m^2+2km)g^2+[Z] \\ c_3(E\otimes T)=2((k+1)f+(m+1)g)[Z]-2\chi(\mathcal{O}_Z) \end{cases}$$

where:

$$\begin{cases} c_1(E\otimes T)=c_1+3c_1(T) \\ c_2(E\otimes T)=c_2+2c_1.c_1(T)+3c_1(T)^2 \\ c_3(E\otimes T)=c_3+c_2.c_1(T)+c_1.c_1(T)^2+c_1(T)^3 \end{cases}$$

We have :

$$\begin{cases} c_1(T)=tf+ug \\ c_1(T)^2=(t^2+2tu)f^2+(u^2+2tu)g^2 \\ c_1(T)^3=3tu(t+u)f^2g \end{cases}$$

Suppose $c_1=c_1'f+c_1''g$. We choose the class of $t \pmod 2$ such that $t\equiv c_1' \pmod 2$ and the class of $u \pmod 2$ such that $u\equiv c_1'' \pmod 2$. Then we put:

$$k=\frac{1}{2}(c_1'+3t),\quad m=\frac{1}{2}(c_1''+3u).$$

From $c_1(E\otimes T)=2kf+2mg$ and $c_1(E\otimes T).c_2(E\otimes T)\equiv c_3(E\otimes T) \pmod 2$ it follows that $c_3(E\otimes T)\equiv 0 \pmod 2$. The last equation of the system can be written:

$$\frac{1}{2}c_3(E\otimes T)=((k+1)f+(m+1)g)[Z]-\chi(\mathcal{O}_Z).$$

Let Z be a disjoint union of x curves of type $\pi^{-1}(z)$ doubled (as above), y curves of type $\pi^{-1}(z)$, x curves of type $\rho^{-1}(\ell)$ doubled and y curves of type $\rho^{-1}(\ell)$ such that $Z\cap Y=\emptyset$. We get

$$\begin{cases} [Z]=(2x+y)f^2+(2x+y)g^2 \\ \chi(\mathcal{O}_Z)=2(3x+y). \end{cases}$$

We must find x and y such that:

(1) $\quad \frac{1}{2}c_3(E\otimes T)=(k+m+2)(2x+y)-2(3x+y)$

To do this, we use:

<u>Lemma 3.</u> Let M and $N>3$ be natural numbers such that N is odd

and $M \geqslant 2N^2$ .Then there exist natural numbers x,y so that

$$M=N(2x+y)-2(3x+y).$$

Moreover,we have $\frac{M}{N-2} \leq 2x+y \leq \frac{M}{N-3}$.

Choosing the class of t(mod 4) such that $t \equiv c_1'+2 \pmod 4$ and the class of u(mod 4) such that $u \equiv c_1'' \pmod 4$ we have that k+m+2 is odd. If $t \gg 0$ and $u \gg 0$ then $\frac{1}{2}c_3(E\otimes T) \geqslant 2(k+m+2)^2$. Hence we can find x and y such that (1) holds.

Now, suppose $c_2(E\otimes T)=c_2'(E\otimes T)f^2+c_2''(E\otimes T)g^2$.

If $t \gg 0$ and $u \gg 0$ then:

$$c_2'(E\otimes T)-(k^2+2km)-(2x+y) \sim 3(t^2+2tu)-\frac{9}{4}(t^2+2tu)-tu=\frac{3}{4}t^2+\frac{1}{2}tu$$

$$c_2''(E\otimes T)-(m^2+2km)-(2x+y) \sim 3(u^2+2tu)-\frac{9}{4}(u^2+2tu)-tu=\frac{3}{4}u^2+\frac{1}{2}tu.$$

It follows that if we define p and q by the formulae:

$$p=c_2'(E\otimes T)-(k^2+2km)-(2x+y)$$

$$q=c_2''(E\otimes T)-(m^2+2km)-(2x+y)$$

then for $t \gg 0$ and $u \gg 0$ we have $p \geqslant 0, q \geqslant 0$.

2.4. The remaining cases $X=\mathbb{P}_2\times\mathbb{P}_1$ and $X=\mathbb{P}_1\times\mathbb{P}_1\times\mathbb{P}_1$ are analogous and we omit the details.

3.Remarks on the method of extensions

3.1. Let X be a nonsingular variety, $r \geqslant 2$, $Y \subset X$ a closed subscheme of codimension 2 which is Cohen-Macaulay and such that $\mathcal{J}_Y$ is locally generated by r elements, F a vector bundle on X of rank r-1. Let ξ be an element of $\mathrm{Ext}^1(\mathcal{J}_Y,F)$ and let

$$0 \longrightarrow F \longrightarrow E_\xi \longrightarrow \mathcal{J}_Y \longrightarrow 0$$

be the extension determined by ξ. The proof of ([10], Chap.I, Lemma 6.4.3.) shows that E_ξ is locally free if and only if the morphism $F^*\otimes \omega_X|Y \longrightarrow \omega_Y$ associated to ξ via the map

$Ext^1(\mathcal{J}_Y,\mathcal{F}) \longrightarrow H^0(X,\mathcal{E}xt^1(\mathcal{J}_Y,\mathcal{F}))$ and the isomorphisms $\mathcal{E}xt^1(\mathcal{J}_Y,F)\simeq\mathcal{E}xt^2(\mathcal{O}_Y,F)\simeq\mathcal{H}om(F^*\otimes\omega_X|Y,\omega_Y)$ is surjective. If $H^2(X,F)=0$ then every epimorphism $F^*\otimes\omega_X|Y \longrightarrow \omega_Y$ defines a rank r bundle as an extension of the form considered above. We also remark that $\mathcal{J}_Y$ is locally generated by r elements if and only if ω_Y is locally generated by r-1 elements.

Now, suppose that Y is l.c.i. To give an epimorphism $F^*\otimes\omega_X|Y \longrightarrow \omega_Y$ is equivalent to giving a global section of the vector bundle $F|Y\otimes\det N_{Y|X}$ vanishing at no point. If dim $Y \geqslant r-1$ this is always possible after tensoring F with a power of an ample invertible sheaf. On the contrary, if dim $Y \geqslant r-1$ this condition is fullfiled only in exceptional cases. However, one can construct in this situation, starting with sections in $F|Y\otimes\det N_{Y|X}$, rank 3 reflexive sheaves which appear as extensions of the form considered above and whose singular sets correspond to the zeroes of the sections we have started with.

3.2. Conversely, any rank 2 vector bundle E on X can be realized as an extension. One tensors E with a power of an ample invertible sheaf and one consider the extension determined by the generic section, or, in the case $X=\mathbb{P}_n$, one considers the least integer k such that $H^0(E(k))\neq 0$ and one takes the extension determined by a nonzero section of E(k).

The same is true for any rank:

Proposition 1. Let X be a nonsingular projective variety of dimension $n\geqslant 2$, L an ample invertible sheaf on X, E and F vector bundles on X of rank r and r-1, respectively.

Then for $k \gg 0$ there exists an extension:

$$0 \longrightarrow F \longrightarrow E\otimes L^k \longrightarrow \mathcal{J}_Y\otimes R \longrightarrow 0$$

where Y is a Cohen-Macaulay subscheme of X of codimension 2 and R is an invertible sheaf.

Moreover, if $\dim X \leq 5$ and $r=3$ one can find such extensions with Y nonsingular.

<u>Proof</u> 1. Firstly, suppose $\mathcal{H}om(F,E)$ is generated by global sections. We show that for the generic morphism $u:F \longrightarrow E$, $\overset{r-1}{\wedge} u$ vanishes along a Cohen-Macaulay subscheme of codimension 2.

We use, as in ([5], Prop. 1.4 or [7] proof of 5.1.), the incidency set:

$Z = \{(u,x) \mid u \in Hom(F,E), x \in X, (\overset{r-1}{\wedge} u)(x) = 0\}$.

We consider the morphism of varieties:

$$\mathcal{H}om(F,E) \longrightarrow \mathcal{H}om(E, \det E \otimes \det F^*)$$

defined by $u \longmapsto (\overset{r-1}{\wedge} u)$. We denote by Q the inverse image of the zero section by this morphism. Q is a fiber bundle over X with fiber the subscheme of $M(r \times (r-1), \mathbb{C})$ determined by the ideal generated by the (r-1)-minors. A classical result in commutative algebra asserts that this sübscheme is Cohen-Macaulay of codimension 2 .

Consequently, Q is a Cohen-Macaulay subscheme of $\mathcal{H}om(F,E)$ of codimension 2. Z is the inverse image of Q by the morphism of evaluation:

$$Hom(F,E) \times X \longrightarrow \mathcal{H}om(F,E)$$

which is a submersion. Therefore Z is a Cohen-Macaulay subscheme of $Hom(F,E) \times X$ of codimension 2. Consider the projection map $\pi : Z \longrightarrow Hom(F,E)$. For $u \in Hom(F,E)$, $\pi^{-1}(u)$ is just the scheme of zeroes Y_u of $\overset{r-1}{\wedge} u$. If $\pi(Z) \neq Hom(F,E)$ then, for generic u, we have $Y_u = \emptyset$. Now, suppose π surjective. From the generic flatness of π it follows that Y_u is Cohen-Macaulay of codimension 2 for generic u. Let K be the kernel of the epimorphism:

$$E \longrightarrow \mathcal{J}_{Y_u} \otimes \det E \otimes \det F^* \longrightarrow 0$$

K is locally free of rank r-1 and there exists a natural morphism $F \longrightarrow K$ which is isomorphism on $X \setminus Y_u$. Therefore $F \simeq K$

and we have an extension:

$$0 \longrightarrow F \longrightarrow E \longrightarrow \mathcal{J}_Y \otimes \det E \otimes \det F^* \longrightarrow 0$$

where Y is empty or is a Cohen-Macaulay subscheme of X of codimension 2.

2. Now, suppose that we are in the general case. Using 1. for F and $F \otimes L^k$, $k \gg 0$, we find extensions:

$$0 \longrightarrow F \longrightarrow E \otimes L^k \longrightarrow \mathcal{J}_Y \otimes R \longrightarrow 0$$

where Y is empty or is a Cohen-Macaulay subscheme of codimension 2 and R is an invertible sheaf.

We show that if $k \gg 0$ then it necessarily follows $Y \neq \emptyset$.

Indeed, suppose we have extensions:

$$0 \longrightarrow F \longrightarrow E \otimes L^k \longrightarrow R \longrightarrow 0$$

for k arbitrarily large. Then an easy computation shows that:

$$c_2(E) \cdot c_1(L)^{n-2} \sim k^2 \left[\binom{r-1}{2} - (r-1)^2 \right] c_1(L)^n$$

which is a contradiction.

3. Recall the arguments in 1. and suppose r=3. The subscheme of $M(3 \times 2, \mathbb{C})$ defined by the 2-minors has no singular point, except 0. On the other hand, if $\dim X \leq 5$ and if the rank 6 vector bundle $\mathcal{H}om(F,E)$ is generated by global sections, the generic morphism $F \longrightarrow E$ vanishes nowhere. Now the last statement in proposition is clear.

Corrolary Let X be a nonsingular projective variety of dimension smaller than 5 and F a rank 2 vector bundle on X. Then there exists a nonsingular subscheme $Y \subset X$ of codimension 2 such that $F|Y$ is an extension of invertible $\mathcal{O}_Y$-modules.

Proof. Let L be am ample invertible sheaf on X. We use Proposition 1 for F and $E = 3\mathcal{O}_X$. For $k \gg 0$ there exists an extension:

$$0 \longrightarrow F \longrightarrow 3L^k \longrightarrow \mathcal{J}_Y \otimes R \longrightarrow 0$$

with Y nonsingular subscheme of X of codimension 2 and R an invertible sheaf on X. This extension defines a section of the vector bundle $\det N_{Y|X} \otimes F \otimes R^*$ vanishing at no point.

Remark. Using a recent result of Lazarsfeld [9] one can make more precise the assertions in Proposition 1: namely, if $\mathcal{H}om(F,E)\otimes L^{k-1}$ is generated by global sections then the conclusions of Proposition 1 hold.

3.3. Now, let P be the projective space $\mathbb{P}_n$ and E a rank r vector bundle on P. We have $\overset{r-1}{\wedge}E \simeq E^*(c_1(E))$. Put:

$d=-\min\{k \mid H^0((\overset{r-1}{\wedge}E)(k))\neq 0\}$

Then there exist exact sequences:

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow \mathcal{J}_Y(c_1(E)-d) \longrightarrow 0$$

where Y is a closed subscheme of P of codimension $\geqslant 2$ and $\mathcal{F}$ is a reflexive subsheaf of rank r-1 with $c_1(\mathcal{F})=d$. For every rank r-1 reflexive subsheaf $\mathcal{F}$ of E we have $c_1(\mathcal{F})\leq d$ and if the equality holds then there exists an exact sequence as above.

Y is not necessarily Cohen-Macaulay of codimension 2 and the set of points where this fails is exactly the singular set of $\mathcal{F}$.

If r=2 then $\mathcal{F}$ is invertible and one obtains the construction mentioned above.

If n=2 (and r arbitrary) then $\mathcal{F}$ is locally free, but if $n\geqslant 3$ and $r\geqslant 3$ then one can not avoid the reflexive sheaves. We shall illustrate this assertion by an example from [10] (Ch.II, Ex.1.1.13). Let $E=\Omega_{\mathbb{P}_3}(1)$. The scheme of zeroes of a nonzero section $s\in H^0(T_{\mathbb{P}_3}(-1))$ consist of a simple point x and we obtain an exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow \Omega^1_{\mathbb{P}_3}(1) \xrightarrow{\ s^*\ } \mathcal{J}_x \longrightarrow 0$$

$\mathcal{F}$ is reflexive, its singular set is $\{x\}$. Moreover, $c_1(\mathcal{F})$ is maximal because $H^0(T_{\mathbb{P}_3}(-2))=0$.

3.4. The remarks above show that it is necessary to consider extensions in the frame of reflexive sheaves, i.e., to connect by extensions rank r and r-1 reflexive sheaves with subspaces of codimension ≥ 2 . We restrict ourselves to the case of extensions of ideals of curves and points by rank 2 reflexive sheaves on nonsingular 3-folds intending to obtain vector bundles as such extensions.

Before stating the result we need some considerations of local algebra:

Proposition 2. Let $(A,\mathfrak{m})$ be a regular local ring of dimension 3, $k=A/\mathfrak{m}$ its residue field, F a reflexive A-module of rank 2.

The following conditions on F are equivalent:

(i) $Ext^1(F,A)$ is a nonzero A-module generated by a single element

(ii) $\dim_k F/\mathfrak{m}F=3$

(iii) There exists an exact sequence:

$$0 \longrightarrow F \longrightarrow A^3 \longrightarrow I \longrightarrow 0.$$

with $I \subsetneq \mathfrak{m}$ ideal of height 3. Moreover, the isomorphism class of F is uniquely determined by I.

Proof. (i) $\Longrightarrow$ (ii) and (iii)

Let e be a generator of $Ext^1(F,A)$. Consider the extension determined by e:

$$(1) \qquad 0 \longrightarrow A \longrightarrow E \longrightarrow F \longrightarrow 0$$

We assert that E is free. Indeed, F being reflexive it is of homological dimension ≤ 1, hence E is of homological dimension ≤ 1. It suffices to prove that $Ext^1(E,A)=0$.

Applying Hom(.,A) to (1) one gets an exact sequence:

$$0 \longrightarrow F^* \longrightarrow E^* \longrightarrow A \xrightarrow{\partial} Ext^1(F,A) \longrightarrow Ext^1(E,A) \longrightarrow 0$$

$\partial(1)=e$, hence ∂ is surjective. Consequently $Ext^1(E,A)=0$. Thus we have an exact sequence:

$$(2) \qquad 0 \longrightarrow A \xrightarrow{u} A^3 \longrightarrow F \longrightarrow 0$$

Suppose u is given by $u(1)=(a_1,-a_2,a_3)$. Put $I=Aa_1+Aa_2+Aa_3$. Applying Hom(.,A) to the exact sequence (2) we derive that:

$$Ext^1(F,A) \simeq A/I$$

$Ext^1(F,A)$ is a nonzero A-module of dimension 0, hence $I \subseteq \mathfrak{m}$ and $ht(I)=3$. It follows that a_1, a_2, a_3 is an A-sequence.

Applying $\otimes_A k$ to the exact sequence (2) we deduce that $\dim_k F/\mathfrak{m}F=3$. On the other hand, from (2) and from the Koszul complex associated to a_1, a_2, a_3 it follows that we have an exact sequence

$$0 \longrightarrow F \longrightarrow A^3 \longrightarrow I \longrightarrow 0.$$

We note, for further reference, that F is isomorphic to the submodule of A^3 generated by $(0,-a_3,a_2),(-a_3,0,a_1),(-a_2,a_1,0)$.

(ii) $\Longrightarrow$ (i).

By Nakayama's lemma there is an epimorphism $A^3 \xrightarrow{p} F \longrightarrow 0$. Ker p is a reflexive A-module of rank 1, hence $\text{Ker } p \simeq A$. Thus we have an exact sequence:

$$0 \longrightarrow A \longrightarrow A^3 \xrightarrow{p} F \longrightarrow 0.$$

Applying Hom(.,A) to this sequence we see that $Ext^1(F,A)$ is generated by a single element.

(iii) $\Longrightarrow$ (i).

Apllying Hom(.,A) to the exact sequence:

$$0 \longrightarrow F \longrightarrow A^3 \longrightarrow I \longrightarrow 0$$

we find:

$$Ext^1(F,A) \simeq Ext^2(I,A) \simeq A/I$$

hence $Ext^1(F,A)$ is generated by a single element.

Concerning the last assertion we observe that $I=Ann(Ext^1(F,A))$, hence $F \simeq F' \Longrightarrow I=I'$.

Conversely, suppose $I=I'$. There are exact sequences:

$$0 \longrightarrow F \longrightarrow A^3 \xrightarrow{p} I \longrightarrow 0, \quad 0 \longrightarrow F' \longrightarrow A^3 \xrightarrow{p'} I \longrightarrow 0$$

Let $\varphi: A^3 \longrightarrow A^3$ be a morphism such that $p' \circ \varphi = p$. Applying $\otimes_A k$ it follows that φ is isomorphism. Therefore $F \simeq F'$.

Now, let X be a nonsingular 3-fold, $\mathcal{F}$ a rank 2 reflexive sheaf on X. Suppose $\mathcal{F}$ verifies the condition:

(*) $Ext^1(\mathcal{F}_x, \mathcal{O}_x)$ is $\mathcal{O}_x$-module generated by a single element, for all $x \in X$. Then $\mathrm{Ann}(\mathcal{E}xt^1(\mathcal{F},\mathcal{O}_X))$ defines a closed subscheme S of X supported by the singular set of $\mathcal{F}$ such that $\mathcal{E}xt^1(\mathcal{F},\mathcal{O}_X) \simeq$ $\simeq \mathcal{O}_S$.

<u>Proposition 3.</u> In the above hypothesis, let $Y \subset X$ be a l.c.i. closed subscheme of codimension 2 such that $Y \cap S = \emptyset$. Suppose:

(i) $H^2(X,\mathcal{F})=0$

(ii) $\det N_{Y|X} \otimes \mathcal{F}$ has a global section vanishing at no point.

Then there exists an extension:

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow \mathcal{I}_{Y\cup S} \longrightarrow 0$$

with E rank 3 vector bundle on X. The Chern classes of E are given by the formulae:

$$c_1(E)=c_1(\mathcal{F})$$
$$c_2(E)=c_2(\mathcal{F})+[Y]$$
$$c_3(E)=(c_1(\mathcal{F})+c_1(X)).[Y]-2\chi(\mathcal{O}_Y)-c_3(\mathcal{F}).$$

<u>Proof.</u> We have isomorphisms:

$$\mathcal{H}om(\mathcal{I}_{Y\cup S},\mathcal{F}) \simeq \mathcal{F} \quad , \mathcal{E}xt^1(\mathcal{I}_{Y\cup S},\mathcal{F}) \simeq (\det N_{Y|X}\otimes \mathcal{F}) \oplus \mathcal{E}xt^1(\mathcal{I}_S,$$

and an exact sequence:

$$0 \longrightarrow H^1(X,\mathcal{F}) \longrightarrow \mathrm{Ext}^1(\mathcal{I}_{Y\cup S},\mathcal{F}) \longrightarrow H^0(X,\mathcal{E}xt^1(\mathcal{I}_{Y\cup S},\mathcal{F})) \longrightarrow H^2(X,\mathcal{F}).$$

We start with extensions:

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{O}_x^3 \longrightarrow \mathcal{I}_{S,x} \longrightarrow 0, x \in S$$

given by Proposition 2, and with a section of $\det N_{Y|X} \otimes \mathcal{F}$ vanishing at no point. These ones define an element of $H^0(X,\mathcal{E}xt^1(\mathcal{I}_{Y\cup S},\mathcal{F}))$ which lifts to an element $e \in \mathrm{Ext}^1(\mathcal{I}_{Y\cup S},\mathcal{F})$. The element e determines the extension we are looking for.

One computes the first two Chern classes by removing S and the third one using Riemann-Roch (recall that $c_3(\mathcal{F})=$ $=\chi(\mathcal{E}xt^1(\mathcal{F},\mathcal{O}_X))=\chi(\mathcal{O}_S)$).

Following ([4], §4) one can construct rank 2 reflexive sheaves $\mathcal{F}$ on a nonsingular 3-fold X as extensions:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_Y \otimes L \longrightarrow 0$$

where L is an invertible sheaf and Y a Cohen-Macaulay subscheme of X of codimension 2, generically l.c.i. The extension is determined by a global section $\xi \in H^0(\omega_Y \otimes \omega_X^* \otimes L^{-1})$ which generates $\omega_Y \otimes \omega_X^* \otimes L^{-1}$ except at finitely many points.

We have $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O}) \simeq (\omega_Y \otimes \omega_X^* \otimes L^{-1})/\mathcal{O}\xi$.

One sees immediately that if Y is l.c.i. everywhere, then $\mathcal{F}$ verifies condition (*).

Conversely we have:

<u>Proposition 4.</u> Let $\mathcal{F}$ be a rank 2 reflexive sheaf on a non-singular 3-fold X and S the singular set of $\mathcal{F}$. We assume $S \neq \emptyset$. For $s \in H^0(\mathcal{F})$ let Z(s) be the scheme of zeroes of s, determined by the ideal:

$$\mathcal{Y} = \mathrm{Im}(\mathcal{F}^* \xrightarrow{s^*} \mathcal{O}_X).$$

Then:

(a) $S \subseteq |Z(s)|$ for every $s \in H^0(\mathcal{F})$.

(b) Suppose $\mathcal{F}$ verifies (*). If $x \in S$ and $s \in H^0(\mathcal{F})$ then: $s(x) \neq 0 \Longrightarrow Z(s)$ is l.c.i. of codimension 2 at x.

Moreover, if $\mathcal{F}$ is generated by global sections then, for the generic section s in a finite dimensional subspace V of $H^0(\mathcal{F})$ which generates $\mathcal{F}$, Z(s) is l.c.i. of codimension 2.

<u>Proof.</u> (a). Let x be an element of S. Ker s_x^* is a reflexive $\mathcal{O}_x$-module of rank 1, hence Ker $s_x^* \simeq \mathcal{O}_x$. Thus we have an exact sequence :

$$0 \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{F}_x \xrightarrow{s_x^*} \mathcal{O}_x$$

If $x \notin Z(s)$ then s_x^* is surjective, hence $\mathcal{F}_x \simeq \mathcal{O}_x^2$ contradicting the fact that x is a singular point of $\mathcal{F}$.

(b) Put $A = \mathcal{O}_x$, $F = \mathcal{F}_x$.

The proof of Proposition 2 shows that there exists an A-sequence $a_1, a_2, a_3 \in \mathfrak{m}$ such that F is isomorphic to the submodule of A^3 generated by $(0,-a_3,a_2),(-a_3,0,a_1),(-a_2,a_1,0)$.

$s_x \in F$ has the form:

$$s_x = t_1(0,-a_3,a_2)+t_2(-a_3,0,a_1)+t_3(-a_2,a_1,0).$$

Dualizing the exact sequence $(I=Aa_1+Aa_2+Aa_3)$:

$$0 \longrightarrow F \overset{i}{\hookrightarrow} A^3 \longrightarrow I \longrightarrow 0$$

we get an exact sequence:

$$0 \longrightarrow A \longrightarrow A^3 \xrightarrow{i^*} F^* \longrightarrow 0$$

because $Ext^1(I,A)=0$. Hence i^* is surjective. It follows that:

$\mathcal{J}_x = \mathrm{Im}(F^* \xrightarrow{s_x^*} A) = A(t_2a_3+t_3a_2)+A(t_1a_3-t_3a_1)+A(t_1a_2+t_2a_1)$.

If $s(x)\neq 0$ then $t_i \notin \mathfrak{m}$ for some $i\in\{1,2,3\}$. Suppose, for example, $t_3 \notin \mathfrak{m}$. Then $\mathcal{J}_x = A(a_1-\frac{t_1}{t_3}a_3)+A(a_2+\frac{t_2}{t_3}a_3)$. Therefore $Z(s)$ is l.c.i. of codimension 2 at x.

Now, suppose $\mathcal{F}$ is generated by global sections. As $\mathcal{F}|X\setminus S$ is a rank 2 vector bundle generated by global sections it follows that for $s\in V$ generic $Z(s)$ is l.c.i. of codimension 2 at every point of $Z(s)\setminus S$.

The morphism $V \longrightarrow \mathcal{F}(x)$ being surjective it follows that for $s\in V$ generic $s(x)\neq 0$ for every $x\in S$, hence $Z(s)$ is l.c.i. of codimension 2 at every point of S.

The method we have used in order to prove Proposition 4 allows us to answer affirmatively a question stated in ([4], Remark 4.1.1):

<u>Proposition 5</u>. Let $\mathcal{F}$ be a rank 2 reflexive sheaf on a nonsingular 3-fold X and S the singular set of $\mathcal{F}$. We assume $S\neq\emptyset$. Suppose $\mathcal{F}$ verifies the condition:

For every point x in S $Ext^1(\mathcal{F}_x,\mathcal{O}_x)\simeq\mathcal{O}_x/(u,v,w^n)$ for some regular system of parameters u,v,w of $\mathcal{O}_x$ and some $n\geq 1$.

If $\mathcal{F}$ is generated by global sections then, for the generic section s in a finite dimensional subspace V of $H^0(\mathcal{F})$ which generates $\mathcal{F}$, $Z(s)$ is nonsingular of codimension 2.

<u>Proof.</u> For generic $s \in V$ $Z(s)$ is nonsingular of codimension 2 at every point of $Z(s) \setminus S$.

Now, let x be a point of S. We have to show that for generic $s \in V$, $Z(s)$ is nonsingular of codimension 2 at x.

One identifies $\mathcal{F}_x$ with a submodule of $\mathcal{O}_x^3$. Then $s_x \in \mathcal{F}_x$ has the form:

$$s_x = t_1(0, -w^n, v) + t_2(-w^n, 0, u) + t_3(-v, u, 0), \quad t_i \in \mathcal{O}_x$$

Let $\mathcal{J}_x$ be the ideal defining $Z(s)$ at x. If $t_3 \notin \mathfrak{m}$ then $\mathcal{J}_x = \mathcal{O}_x(u - \frac{t_1}{t_3} w^n) + \mathcal{O}_x(v + \frac{t_2}{t_3} w^n)$ hence $Z(s)$ is nonsingular of codimension 2 at x. It follows that then there exists a 2-dimensional subspace W of $\mathcal{F}(x) = \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x$ such that:

$$s(x) \notin W \Rightarrow Z(s) \text{ is nonsingular of codimension 2 at } x.$$

Since the morphism $V \longrightarrow \mathcal{F}(x)$ is surjective it follows that for the generic section $s \in V$ we have $s(x) \notin W$.

3.5. The method of extensions stated as in 1.1. or in 3.3 Proposition 3 might be useful in studying stable rank 3 vector bundles on $P = \mathbb{P}_3$. We shall illustrate this by two examples:

<u>Example 1.</u> Let F be a stable rank 2 vector bundle on P with $c_1(F) = -1, c_2(F) = 2d$ $(d \geq 1)$ and minimal spectrum $(-1, -1, \ldots, 0, 0)$. We have $H^2(P, F(-1)) = 0$. Let $Y \subset P$ be a l.c.i. curve not contained in a plane and such that $\det N_{Y|P} \otimes F(-1)$, considered as a vector bundle on Y, has a global section vanishing at no point.

Under these conditions there exists an extension:

$$0 \longrightarrow F \longrightarrow E \longrightarrow \mathcal{J}_Y(1) \longrightarrow 0$$

with E rank 3 vector bundle on P with $c_1(E) = 0$.

1. Suppose $H^0(F(1)) = 0, H^0(F(2)) \neq 0$ (there exist such bundles for every $d \geq 2$, [5] Ex.4.3.2). The first condition implies that E is stable and the second that we have an exact sequence:

$$0 \longrightarrow \mathcal{O}_P(-2) \longrightarrow F \longrightarrow \mathcal{J}_Z(1) \longrightarrow 0$$

Let $Y \subset P$ be a l.c.i. curve not contained in a plane such that $Y \cap Z = \emptyset$, det $N_{Y|P}(-3)$ is generated by global sections and $H^1(Y, \det N_{Y|P}(-3))=0$. Then det $N_{Y|P}(-3)$ is generated by global sections and one can realize the above construction.

On the other hand, for the generic line L we have $\det N_{L|P} \otimes F(-1) \simeq \mathcal{O}_L \oplus \mathcal{O}_L(1)$. Therefore if Y is a union of disjoint nonsingular curves of degree $\geqslant 2$ and of generic lines for F then the construction is possible.

2. Suppose $h^0(F(1))=1$. There exist such bundles for every $d \geqslant 1$. They can be realized, for example, as extensions:

$$0 \longrightarrow \mathcal{O}_P(-1) \longrightarrow F \longrightarrow \mathcal{I}_Z \longrightarrow 0$$

where Z is a union of d disjoint lines doubled such that $\omega_Z \simeq \mathcal{O}_Z(-3)$.

If Y is a union of disjoint nonsingular curves such that $Y \cap Z = \emptyset$, then the generic section of det $N_{Y|P} \otimes F(-1)$ determines a stable rank 3 vector bundle E with $c_1(E)=0$ ("generic" is needed to prove that $H^0(E^*)=0$).

<u>Example 2.</u> Let $\mathcal{F}$ be a stable rank 2 reflexive sheaf on P given by an exact sequence

$$0 \longrightarrow \mathcal{O}_P \xrightarrow{\ s\ } T_P(-1) \longrightarrow \mathcal{F}^* \longrightarrow 0$$

where s is a nonzero section of $T_P(-1)$. It follows that $h^0(\mathcal{F}^*)=3$ and if $t \in H^0(\mathcal{F}^*)$, $t \neq 0$, then the scheme of zeroes Z(t) is a line passing through the point x where s vanishes. The Chern classes of $\mathcal{F}$ are $c_1(\mathcal{F})=-1$, $c_2(\mathcal{F})=1$, $c_3(\mathcal{F})=1$ and the singular set of $\mathcal{F}$ is $\{x\}$.

Let $Y \subset P$ be a l.c.i. curve such that $\omega_Y(2)$ is generated by global sections and such that $x \notin Y$.

$H^2(P, \mathcal{F}(-1))=0$ and det $N_{Y|P}(-1) \otimes \mathcal{F} \simeq \omega_Y(3) \otimes \mathcal{F}$ is generated by global sections (because $\mathcal{F}(1)$ is) hence it has a global section vanishing at no point. By Proposition 3 there exists an extension:

(1) $$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow \mathcal{J}_{Y\cup x}(1) \longrightarrow 0$$

with E rank 3 vector bundle on P. The Chern classes of E are:

$$c_1(E)=0,\ c_2(E)=\deg Y,\ c_3(E)=2\deg Y-2\chi(\mathcal{O}_Y).$$

If $\deg Y \geqslant 2$ then we may suppose that $H^0(\mathcal{J}_{Y\cup x}(1))=0$, moving Y in case of need. It follows that $H^0(E)\neq 0$.

Dualizing (1) we get an exact sequence:

$$0 \to \mathcal{O}_P(-1) \longrightarrow E^* \longrightarrow \mathcal{F}^* \xrightarrow{\ \mathcal{E}\ } \omega_Y(3) \longrightarrow 0$$

$\mathcal{E}$ being the epimorphism which determines the extension (1).

$$H^0(E^*)=0 \iff H^0(\mathcal{E}) \text{ injective.}$$

Let $Y_1,\dots,Y_n$ be the connected components of Y. $\mathcal{E}$ is determined by epimorphisms $\mathcal{E}_i : \mathcal{F}^*|Y_i \longrightarrow \omega_{Y_i}(3)$, $i=1,\dots,n$.

$\det \mathcal{F}^* \simeq \mathcal{O}_P(1)$ hence we have exact sequences:

$$0 \to \omega_{Y_i}(2)^* \longrightarrow \mathcal{F}^*|Y_i \xrightarrow{\ \mathcal{E}_i\ } \omega_{Y_i}(3) \longrightarrow 0,\ i=1,\dots,n.$$

The morphisms $H^0(\mathcal{F}^*) \longrightarrow H^0(\mathcal{F}^*|Y_i)$ are injective, because $x \notin Y_i$.

If $H^0(\omega_{Y_i}(2)^*)=0$ for at least one $i\in\{1,\dots,n\}$ then $H^0(\mathcal{E}_i)$ is injective, hence $H^0(\mathcal{E})$ is injective.

If not, suppose $h^0(\omega_{Y_1}(2)^*)=1$ and $n\geqslant 2$. It follows that the kernel of the composite map:

$$H^0(\mathcal{F}^*) \longrightarrow H^0(\mathcal{F}^*|Y_1) \xrightarrow{\ H^0(\mathcal{E}_1)\ } H^0(\omega_{Y_1}(3))$$

has dimension $\leqslant 1$. Let $t\in H^0(\mathcal{F}^*)$ be a generator of this kernel. If $t\neq 0$ we can choose the epimorphism $\mathcal{E}_2 : \mathcal{F}^*|Y_2 \longrightarrow \omega_{Y_2}(3)$ such that $\mathcal{E}_2(t|Y_2)\neq 0$. It follows that $H^0(\mathcal{E})$ is injective.

Thus, in anyone of these two cases $H^0(E^*)=0$ and E is stable.

There are many families of curves Y satisfying the above conditions and one obtains in this way families of stable rank 3 vector bundles.

We mention here only two extreme cases:

If Y is an union of $d\geqslant 2$ disjoint lines not passing through x we obtain a stable rank 3 vector bundle E with Chern classes:

$c_1(E)=0, c_2(E)=d$, $c_3(E)=0$.

If Y is a nonsingular plane curve of degree $d\geq 2$ not passing through x we obtain a stable rank 3 vector bundle E in the family studied by Spindler[13], with Chern classes:

$c_1(E)=0$, $c_2(E)=d$, $c_3(E)=d^2-d$.

3.6. We return to the general discussion for a final remark. To be precise, we want to find out, under the conditions of 3.4., what it happens when $S\cap Y\neq \emptyset$.

Let X be a nonsingular 3-fold and $\mathcal{F}$ a rank 2 reflexive sheaf with $\mathcal{E}xt^1(\mathcal{F},\mathcal{O}_X)\simeq \mathcal{O}_S$ where S is a 0-dimensional subscheme of X. Let Z be a subscheme of X with $|Z|=|S|$.

We want to describe the elements ξ of $\mathrm{Ext}^1(\mathcal{J}_{Y\cup Z},\mathcal{F})$ which determines extensions:

$$0 \longrightarrow \mathcal{F} \longrightarrow E_\xi \longrightarrow \mathcal{J}_{Y\cup Z} \longrightarrow 0$$

with E_ξ locally free and to find conditions on F,Y,Z under which such extensions exist.

For the simplicity of the statements we assume that Y and Z are locally complete intersection. Firstly, we need a local result:

Lemma Let A be a regular local ring of dimension 3 and F a reflexive A-module such that $\mathrm{Ext}^1(F,A)\simeq A/I$ with I ideal of height 3. Let I_o and I_1 be complete intersection ideals of height 3 and 2, respectively. Then there exists an extension:

$$0 \longrightarrow F \longrightarrow A^3 \longrightarrow I_o\cap I_1 \longrightarrow 0$$

if and only if : $I_o\cap I_1$ is generated by 3 elements, $II_1\subseteq I_o$ and

$$\ell(A/I_o)=\ell(A/I_o+I_1)+\ell(A/I)$$

Proof. Suppose there exists an extension as above. We obtain an isomorphism

$$\mathrm{Ext}^1(F,A)\simeq \mathrm{Ext}^2(I_o\cap I_1,A).$$

On the other hand, we have:

$$\mathrm{Ext}^2(I_o\cap I_1,A)\simeq \mathrm{Ext}^3(I_1/I_o\cap I_1,A)\simeq \mathrm{Ext}^3(I_o+I_1/I_o,A)$$

and an exact sequence relating the dualizing modules:

$$0 \longrightarrow \omega_{A/I_o+I_1} \longrightarrow \omega_{A/I_o} \simeq A/I_o \longrightarrow \mathrm{Ext}^3(I_o+I_1/I_o, A) \longrightarrow 0$$

Furthermore, $\omega_{A/I_o+I_1} = (0:I_1)_{A/I_o}$ and $\ell(\omega_{A/I_o+I_1}) = \ell(A/I_o+I_1)$. One checks easily the three conditions in the statement.

Now, we prove the converse. $I_o \cap I_1$ is generated by 3 elements hence we have an exact sequence:

$$0 \longrightarrow G \longrightarrow A^3 \longrightarrow I_o \cap I_1 \longrightarrow 0$$

with G reflexive module of rank 2. We have $\mathrm{Ext}^1(G,A) \simeq \mathrm{Ext}^2(I_o \cap I_1, A) \simeq \mathrm{Ext}^3(I_o+I_1/I_o, A)$. The assumptions we made implies that $\mathrm{Ext}^3(I_o+I_1/I_o, A)$ is an A/I-module of length $\ell(A/I)$ generated by a single element, hence $\mathrm{Ext}^1(G,A) \simeq A/I$. From Proposition 2 it follows that $G \simeq F$.

We go back to geometry. From the lemma it follows that, in order to solve our problem, it is necessary to assume that $Y \cup Z$ is locally generated by 3 elements, $\mathcal{J}_S \mathcal{J}_Y \subseteq \mathcal{J}_Z$ and

$$\ell_x(Z) = \ell_x(Y \cap Z) + \ell_x(S) \quad \text{for every } x \in X.$$

For every $x \in S$ we choose an extension:

$$(\mathcal{E}_x) \quad 0 \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}_x^3 \longrightarrow \mathcal{F}_x \longrightarrow 0$$

and we consider the Yoneda map associated to it:

$$\mathcal{E}xt^1(\mathcal{J}_{Y \cup Z}, \mathcal{F})_x \longrightarrow \mathcal{E}xt^2(\mathcal{J}_{Y \cup Z}, \mathcal{O}_X)_x \simeq \mathcal{O}_{S,x}$$

We denote by $\mathcal{E}(x): \mathrm{Ext}^1(\mathcal{J}_{Y \cup Z}, \mathcal{F}) \longrightarrow k(x)$ the induced map. On the other hand we choose for every $x \in Y$ three generators τ_1, τ_2, τ_3 of $\mathcal{F}_x^*$. We have $\mathcal{E}xt^1(\mathcal{J}_{Y \cup Z}, \mathcal{O}_X) \simeq \mathcal{E}xt^1(\mathcal{J}_Y, \mathcal{O}_X) \simeq \det N_{Y|X}$

As above, we can define morphisms

$$\tau(x): \mathrm{Ext}^1(\mathcal{J}_{Y \cup Z}, \mathcal{F}) \longrightarrow k(x)^3.$$

Now, let ξ be an element of $\mathrm{Ext}^1(\mathcal{J}_{Y \cup Z}, \mathcal{F})$. E_ξ is locally free if and only if $\mathcal{E}(x)(\xi) \neq 0$ for every $x \in S$ and $\tau(x)(\xi) \neq 0$ for every $x \in Y$. This assertion results analysing the long exact sequence of $\mathcal{E}xt^{\circ}(.,\mathcal{O}_X)$ associated to the extension.

If $x \in Y \setminus S$ the condition $\tau(x)(\xi)\neq 0$ is equivalent to the fact that x is not a zero of the section of $(F|Y\setminus S) \otimes \det N_{Y\setminus S|X\setminus S}$ determined by ξ. We also observe that the morphisms $\varepsilon(x)$ and $\tau(x)$ are different from 0 because there exist extensions:

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{O}_x^3 \longrightarrow \mathcal{I}_{Y\cup Z,x} \longrightarrow 0$$

Consequently, if $X,\mathcal{F},Y,Z$ are as above, $H^2(X,\mathcal{F})=0$ and $\mathcal{E}xt^2(\mathcal{O}_{Y\cup Z},\mathcal{F})$ is generated by global sections then there exist extensions:

$$0 \longrightarrow \mathcal{F} \longrightarrow E \longrightarrow \mathcal{I}_{Y\cup Z} \longrightarrow 0$$

with E locally free.

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