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ABSTRACT

For $f:\{0,1\}^n \to R=(-\infty,+\infty)$, in §2 we introduce and study "tight extensions" $\bar{f}^{\mathcal{D}}:[0,1]^n \to R$, defined on each n-simplex D_i of a triangulation $\mathcal{D}$ of $[0,1]^n$, with all vertices in $\{0,1\}^n$, as the unique affine function which interpolates f at the vertices of D_i. In §3 we study convexity of tight extensions. In §4 we show the existence of polyhedral convex (generally, non-tight) extensions. As applications, in §5 we give some duality theorems for minimization and maximization of submodular functions and in §6 (Appendix) we obtain new insight into the "greedy solutions" of a certain linear maximization problem.

1. INTRODUCTION

A combinatorial optimization problem can be formulated (see e.g. [2]) as the problem of minimizing or maximizing a real-valued function f defined on the family $2^{\{1,\dots,n\}}$ of all subsets of the set $\{1,\dots,n\}$ (where $n<+\infty$), including the empty set $\emptyset$. Since there is a natural one-to-one mapping

$$S \to \sum_{i\in S} e_i \qquad (S\subseteq\{1,\dots,n\}), \qquad (1.1)$$

where $e_i=\{\underbrace{0,\dots,0}_{i-1},1,0,\dots,0\}$ $(i=1,\dots,n)$ and $\sum_{i\in\emptyset} e_i=0$ (thus, $\sum_{i\in S} e_i$ is the "incidence vector", or "characteristic vector", of the set S), of the set $2^{\{1,\dots,n\}}$ onto the set C_n of all vertices of the "0 1 cube" in R^n, i.e., onto

$$C_n=\{0,1\}^n=\{x=\{\xi_i\}\in R^n \mid \xi_i=0 \text{ or } 1 \ (i=1,\dots,n)\}, \tag{1.2}$$

one can regard (see e.g. [10], p.248) each function $f:2^{\{1,\dots,n\}}\to R=(-\infty,+\infty)$ as a function $f:C_n\to R$ (a "pseudo-Boolean function", in the sense of [9]), defining

$$f(\sum_{i\in S} e_i)=f(S) \qquad (S\subseteq\{1,\dots,n\}); \tag{1.3}$$

in this way, each combinatorial optimization problem (in the above sense) is identified with a "0-1 optimization problem", i.e., a problem of minimizing or maximizing a function $f:C_n\to R$.

In the present paper we shall obtain some results on extensions of functions $f:C_n\to R$ to functions $\overline{f}:\text{co } C_n\to R$, where co C_n is the convex hull of C_n, i.e., the 0-1 cube

$$\text{co } C_n=[0,1]^n=\{x=\{\xi_i\}\in R^n \mid 0\leq\xi_i\leq 1 \ (i=1,\dots,n)\}, \tag{1.4}$$

and we shall give some applications to minimization and maximization of functions $f:C_n\to R$ which admit extensions with certain properties (in particular, of submodular functions). Thus, these are "non-Boolean approaches", in the sense of [9], p.IX, to some extension problems and some combinatorial optimization problems. Our results can be also extended to functions with values in $\overline{R}=R\cup\{-\infty,+\infty\}$, with the usual methods, but here we shall consider only R-valued functions.

In §2 we shall define and study "tight extensions" of $f:C_n\to R$ which can be described, roughly speaking, as follows: Take a "v-triangulation" of co C_n, i.e., a subdivision $\mathcal{D}=\{D_1,\dots,D_p\}$ of co C_n into simplices of dimension n, with all vertices belonging to C_n and such that the intersections $D_i\cap D_j$ are either empty, or a face of D_i and D_j. Next, for each simplex D_i of $\mathcal{D}$, extend f to the unique affine function Ψ_i on D_i, which interpolates the values of f at the n+1 vertices of D_i , and then "glue together" these pieces of extensions to a function $\bar{f}=\bar{f}^{\mathcal{D}}$:co $C_n \to R$, which we shall call the "tight extension" of f associated to the v-triangulation $\{D_1,\ldots,D_p\}$ of co C_n. We shall show that each such $\bar{f}$ is continuous and that, for a certain extension $\hat{f}:R_+^n \to R$ of a function $f:C_n \to R$, defined by Lovász [10], $\hat{f}|_{\text{co } C_n}$ coincides with the tight extension of f associated to the "standard" v-triangulation of co C_n into n! simplices, given in [11].

In §3, in order to study convex $\bar{f}^{\mathcal{D}}$'s , to any tight extension $\bar{f}^{\mathcal{D}}$:co $C_n \to R$ of $f:C_n \to R$ we shall associate a polyhedral convex function $\tilde{f}^{\mathcal{D}}:R^n \to R$ (namely, the maximum, at each $x \in R^n$, of the unique interpolating affine functions Ψ_i occuring in the definition of $\bar{f}^{\mathcal{D}}$), such that $\bar{f}^{\mathcal{D}} \leq \tilde{f}^{\mathcal{D}}$ on co C_n. We shall show that $\bar{f}^{\mathcal{D}}$ is convex if and only if $\bar{f}^{\mathcal{D}}=\tilde{f}^{\mathcal{D}}$ on co C_n , or, equivalently, $f=\tilde{f}^{\mathcal{D}}$ on C_n , or, equivalently, each $\Psi_i \leq f$ on C_n . Furthermore, for the "Lovász extension" $\hat{f}$ of f, mentioned above, we shall show that these properties hold if and only if f is submodular; this yields a sharpening (and a proof) of a result stated by Lovász ([10], p.249), according to which $\hat{f}$ is convex if and only if f is submodular. Moreover, we shall show that the standard v-triangulation of co C_n has a subset of cardinality $\binom{n}{[\frac{n}{2}]}$, such that if f is submodular, then it is the maximum of the unique affine functions which interpolate f at the vertices of the simplices belonging to this subset. Finally, we shall show that other v-triangulations of co C_n are also of interest, e.g., when n=2, the tight extension $\bar{f}^{\mathcal{D}}$ of f associated to the only non-standard v-triangulation $\mathcal{D}$ of co C_2 is (polyhedral) convex if and only if f is supermodular. From our results on submodular functions and convex functions there follow, mutatis mutandis, corresponding results on supermodular functions and on concave functions, and vice versa. We shall show that these results also throw some more light on an interesting phenomenon, described by Lovász in [10], p.249, in the following terms: "Are submodular set-functions more like convex of like concave functions? In this section we discuss some properties of them which are analogous to properties of convex functions; in the next section we shall survey some aspects of submodularity which relate it to concavity. The reader may then decide how he or she would answer the question above".

In §4 we shall show that every function $f:C_n \to R$ admits a polyhedral convex extension $\overline{f}:\text{co } C_n \to R$ (which need not be tight). Actually, we shall express this extension result in terms of a "finite" strengthening of the concept of "convexity of f with respect to a family W of functions", in the sense of [5], namely, in our case, the family $W=\mathcal{M}_n$ of all modular functions on C_n. Furthermore, we shall show that every supermodular function $f:C_3 \to R$ can be represented as the maximum of two affine functions.

In §5 we shall give some results on minimization and maximization of functions $f:C_n \to R$. Firstly, we shall show that the minimum (maximum) of f on C_n coincides with that of any tight extension $\overline{f}=\overline{f}^{\mathcal{D}}$ on co C_n; this will make it possible, in some cases, to study the problem of minimizing (maximizing) f on C_n with the aid of the problem of minimizing (maximizing) $\overline{f}$ on the convex set co C_n. Thus, using this result and an extension theorem of §3, and applying some of our results of [16]-[18] on convex optimization, to $\tilde{f}^{\mathcal{D}}:R^n \to R$, we shall obtain some new "duality theorems" for minimization and maximization of functions $f:C_n \to R$ which admit a convex tight extension, e.g., of submodular functions (these have the additional interest that the problem of maximization of submodular functions $f:C_n \to R$ is known to be NP-hard).

Finally, in §6 (Appendix), using some of the preceding results, we shall give a new geometric interpretation of the "greedy solutions" of the linear maximization problem on the polyhedron associated to a function $f:C_n \to R$ with $f(0)=0$, studied in [10], and a result on greedy solutions (which yields also a new proof of their optimality in the submodular case, without using the dual linear minimization problem).

Let us recall now some terminology and notations which are used in this paper. A function $f:C_n \to R$ is called a) submodular, if for all $S,T \subseteq \{1,\dots,n\}$ we have

$$f(\sum_{i\in S\cap T} e_i)+f(\sum_{i\in S\cup T} e_i) \leqslant f(\sum_{i\in S} e_i)+f(\sum_{i\in T} e_i); \qquad (1.5)$$

b) subadditive, if for all $S,T \subseteq \{1,\dots,n\}$ with $S\cap T \neq \emptyset$, we have

$$f(\sum_{i\in S\cup T} e_i) \leqslant f(\sum_{i\in S} e_i)+f(\sum_{i\in T} e_i); \qquad (1.6)$$

c) supermodular, if for all $S,T \subseteq \{1,\dots,n\}$ we have the opposite inequality $\geqslant$ to (1.5); d) modular, if it is simultaneuously submodular and supermodular (i.e., if for any $S,T \subseteq \{1,\dots,n\}$ the equality holds in (1.5)). A function $g:R^n \to R$ (respectively, $g:R^n_+ \to R$, where $R^n_+=\{x=\{\xi_i\}\in R^n \mid \xi_1,\dots,\xi_n \geqslant 0\}$) is called a) subadditive, if for all $x,y \in R^n$ (respectivelly, all $x,y \in R^n_+$) we have

$$g(x+y) \leqslant g(x)+g(y); \qquad (1.7)$$

b) positively homogeneous, if for all $x \in R^n$ (respectively, $x \in R^n_+$) and $\lambda \geqslant 0$,

$$g(\lambda x)=\lambda g(x); \qquad (1.8)$$

c) linear, if for all $x,y \in R^n$ the equality sign holds in (1.7) and if we have (1.8) for all $x \in R^n$ and $\lambda \in R$; d) affine, if $g=\Phi+c$, with $\Phi \in (R^n)^*$ (the linear space of all linear functions on R^n) and $c \in R$, where we identify R with the family of all constant functions on R and where $+$ means pointwise addition on R^n, i.e., $g(x)=\Phi(x)+c$ $(x\in R^n)$. For any functions $f_i:R^n \to R$ $(i \in I)$, the function $\sup_{i\in I} f_i$ is defined pointwise on R^n, i.e., $(\sup_{i\in I} f_i)(x)=\sup_{i\in I} f_i(x)$ $(x \in R^n)$ (and, similarly for inf, max and min, where max and min denote a sup, respectively inf, which are attained). We assume familiarity with the terms and results of convexity theory (of sets and functions), which we shall employ freely.

We shall denote by $g|_A$ the restriction of a function g to a set A. For any set $A \subseteq R^n$, co A denotes the convex hull of A. For any set S, $|S|$ and S^n denote, respectively, the cardinality of S and the cartesian product $\underbrace{S\times\dots\times S}_{n}$. As usual, Z denotes the set of all integers (so Z^n is the set of all $x=\{\xi_i\}\in R^n$ with integer coordinates $\xi_1,\ldots,\xi_n$), $\binom{n}{k}=\frac{n!}{k!(n-k)!}$, where $n!=1.2\ \ldots\ (n-1)n$, and, for any $\alpha\in R$, $[\alpha]$ denotes the integer part of α. Finally, we shall denote by Π_n the family of all permutations π of the set $\{1,\ldots,n\}$.

2. TIGHT EXTENSIONS OF A FUNCTION $f:C_n\to R$

Lovász ([10], § 3) has observed that any $x\in R_+^n\setminus\{0\}$ can be written uniquely in the form

$$x=\sum_{k=1}^{\ell}\lambda_k\sum_{j\in S_k}e_j\ , \qquad (2.1)$$

where $\lambda_1,\ldots,\lambda_\ell>0$ and $S_1\subset\ldots\subset S_\ell\subseteq\{1,\ldots,n\}$, and hence any function $f:C_n\to R$ with $f(0)=0$ can be extended to $\hat{f}:R_+^n\to R$, by $\hat{f}(0)=0$ and

$$\hat{f}(x)=\sum_{k=1}^{\ell}\lambda_k f(\sum_{j\in S_k}e_j)\quad (0\neq x=\sum_{k=1}^{\ell}\lambda_k\sum_{j\in S_k}e_j\in R_+^n); \qquad (2.2)$$

indeed, $\hat{f}$ is well defined (due to the uniqueness of (2.1)) and $\hat{f}(x)=f(x)$ for all $x\in C_n$. The function $\hat{f}$ is called [8] the Lovász extension of f. As has been stated by Lovász ([10], proposition 4.1), $\hat{f}$ is convex (concave) if and only if f is submodular (supermodular). As a consequence [10], $\hat{f}$ is linear if and only if f is modular. Conversely, the restriction of a linear function to C_n is modular, but the restriction of a convex function to C_n need not be submodular [10].

In the present section we shall define and study piecewise affine "tight extensions" $\bar{f}:\text{co}\ C_n\to R$ of a function $f:C_n\to R$, which will encompass $\hat{f}|_{\text{co}\ C_n}$ as a special case and which will be useful in the sequel. To this end, we shall use another approach, via v-triangulations of co C_n and interpolating affine functionals. This will also yield a new representation of the Lovász extension $\hat{f}$.

Definition 2.1. A finite family $\mathcal{D}=\{D_i\}_{i=1}^p$ of n-simplices $D_i\subset \operatorname{co} C_n$ is called

a) a triangulation (see e.g. [15]) of co C_n, if (i) $\operatorname{co} C_n = \bigcup_{i=1}^p D_i$; (ii) for $i\neq j$, either $D_i\cap D_j=\emptyset$, or $D_i\cap D_j$ is a face of both D_i and D_j.

b) a v-triangulation of co C_n if, in addition, its vertices belong to C_n, i.e., $D_i=\operatorname{co}\{x_1^i,\dots,x_{n+1}^i\}$, with (affinely independent) $x_1^i,\dots,x_{n+1}^i\in C_n$ $(i=1,\dots,p)$.

In the sequel we shall consider only v-triangulations of co C_n.

The "standard v-triangulation" of co C_n is given in

Lemma 2.1. The family $\mathcal{B}=\{B_\pi\}_{\pi\in\Pi_n}$ of the n! simplices of the form

$$B_\pi=\operatorname{co}\left\{0, \sum_{i=1}^{k} e_{\pi(i)}\right\}_{k=1}^n \qquad (\pi\in\Pi_n), \qquad (2.3)$$

is a v-triangulation of co C_n.

Proof. We claim that

$$B_\pi=\{x=\{\xi_i\}\in R^n \mid 0\leq\xi_{\pi(n)}\leq\dots\leq\xi_{\pi(1)}\leq 1\} \qquad (\pi\in\Pi_n), \qquad (2.4)$$

so $\mathcal{B}=\{B_\pi\}_{\pi\in\Pi_n}$ coincides with the "standard triangulation" of co C_n, given (without proof) in [11]. Indeed, for each $x\in B_\pi$ we have $x=\sum_{k=1}^n\lambda_k\sum_{i=1}^k e_{\pi(i)}=\sum_{k=1}^n(\sum_{i=k}^n\lambda_i)e_{\pi(k)}=\sum_{k=1}^n \xi_{\pi(k)}e_{\pi(k)}$, where $\lambda_1,\dots,\lambda_n\geq 0$, $\sum_{k=1}^n\lambda_k\leq 1$, which proves the inclusion $\subseteq$ in (2.4). Conversely, if $x=\{\xi_i\}\in R^n$, $0\leq\xi_{\pi(n)}\leq\dots\leq\xi_{\pi(1)}\leq 1$, where $\pi\in\Pi_n$, then for $\lambda_k=\xi_{\pi(k)}-\xi_{\pi(k+1)}$ $(k=1,\dots,n-1)$, $\lambda_n=\xi_{\pi(n)}$, we have $\lambda_1,\dots,\lambda_n\geq 0$, $\sum_{k=1}^n\lambda_k=\xi_{\pi(1)}\leq 1$ and $x=\sum_{k=1}^n\xi_{\pi(k)}e_{\pi(k)}=\sum_{k=1}^n(\sum_{i=k}^n\lambda_i)e_{\pi(k)}=\sum_{k=1}^n\lambda_k\sum_{i=1}^k e_{\pi(i)}\in B_\pi$, proving (2.4).

Now, clearly, $\mathcal{B}$ satisfies b) of definition 2.1 and $\bigcup_{\pi\in\Pi_n} B_\pi \subseteq$ $\subseteq \text{co } C_n$. Conversely, if $x=\{\xi_i\} \in (\text{co } C_n)\setminus\{0\}$, then for any $\pi\in\Pi_n$ such that $0 \leq \xi_{\pi(n)} \leq \ldots \leq \xi_{\pi(1)} \leq 1$, we have, by (2.4), $x \in B_\pi$, which proves (i). Furthermore, each $B_\pi \in \mathcal{B}$ contains the "main diagonal" of co C_n, i.e.,

$$\left\{\lambda \sum_{i=1}^{n} e_i \,\middle|\, 0 \leq \lambda \leq 1\right\} \subset B_\pi \qquad (\pi\in\Pi_n), \tag{2.5}$$

and thus $B_\pi \cap B_{\pi'} \neq \emptyset$ $(\pi\neq\pi')$. Also, for any $\pi, \pi' \in \Pi_n$ we have, by (2.3) and since $\{e_j\}$ is a basis of R^n,

$$B_\pi \cap B_{\pi'} = \text{co}\left\{0, \sum_{i=1}^{k} e_{\pi(i)}\right\}_{k\in M} = \text{co}\left\{0, \sum_{i=1}^{k} e_{\pi'(i)}\right\}_{k\in M}, \tag{2.6}$$

where $M=\left\{k\in\{1,\ldots,n\} \,\middle|\, \sum_{i=1}^{k} e_{\pi(i)} = \sum_{i=1}^{k} e_{\pi'(i)}\right\}$. But, since B_π and $B_{\pi'}$ of (2.3) are simplices, (2.6) is a face of both B_π and $B_{\pi'}$, which proves (ii).

Remark 2.1.a) For any x of the form (2.1), where $\lambda_1, \ldots, \lambda_\ell >$ > 0, $\sum_{k=1}^{\ell} \lambda_k \leq 1$ and $S_1 \subset \ldots \subset S_\ell \subseteq \{1,\ldots,n\}$, and any $\pi\in\Pi_n$ such that $S_k = \{\pi(1),\ldots,\pi(|S_k|)\}$ $(k=1,\ldots,\ell)$, we have $x\in B_\pi$. Conversely, for the unique representation (2.1) of any $x \in B_\pi$, see (2.16) below.

b) For $\mathcal{B}$ above, expressed in the equivalent language of maximal chains (3.35), it has been stated in [8], lemma 4.4, that $\mathcal{B}$ is a "simplicial subdivision" of co C_n, without any definition of this term.

Lemma 2.2. For any $f: C_n \to R$ and any affinely independent $x_1,\ldots,x_{n+1} \in R^n$ there exists a unique affine function $\Psi = \Phi + c: R^n \to$ $\to R$ (where $\Phi \in (R^n)^*$, $c\in R$), such that

$$\Psi(x_k) = \Phi(x_k) + c = f(x_k) \qquad (k=1,\ldots,n+1). \tag{2.7}$$

Proof. Since $x_1,\ldots,x_{n+1}$ are affinely independent in R^n, each

$x \in R^n$ can be written, uniquely (see e.g. [13], p.7), in the form $x=\sum_{k=1}^{n+1} \lambda_k x_k$, where $\lambda_1,\ldots,\lambda_{n+1} \in R$, $\sum_{k=1}^{n+1} \lambda_k=1$ (the "barycentric coordinates" of x). Define $\Psi: R^n \to R$ by

$$\Psi(x)=\sum_{k=1}^{n+1} \lambda_k f(x_k) \qquad (x=\sum_{k=1}^{n+1} \lambda_k x_k \in R^n, \sum_{k=1}^{n+1} \lambda_k=1). \tag{2.8}$$

Then, it is easy to check that Ψ is affine and, clearly, Ψ satisfies (2.7). Finally, if $\Psi_i=\Phi_i+c_i: R^n \to R$ $(i=1,2)$ are two affine functions satisfying (2.7) (where $\Phi_i \in (R^n)^*$, $c_i \in R$), then for any $x=\sum_{k=1}^{n+1} \lambda_k x_k \in R^n$, where $\sum_{k=1}^{n+1} \lambda_k=1$, we have

$$\Psi_i(x)=\Phi_i(\sum_{k=1}^{n+1} \lambda_k x_k)+\sum_{k=1}^{n+1} \lambda_k c_i=\sum_{k=1}^{n+1} \lambda_k \Psi_i(x_k)=\sum_{k=1}^{n+1} \lambda_k f(x_k) \qquad (i=1,2),$$

whence $\Psi_1=\Psi_2$, which completes the proof.

<u>Remark 2.2</u>. If $0 \in \{x_k\}_{k=1}^{n+1}$, then, by (2.7), $c=f(0)$. Hence, if also <u>$f(0)=0$, then</u> $c=0$, so $\Psi=\Phi \in (R^n)^*$.

<u>Definition 2.2</u>. a) For any function $f: C_n \to R$ and any v-triangulation $\mathcal{D}=\{D_i\}_{i=1}^p$ of co C_n, we define <u>the tight extension $\bar{f}^{\mathcal{D}}$ of f associated to $\mathcal{D}$</u>, by

$$\bar{f}^{\mathcal{D}}(x)=\Psi_i(x) \qquad (x \in D_i;\ i=1,\ldots,p), \tag{2.9}$$

where $\Psi_i=\Phi_i+c_i$ (with $\Phi_i \in (R^n)^*$, $c_i \in R$) is the unique affine function on R^n which coincides with f at the vertices of D_i $(i=1,\ldots,p)$; that is, if $D_i=\text{co}\,\{x_1^i,\ldots,x_{n+1}^i\}$, then

$$\Psi_i(x_k^i)=f(x_k^i) \qquad (k=1,\ldots,n+1;\ i=1,\ldots,p); \tag{2.10}$$

since D_i has n+1 vertices $x_1^i,\ldots,x_{n+1}^i$, we shall say that Ψ_i <u>extends tightly</u> f <u>to</u> D_i, or that Ψ_i <u>interpolates</u> f <u>at</u> $x_1^i,\ldots,x_{n+1}^i$.

b) We shall say that $\bar{f}: \text{co}\, C_n \to R$ is a <u>tight extension</u> of f, if there exists a v-triangulation $\mathcal{D}$ of co C_n, such that $\bar{f}=\bar{f}^{\mathcal{D}}$.

Remark 2.3. a) Since $\mathrm{co}\, C_n = \bigcup_{i=1}^{p} D_i$, $\overline{f}^{\mathcal{D}}$ is well defined at each $x \in \mathrm{co}\, C_n$ which belongs to only one of the D_i's. On the other hand, assume now that $x \in D_i \cap D_j$, where $i \neq j$. Then, by (ii) of definition 2.1, x is a convex combination of the vertices of $D_i \cap D_j$, hence of the common vertices of D_i and D_j, say $x = \sum_{k=1}^{\ell} \lambda_k x_k$, where $1 \leqslant \ell \leqslant n+1$, $\lambda_1, \dots, \lambda_\ell > 0$, $\sum_{k=1}^{\ell} \lambda_k = 1$ and where each x_k is a common vertex of $D_i = \mathrm{co}\, \{x_1^i, \dots, x_{n+1}^i\}$ and $D_j = \mathrm{co}\, \{x_1^j, \dots, x_{n+1}^j\}$. Thus, by (2.10), $\Psi_i(x_k) = f(x_k) = \Psi_j(x_k)$ $(k=1,\dots,\ell)$, whence, writing $\Psi_i = \Phi_i + c_i$, where $\Phi_i \in (R^n)^*$, $c_i \in R$ (and similarly for Ψ_j), and using $\sum_{k=1}^{\ell} \lambda_k = 1$, we obtain

$$\Psi_i(x) = \Phi_i\Big(\sum_{k=1}^{\ell} \lambda_k x_k\Big) + c_i = \sum_{k=1}^{\ell} \lambda_k \Psi_i(x_k) = \sum_{k=1}^{\ell} \lambda_k f(x_k) = \sum_{k=1}^{\ell} \lambda_k \Psi_j(x_k) = \Psi_j(x),$$

which proves that $\overline{f}^{\mathcal{D}}$ of (2.9) is well defined on $\mathrm{co}\, C_n$. Note that, by (2.9), $\overline{f}^{\mathcal{D}}$ is a piecewise affine function on $\mathrm{co}\, C_n$. Also, by (2.9) and (2.10), we have $\overline{f}^{\mathcal{D}}(x) = f(x)$ $(x \in C_n)$, so $\overline{f}^{\mathcal{D}}$ is indeed an extension of f. Finally, let us observe that some of the Ψ_i's in (2.9) may coincide (e.g., when $\overline{f}^{\mathcal{D}}$ is affine).

b) Geometrically, definition 2.2 a) means that the graph (in R^{n+1}) of $\overline{f}^{\mathcal{D}}$ is obtained as follows: On each simplex $D_i = \mathrm{co}\, \{x_1^i, \dots, x_{n+1}^i\}$ of the v-triangulation $\mathcal{D}$ of $\mathrm{co}\, C_n$, Graph $\overline{f}^{\mathcal{D}}$ is the portion of the unique hyperplane passing through $(x_1^i, f(x_1^i)), \dots, (x_{n+1}^i, f(x_{n+1}^i))$ (which is, actually, their convex hull).

Some properties of tight extensions are given in

Theorem 2.1. a) For any $f: C_n \to R$ and any v-triangulation $\mathcal{D} = \{D_i\}_{i=1}^{p}$ of $\mathrm{co}\, C_n$, say $D_i = \mathrm{co}\, \{x_1^i, \dots, x_{n+1}^i\}$ $(i=1,\dots,p)$, $\overline{f}^{\mathcal{D}}$ satisfies (where $\lambda_1, \dots, \lambda_{n+1} \geqslant 0$, $\sum_{k=1}^{n+1} \lambda_k = 1$)

$$\overline{f}^{\mathcal{D}}(x) = \sum_{k=1}^{n+1} \lambda_k f(x_k^i) \qquad \Big(x = \sum_{k=1}^{n+1} \lambda_k x_k^i \in D_i\ ;\ i=1,\dots,p\Big). \tag{2.11}$$

b) Each tight extension $\overline{f}=\overline{f}^{\mathcal{D}}$ is continuous on co C_n.

Proof. a) follows from (2.9) and (2.8).

b) Let $x \in$ co C_n, $\{x_k\} \subset$ co C_n, $x_k \to x$. Then, by co $C_n = \bigcup_{i=1}^{p} D_i$, there exists $i \leq p$ such that D_i contains an infinite subsequence of $\{x_k\}$, say $\{x_{k_m}\}$; then $x_{k_m} \to x$, whence, since D_i is closed, we get $x \in D_i$. Hence, by (2.9) and since each affine function $\Psi_i : R^n \to R$ is continuous,

$$\overline{f}(x_{k_m}) = \Psi_i(x_{k_m}) \to \Psi_i(x) = \overline{f}(x).$$

Now we give a new representation of the Lovász extension.

Theorem 2.2. Let $f: C_n \to R$ be a function with $f(0)=0$ and with Lovász extension $\hat{f}: R^n_+ \to R$ and let $\mathcal{B} = \{B_\pi\}_{\pi \in \Pi_n}$ be as in lemma 2.1. Then

$$\hat{f}|_{\text{co } C_n} = \overline{f}^{\mathcal{B}} \tag{2.12}$$

and, moreover,

$$\hat{f}(x) = \Phi_\pi(x) \qquad (x \in K_\pi = \{\lambda B_\pi \mid 0 \leq \lambda < +\infty\},\ \pi \in \Pi_n), \tag{2.13}$$

where $\Phi_\pi \in (R^n)^*$ is the unique linear function such that

$$\Phi_\pi\left(\sum_{i=1}^{k} e_{\pi(i)}\right) = f\left(\sum_{i=1}^{k} e_{\pi(i)}\right) \qquad (k=1,\ldots,n). \tag{2.14}$$

Proof. By (2.3) and remark 2.2, for each $\pi \in \Pi_n$, the affine function Ψ_π occurring in the definition of $\overline{f}^{\mathcal{B}}$ is nothing else than the unique linear function $\Phi_\pi \in (R^n)^*$ satisfying (2.14) and, by definition 2.2 a), we have

$$\overline{f}^{\mathcal{B}}(x) = \Phi_\pi(x) \qquad (x \in B_\pi,\ \pi \in \Pi_n). \tag{2.15}$$

Now, if $\pi \in \Pi_n$ and $x \in \lambda B_\pi = \lambda \,\text{co}\,\{0, \sum_{i=1}^{k} e_{\pi(i)}\}_{k=1}^{n}$, where $0 \leq \lambda < +\infty$, then there exist $\lambda_1, \ldots, \lambda_n \geq 0$ such that

$$x = \sum_{k=1}^{n} \lambda_k \sum_{i=1}^{k} e_{\pi(i)} = \sum_{k \in I_x} \lambda_k \sum_{i=1}^{k} e_{\pi(i)}, \tag{2.16}$$

where $I_x = \{k \leq n \mid \lambda_k > 0\}$. Then, the second part of (2.16) is nothing else than the unique representation (2.1) of x, with $\ell=|I_x|$ and $S_k=\{\pi(1),\ldots,\pi(k)\}$ $(k=1,\ldots,\ell)$. Hence, by (2.2) (for this case), (2.14) and $\Phi_\pi\in(R^n)^*$, we obtain

$$\hat{f}(x)=\sum_{k\in I_x}\lambda_k f(\sum_{i=1}^{k}e_{\pi(i)})=\sum_{k\in I_x}\lambda_k\Phi_\pi(\sum_{i=1}^{k}e_{\pi(i)})=$$

$$=\Phi_\pi(\sum_{i\in I_x}\lambda_k\sum_{i=1}^{k}e_{\pi(i)})=\Phi_\pi(x),$$

which proves (2.13).

Remark 2.4. a) The function $\Phi_\pi\in(R^n)^*$ defined by

$$\Phi_\pi(x)=\xi_{\pi(1)}f(e_{\pi(1)})+\sum_{k=2}^{n}\xi_{\pi(k)}\{f(\sum_{i=1}^{k}e_{\pi(i)})-f(\sum_{i=1}^{k-1}e_{\pi(i)})\}$$

$$(x=\sum_{i=1}^{n}\xi_i e_i\in R^n), \tag{2.17}$$

satisfies (2.14) (even when $f(0)\neq 0$); its uniqueness follows also from the fact that, for $\Phi_\pi\in(R^n)^*$, the equalities (2.14) are equivalent to

$$\Phi_\pi(e_{\pi(1)})=f(e_{\pi(1)}), \tag{2.18}$$

$$\Phi_\pi(e_{\pi(k)})=f(\sum_{i=1}^{k}e_{\pi(i)})-f(\sum_{i=1}^{k-1}e_{\pi(i)})\quad(k=2,\ldots,n). \tag{2.19}$$

b) By (2.2) (or (2.13)), $\hat{f}$ is positively homogeneous. Moreover, if we use lemma 2.1 to subdivide R^n_+ into the cones K_π defined in (2.13), then on each cone K_π, Graph $\hat{f}$ is obtained by the "continuations" of the portions of hyperplanes over B_π (which pass now through 0), described in remark 2.3 b). Also, $\hat{f}$ is continuous on R^n_+ (by (2.13) and the proof of theorem 2.1 b)).

c) Although the restriction of a convex function $g:R^n_+\to R$ to C_n need not be submodular, one can make the following observation, which implies the "only if" part in the result of Lovász on $\hat{f}$, mentioned above, in the particular case when n=2: If $g:R^n_+\to R$ is convex and positively homogeneous, then $g|_{C_n}$ is subadditive (and hence, when n=2, submodular). Indeed, if $S,T \subseteq \{1,\dots,n\}$ and $S \cap T = \emptyset$, then

$$g(\sum_{i \in S \cup T} e_i) = g(\sum_{i \in S} e_i + \sum_{i \in T} e_i) =$$

$$= 2g(\frac{1}{2}(\sum_{i \in S} e_i + \sum_{i \in T} e_i)) \leq g(\sum_{i \in S} e_i) + g(\sum_{i \in T} e_i) .$$

However, for n=3, $g|_{C_n}$ need not be submodular, as shown by the same counterexample of [10], p.250, which is not only convex, but also positively homogeneous: $g(x) = \max(\xi_1, \xi_2 + \xi_3)$ $(x = \sum_{i=1}^{3} \xi_i e_i \in R_+^3)$.

d) Theorem 2.2 permits to give the following proof of the "only if" part in the result of Lovász, mentioned above: Let $\hat{f}: R_+^n \to R$ be convex and let $S,T \subseteq \{1,\dots,n\}$. If $S \cap T \neq \emptyset$, then there exists $\pi \in \Pi_n$ such that

$$S \cap T = \{\pi(1),\dots,\pi(|S \cap T|)\} \subseteq S \cup T = \{\pi(1),\dots,\pi(|S \cup T|)\}; \quad (2.20)$$

if $S \cap T = \emptyset$, we shall consider any π such that the second equality holds and we shall use that $\sum_{i \in \emptyset} e_i = 0$ (or, alternatively, one can use c) above). Then $\sum_{i \in S \cap T} e_i$, $\sum_{i \in S \cup T} e_i \in B_\pi$, whence, since B_π is convex, also $\frac{1}{2}(\sum_{i \in S \cap T} e_i + \sum_{i \in S \cup T} e_i) \in B_\pi$. Hence, using (2.14), $\Phi_\pi \in (R^n)^*$, (2.13), the convexity of $\hat{f}$ and the equality $f = \hat{f}|_{C_n}$, we obtain

$$\frac{1}{2}\{f(\sum_{i \in S \cap T} e_i) + f(\sum_{i \in S \cup T} e_i)\} = \frac{1}{2}\{\Phi_\pi(\sum_{i \in S \cap T} e_i) + \Phi_\pi(\sum_{i \in S \cup T} e_i)\} =$$

$$= \Phi_\pi(\frac{1}{2}(\sum_{i \in S} e_i + \sum_{i \in T} e_i)) = \hat{f}(\frac{1}{2}(\sum_{i \in S} e_i + \sum_{i \in T} e_i)) \leq$$

$$\leq \frac{1}{2}\{\hat{f}(\sum_{i \in S} e_i) + \hat{f}(\sum_{i \in T} e_i)\} = \frac{1}{2}\{f(\sum_{i \in S} e_i) + f(\sum_{i \in T} e_i)\} .$$

e) The preceding geometric observations also suggest how to remove the assumption $f(0)=0$, i.e., how to define the Lovász extension of an arbitrary function $f: C_n \to R$ (of course, when $f(0) \neq 0$, this extension will no longer be positively homogeneous): namely, $f_o = f - f(0)$ satisfies $f_o(0) = 0$ and the Lovász extension $\hat{f}$ of f should be defined by

$$\hat{f}(x)=f(0)+\widehat{f_o}(x) \qquad (x\in R^n_+), \tag{2.21}$$

where $\widehat{f_o}$ is the Lovász extension of f_o. Then

$$\hat{f}(x)=f(0)+\widehat{f_o}(x)=f(0)+f_o(x)=f(x) \qquad (x\in C_n), \tag{2.22}$$

so $\hat{f}$ is indeed an extension of f. Furthermore, by theorem 2.2 for f_o, we have

$$\hat{f}(x)=f(0)+\Phi^o_\pi(x) \qquad (x\in B_\pi, \pi\in\Pi_n), \tag{2.23}$$

where, for each $\pi\in\Pi_n$, $\Phi^o_\pi\in(R^n)^*$ corresponds to $f_o=f-f(0)$ as in theorem 2.2, i.e., Φ^o_π is the unique linear function satisfying

$$\Phi^o_\pi\Big(\sum_{i=1}^k e_{\pi(i)}\Big)=f\Big(\sum_{i=1}^k e_{\pi(i)}\Big)-f(0) \qquad (k=1,\dots,n). \tag{2.24}$$

Thus, the unique affine function $\Psi_\pi:R^n\to R$ interpolating f at the vertices of B_π is

$$\Psi_\pi=\Phi^o_\pi+f(0) \qquad (\pi\in\Pi_n), \tag{2.25}$$

and formula (2.23) shows that (2.12) remains valid for any $f:C_n\to\to R$ (with f(0) arbitrary); hence (see remark 2.3 b)), for any $f:C_n\to R$, on each simplex B_π, Graph $\hat{f}$ is obtained by "filling in" the ordinates of f at the vertices of B_π by the portion of the unique hyperplane passing through them. Let us also note that, by (2.17), the functions $\Phi_\pi\in(R^n)^*$ satisfying (2.14) (which do not enter into the definition of $\hat{f}$ when $f(0)\neq 0$) and $\Phi^o_\pi\in(R^n)^*$ are related by

$$\Phi_\pi(x)-\Phi^o_\pi(x)=f(0)\,\xi_{\pi(1)} \qquad \Big(x=\sum_{i=1}^n \xi_i e_i\in B_\pi, \pi\in\Pi_n\Big); \tag{2.26}$$

hence, for each $\pi\in\Pi_n$ and each $c\in R$, the difference $\Phi_\pi-\Phi^o_\pi$ is constant on the intersection of the simplex B_π with the hyperplane $\{x=\sum_{i=1}^n \xi_i e_i\in R^n \mid \xi_{\pi(1)}=c\}$ (this intersection may be also empty or the singleton $\{0\}$).

3. CONVEX (CONCAVE) TIGHT EXTENSIONS

We recall (see [13], p.172) that a function $g:D \to R$, where $D \subseteq R^n$, is called polyhedral convex, if there exist affine functions $\Psi_1', \dots, \Psi_r':R^n \to R$ (where $r < +\infty$) such that $g(x) = \max_{1 \leq i \leq r} \Psi_i'(x)$ $(x \in D)$; clearly, every such function admits a natural extension to the polyhedral convex function $\tilde{g} = \max_{1 \leq i \leq r} \Psi_i':R^n \to R$. A useful tool will be

Definition 3.1. For any function $f:C_n \to R$ and any v-triangulation $\mathcal{D} = \{D_i\}_{i=1}^p$ of co C_n, we define the polyhedral convex function $\tilde{f}^{\mathcal{D}}:R^n \to R$ associated to $\bar{f}^{\mathcal{D}}$ (or, to f and $\mathcal{D}$), by

$$\tilde{f}^{\mathcal{D}}(x) = \max_{1 \leq i \leq p} \Psi_i(x) \qquad (x \in R^n), \tag{3.1}$$

where $\Psi_1, \dots, \Psi_p:R^n \to R$ are the affine functions occurring in definition 2.2 a).

Theorem 3.1. For any $f:C_n \to R$ and any v-triangulation $\mathcal{D} = \{D_i\}_{i=1}^p$ of co C_n, we have

$$\bar{f}^{\mathcal{D}}(x) \leq \tilde{f}^{\mathcal{D}}(x) \qquad (x \in \text{co } C_n), \tag{3.2}$$

$$f(x) \leq \tilde{f}^{\mathcal{D}}(x) \qquad (x \in C_n). \tag{3.3}$$

Proof. By (3.1), we have

$$\Psi_i(x) \leq \tilde{f}^{\mathcal{D}}(x) \qquad (x \in R^n,\ i=1,\dots,p). \tag{3.4}$$

Now let $x \in$ co C_n. Then, by $\bigcup_{i=1}^p D_i = \text{co } C_n$, there exists $i \leq p$ such that $x \in D_i$. Hence, by (2.9) and (3.4), we obtain $\bar{f}^{\mathcal{D}}(x) = \Psi_i(x) \leq \tilde{f}^{\mathcal{D}}(x)$, which proves (3.2). Finally, by $\bar{f}^{\mathcal{D}}|_{C_n} = f$ and (3.2), we have (3.3).

In order to characterize the functions $f:C_n \to R$ for which the equality sign holds in (3.2), (3.3), we shall use

Lemma 3.1. If $D = \text{co}\{x_1, \dots, x_{n+1}\} \subset R^n$ is an n-simplex with $x_1, \dots, x_{n+1} \in C_n$ and if $\Psi_i':R^n \to R$ $(i \in I)$ and $\Psi:R^n \to R$ are affine functions such that

$$\sup_{i\in I} \Psi_i'(x) = \Psi(x) \qquad (x \in D), \qquad (3.5)$$

then for each $\varepsilon > 0$ there exists $i(\varepsilon) \in I$ such that

$$\Psi'_{i(\varepsilon)}(x) > \Psi(x) - \varepsilon \qquad (x \in \text{co } C_n). \qquad (3.6)$$

Proof. Let us first show that for each $\varepsilon > 0$ there exists $i(\varepsilon) \in I$ such that

$$\Psi'_{i(\varepsilon)}(x_k) > \Psi(x_k) - \varepsilon \qquad (k=1,\dots,n+1). \quad (3.7)$$

Indeed, if for some $\varepsilon_o > 0$ there is no such $i(\varepsilon_o)$, then for each $i \in I$ there exists $k_i \in \{1,\dots,n+1\}$ such that $\Psi_i'(x_{k_i}) \leq \Psi(x_{k_i}) - \varepsilon_o$. Hence, if $\{k_1,\dots,k_p\}$ is the set of all distinct k_i's, then for $x_o = \frac{1}{p}\sum_{j=1}^{p} x_{k_j} \in D$ we obtain, using that Ψ_i' and Ψ are affine functions satisfying (3.5),

$$\Psi_i'(x_o) = \frac{1}{p}\sum_{j=1}^{p} \Psi_i'(x_{k_j}) \leq \frac{1}{p}\sum_{j=1}^{p} \Psi(x_{k_j}) - \varepsilon_o = \Psi(x_o) - \varepsilon_o \qquad (i \in I),$$

which contradicts (3.5). This proves (3.7).

Now let $C_n = \{x_1,\dots,x_{n+1}\} \cup \{x_{n+2},\dots,x_{2^n}\}$. Then, since $x_1,\dots,x_{n+1}$ are affinely independent in R^n, for each $x_j \in C_n$ there exist uniquely determined $\lambda_1^j,\dots,\lambda_{n+1}^j \in R$ with $\sum_{k=1}^{n+1} \lambda_k^j = 1$, such that $x_j = \sum_{k=1}^{n+1} \lambda_k^j x_k$. Let

$$\alpha_n = \max_{1\leq j\leq 2^n} \sum_{k=1}^{n+1} |\lambda_k^j| \qquad (3.8)$$

and let $\varepsilon > 0$. Then, by the above, there exists $i(\varepsilon) \in I$ such that

$$\Psi(x_k) \geq \Psi'_{i(\varepsilon)}(x_k) > \Psi(x_k) - \frac{\varepsilon}{\alpha_n} \qquad (k=1,\dots,n+1),$$

whence

$$|\Psi(x_j) - \Psi'_{i(\varepsilon)}(x_j)| = \left|\sum_{k=1}^{n+1} \lambda_k^j (\Psi(x_k) - \Psi'_{i(\varepsilon)}(x_k))\right| \leq$$

$$\leq \sum_{k=1}^{n+1} |\lambda_k^i| \max_{1\leq k\leq n+1} |\Psi(x_k)-\Psi'_{i(\varepsilon)}(x_k)| < \varepsilon \qquad (j=1,\dots,2^n).$$

Consequently, if $x\in \mathrm{co}\, C_n$, say $x=\sum_{j=1}^{p}\mu_j x_{k_j}$, where $\mu_j\geq 0$ $(j=1,\dots$ $\dots,p)$, $\sum_{j=1}^{p}\mu_j=1$, then

$$|\Psi(x)-\Psi'_{i(\varepsilon)}(x)| = \left|\sum_{j=1}^{p}\mu_j(\Psi(x_{k_j})-\Psi'_{i(\varepsilon)}(x_{k_j}))\right| \leq$$

$$\leq \sum_{j=1}^{p}\mu_j \max_{1\leq j\leq p}|\Psi(x_{k_j})-\Psi'_{i(\varepsilon)}(x_{k_j})| < \varepsilon,$$

which proves (3.6).

Remark 3.1. The argument of the first part, combined with the uniqueness part of lemma 2.2, shows that if $D=\mathrm{co}\,\{x_1,\dots,x_{n+1}\}\subset R^n$ is an n-simplex and $\Psi'_1,\dots,\Psi'_r,\Psi: R^n\to R$ are affine functions such that

$$\max_{1\leq i\leq r}\Psi'_i(x)=\Psi(x) \qquad (x\in D), \qquad (3.9)$$

then there exists $i_o\in\{1,\dots,r\}$ such that

$$\Psi'_{i_o}(x)=\Psi(x) \qquad (x\in R^n). \qquad (3.10)$$

Theorem 3.2. For any $f: C_n\to R$, $\mathcal{D}=\{D_i\}_{i=1}^{p}$ and $\{\Psi_i\}_{i=1}^{p}$ as in definition 3.1, the following statements are equivalent:

1° The tight extension $\overline{f}^{\mathcal{D}}$ is convex.

2° $\overline{f}^{\mathcal{D}}$ is polyhedral convex.

3° $\tilde{f}^{\mathcal{D}}$ is an extension of the tight extension $\overline{f}^{\mathcal{D}}$, i.e.,

$$\overline{f}^{\mathcal{D}}(x)=\tilde{f}^{\mathcal{D}}(x) \qquad (x\in \mathrm{co}\, C_n). \qquad (3.11)$$

4° $\tilde{f}^{\mathcal{D}}$ is an extension of f, i.e.,

$$f(x)=\tilde{f}^{\mathcal{D}}(x) \qquad (x\in C_n). \qquad (3.12)$$

5° We have

$$\Psi_i(x)\leq f(x) \qquad (x\in C_n,\ i=1,\dots,p). \qquad (3.13)$$

Proof. 1°⇒3°. If 1° holds, then, by theorem 2.1 b) and e.g. [13], p.102, theorem 2.1, we have

$$\overline{f}^{\mathcal{D}}(x)=\sup_{i\in I}\Psi'_i(x) \qquad (x\in \mathrm{co}\, C_n), \qquad (3.14)$$

where $\Psi'_i: R^n\to R$ $(i\in I)$ are affine. Then, by (2.9),

$$\sup_{i\in I} \Psi_i'(x)=\Psi_j(x) \qquad (x\in D_j,\ j=1,\dots,p), \qquad (3.15)$$

and hence, by lemma 3.1, for each $\varepsilon>0$ and $j\in\{1,\dots,p\}$ there exists $i_j(\varepsilon)\in I$ such that

$$\Psi'_{i_j(\varepsilon)}(x)>\Psi_j(x)-\varepsilon \qquad (x\in \mathrm{co}\ C_n). \qquad (3.16)$$

Consequently, by (3.1) and (3.14),

$$\tilde{f}^{\mathcal{D}}(x)=\max_{1\leq j\leq p}\Psi_j(x)<\max_{1\leq j\leq p}\Psi'_{i_j(\varepsilon)}(x)+\varepsilon\leq\sup_{i\in I}\Psi_i'(x)+\varepsilon=$$
$$=\overline{f}^{\mathcal{D}}(x)+\varepsilon \qquad (x\in \mathrm{co}\ C_n), \qquad (3.17)$$

whence, since $\varepsilon>0$ has been arbitrary and using (3.2), we obtain (3.11).

Furthermore, the implications $3^{\circ}\Rightarrow 4^{\circ}\Rightarrow 5^{\circ}$ are obvious, by $\overline{f}^{\mathcal{D}}|_{C_n}=f$ and (3.1). The implication $5^{\circ}\Rightarrow 4^{\circ}$ is an immediate consequence of (3.1) and theorem 3.1.

$4^{\circ}\Rightarrow 3^{\circ}$. If 4° holds and $D_i=\mathrm{co}\ \{x_1^i,\dots,x_{n+1}^i\}$ $(i=1,\dots,p)$, let $x\in \mathrm{co}\ C_n=\bigcup_{i=1}^{p} D_i$ and let $i\leq p$ be such that $x\in D_i$, so there exist $\lambda_1,\dots,\lambda_{n+1}\geq 0$ with $\sum_{k=1}^{n+1}\lambda_k=1$, such that $x=\sum_{k=1}^{n+1}\lambda_k x_k^i$. Then, by the convexity of $\tilde{f}^{\mathcal{D}}$, (3.12) and (2.11),

$$\tilde{f}^{\mathcal{D}}(x)=\tilde{f}^{\mathcal{D}}\Big(\sum_{k=1}^{n+1}\lambda_k x_k^i\Big)\leq\sum_{k=1}^{n+1}\lambda_k\tilde{f}^{\mathcal{D}}(x_k^i)=\sum_{k=1}^{n+1}\lambda_k f(x_k^i)=\overline{f}^{\mathcal{D}}(x) ,$$

whence, by (3.2), we obtain (3.11).

Finally, the implications $3^{\circ}\Rightarrow 2^{\circ}\Rightarrow 1^{\circ}$ are obvious.

Corollary 3.1. A function $f:C_n\rightarrow R$ admits a convex tight extension if and only if there exists a v-triangulation $\mathcal{D}=\{D_i\}_{i=1}^{p}$ of co C_n such that (3.13) holds, where $\Psi_i:R^n\rightarrow R$ is the unique affine functional which interpolates f at the vertices of D_i $(i=1,\dots,p)$.

We shall show the interest of such functions $f:C_n\rightarrow R$ for combinatorial optimization in §5.

Let us consider now the standard v-tringulation $\mathcal{B}=\{B_\pi\}_{\pi\in\Pi_n}$ of co C_n (see lemma 2.1). For any $f:C_n\rightarrow R$ with $f(0)=0$ we have, by remark 2.2 and definition 3.1,

$$\tilde{f}^{\mathcal{B}}(x) = \max_{\pi\in\Pi_n} \Phi_\pi(x) \qquad (x\in R^n), \qquad (3.18)$$

where $\Phi_\pi \in (R^n)^*$ is the unique linear function satisfying (2.14) (and thus $\tilde{f}^{\mathcal{B}}$ is subadditive and positively homogeneous). Moreover, for any $f:C_n \to R$ we have, by (2.25) and definition 3.1,

$$\tilde{f}^{\mathcal{B}}(x) = f(0) + \max_{\pi\in\Pi_n} \Phi_\pi^o(x) \qquad (x\in R^n), \qquad (3.19)$$

where $\Phi_\pi^o \in (R^n)^*$ is the unique linear function satisfying (2.24).

Remark 3.2. In general, $\tilde{f}^{\mathcal{B}}$ is not "the smallest" polyhedral convex majorant of $\hat{f}$ on R_+^n, nor the smallest submodular majorant of f on C_n, and such "smallest" majorants need not even exist, as shown by the following example: $n=2$, $f(0)=0$, $f(e_1)=f(e_2)=0$, $f(e_1+e_2)=1$ (so $f:C_2\to R$ is not submodular). Indeed, here $\tilde{f}^{\mathcal{B}}(x) = \max(\Phi_1(x),\Phi_2(x))$, where, e.g. by (2.17), $\Phi_k(\sum_{i=1}^{2}\xi_i e_i)=\xi_{3-k}$ $(\sum_{i=1}^{2}\xi_i e_i \in R^2,\ k=1,2)$, but both Φ_1 and Φ_2 are "minimal" polyhedral convex majorants of $\hat{f}$ on R_+^2 (i.e., if a polyhedral convex majorant is $\leq\Phi_1$, then it is $=\Phi_1$, and similarly for Φ_2) and "minimal" submodular majorants of f on C_2.

Now we shall prove the following sharpening of the theorem of Lovász, mentioned in §2 (i.e., of the equivalence $1^o \Leftrightarrow 2^o$ below):

Theorem 3.3. For any $f:C_n\to R$, with Lovász extension $\hat{f}:R_+^n\to R$, the following statements are equivalent:

1^o. f is submodular.

2^o. $\hat{f}$ is convex.

3^o. $\hat{f}$ is polyhedral convex.

4^o. We have

$$\hat{f}(x) = f(0) + \max_{\pi\in\Pi_n} \Phi_\pi^o(x) \qquad (x\in R_+^n), \qquad (3.20)$$

where $\Phi_\pi^o \in (R^n)^*$ is the unique linear function satisfying (2.24).

5^o. We have

$$f(x) = f(0) + \max_{\pi\in\Pi_n} \Phi_\pi^o(x) \qquad (x\in C_n). \qquad (3.21)$$

6° We have

$$f(0)+\Phi^{o}_{\pi}(x)\leq f(x) \qquad (x\in C_n, \pi\in\Pi_n). \qquad (3.22)$$

Proof. The implication $2^o \Rightarrow 1^o$ follows from remark 2.4 d), e). The equivalences $2^o \Leftrightarrow \ldots \Leftrightarrow 6^o$ (actually, we need here only the implications $6^o \Rightarrow \ldots \Rightarrow 2^o$) follow from theorem 2.2 and theorem 3.2 applied to $\mathcal{D}=\mathcal{B}$ (using Ψ_π of (2.25)).

$1^o \Rightarrow 6^o$. Assume 1^o and assume first that $f(0)=0$, so $\Phi^o_\pi=\Phi_\pi$, the unique linear function satisfying (2.14).

If $n=2$, then for $\pi\in\Pi_2$ we have, by our assumptions,

$$\Phi_\pi(0)=0=f(0), \quad \Phi_\pi(e_{\pi(1)})=f(e_{\pi(1)}), \quad \Phi_\pi(e_1+e_2)=f(e_1+e_2),$$

whence, using $\Phi_\pi\in(R^n)^*$ and the submodularity of f,

$$\Phi_\pi(e_{\pi(2)})=\Phi_\pi(e_1+e_2)+\Phi_\pi(0)-\Phi_\pi(e_{\pi(1)})\leq f(e_1+e_2)+f(0)-f(e_{\pi(1)})\leq$$

$$\leq f(e_{\pi(2)}).$$

Now let $n\geq 3$ be arbitrary and assume that the statement has been proved for n replaced by n-1. Thus, for any $\pi\in\Pi_n$, if

$$C^\pi_{n-1}=\{x=\{\xi_{\pi(i)}\}^{n-1}_{i=1}\in R^{n-1} \mid \xi_{\pi(i)}=0 \text{ or } 1 \ (i=1,\ldots,n-1)\}, \qquad (3.23)$$

then, by our induction assumption, we have $\Phi_\pi(x)\leq f(x)$ for all $x\in C^\pi_{n-1}$. But, any $x\in C_n\setminus C^\pi_{n-1}$ is of the form $x=\sum_{i\in S\cup\{\pi(n)\}} e_i$, where $S\subset\{\pi(1),\ldots,\pi(n-1)\}$. Hence, since

$\{S\cup\{\pi(n)\}\}\cap\{\pi(1),\ldots,\pi(n-1)\}=S, \{S\cup\{\pi(n)\}\}\cup\{\pi(1),\ldots,\pi(n-1)\}=\{1,\ldots,n\}$, and since $\sum_{i\in S} e_i\in C^\pi_{n-1}$, we obtain, using $\Phi_\pi\in(R^n)^*$, (2.14) and the submodularity of f, that

$$\Phi_\pi(x)=\Phi_\pi(\sum_{i\in S} e_i)+\Phi_\pi(\sum_{i=1}^{n} e_i)-\Phi_\pi(\sum_{i=1}^{n-1} e_{\pi(i)})\leq$$

$$\leq f(\sum_{i\in S} e_i)+f(\sum_{i=1}^{n} e_i)-f(\sum_{i=1}^{n-1} e_{\pi(i)})\leq f(x).$$

Assume, finally, that $f(0)\in R$ is arbitrary. Then $f_o=f-f(0)$ is submodular and $f_o(0)=0$, whence, by the above, we have

$$\Phi^o_\pi(x)\leq f_o(x)=f(x)-f(0) \qquad (x\in C_n, \pi\in\Pi_n),$$

i.e., (3.22).

Corollary 3.2. a) Every submodular function $f:C_n \to R$ admits a polyhedral convex tight extension, namely, $\hat{f}|_{co\ C_n}$.

b) If $f:C_n \to R$ is submodular and $f(0)=0$, then $\hat{f}:R^n_+ \to R$ is subadditive.

Proof. a) By theorem 2.2 and remark 2.4 e), $\hat{f}|_{co\ C_n}$ is a tight extension of f, so it is enough to apply theorem 3.3, implication $1^o \Rightarrow 3^o$.

b) follows from theorem 3.3, implication $1^o \Rightarrow 4^o$.

Remark 3.3. a) One can define, for $M \subset R^n$, the subdifferential $\partial g(x_o)$ of a function $g:M \to R$ at a point $x_o \in M$, by

$$\partial g(x_o)=\{\Phi \in (R^n)^* \mid \Phi(x)-\Phi(x_o) \leq g(x)-g(x_o) \quad (x \in M)\}; \tag{3.24}$$

for $M=C_n$, such a concept has been introduced by Fujishige ([7], [8]), who has used functions $\phi:\{1,\dots,n\} \to R$ (clearly, they can be identified with functions $\Phi \in (R^n)^*$). By (3.24) and theorems 2.2 and 5.1 (applied to $f-\Phi$), we have, for any $f:C_n \to R$ and $x_o \in C_n$,

$$\partial \hat{f}|_{co\ C_n}(x_o)=\{\Phi \in (R^n)^* \mid \hat{f}(x_o)-\Phi(x_o)= \min_{x \in co\ C_n} \{\hat{f}(x)-\Phi(x)\}\}=$$

$$=\{\Phi \in (R^n)^* \mid f(x_o)-\Phi(x_o)=\min_{x \in C_n} \{f(x)-\Phi(x)\}\}=\partial f(x_o) \tag{3.25}$$

(for a similar result, see [8], theorem 4.2), whence

$$\max_{\Phi \in \partial f(0)} \Phi(x)= \max_{\Phi \in \partial \hat{f}|_{co\ C_n}(0)} \Phi(x) \leq \hat{f}(x)-f(0) \quad (x \in co\ C_n). \tag{3.26}$$

If $f:C_n \to R$ is submodular, then, by (3.22), we have $\{\Phi^o_\pi\}_{\pi \in \Pi_n} \subseteq \partial f(0)$, whence, by (3.26) (extended by positive homogeneity to all $x \in R^n_+$) and (3.20), we obtain

$$\max_{\pi \in \Pi_n} \Phi^o_\pi(x) \leq \max_{\Phi \in \partial f(0)} \Phi(x) \leq \hat{f}(x)-f(0)=\max_{\pi \in \Pi_n} \Phi^o_\pi(x) \qquad (x \in R^n_+),$$

and thus

$$\hat{f}(x)=f(0)+ \max_{\Phi \in \partial f(0)} \Phi(x) \qquad (x \in R^n_+). \tag{3.27}$$

Formula (3.27) (which also implies 2° of theorem 3.3), has been obtained, using the "greedy algorithm", in [8], formula (4.1) and, actually, in Lovász ([10], p.248), if we take into account that, by (3.24) (or, by [7], lemma 4.3 (a)), the polyhedron P_f of [10], p.246 is nothing else than

$$P_f=\{\Phi\in(R^n)^* \mid \sum_{j\in T}\Phi(e_j)\leq f(\sum_{j\in T}e_j)-f(0) \quad (T\subseteq\{1,\ldots,n\})\}=\partial f(0), \tag{3.28}$$

where $\sum_{j\in\emptyset}\Phi(e_j)=0$. Our methods, used above, for obtaining formula (3.27), are entirely non-algorithmic and, moreover, they lead to a deeper understanding of the greedy algorithm (see §6).

b) Since for each $x\in R^n$ the function $h_x(\Phi)=\Phi(x)$ $(\Phi\in(R^n)^*)$ is linear and since $\partial f(0)$ of (3.25) is a convex polyhedron in $(R^n)^*$, we have

$$\max_{\Phi\in\partial f(0)}\Phi(x)=\max_{\Phi\in\mathscr{E}(\partial f(0))}\Phi(x) \qquad (x\in R^n),$$

where $\mathscr{E}(\partial f(0))$ denotes the set of extreme points (vertices) of $\partial f(0)$, and hence (3.27) is equivalent to

$$\hat{f}(x)=f(0)+\max_{\Phi\in\mathscr{E}(\partial f(0))}\Phi(x) \qquad (x\in R^n_+). \tag{3.29}$$

One can also show that formula (3.29) implies (3.20), and thus they are equivalent. Indeed, using (3.36) below, one can write (2.18), (2.19) of remark 2.4 a) in the form

$$\Phi_\pi(e_{S_1})=f(e_{S_1}), \tag{3.30}$$

$$\Phi_\pi(e_{S_k\setminus S_{k-1}})=f(\sum_{i\in S_k}e_i)-f(\sum_{i\in S_{k-1}}e_i) \qquad (k=2,\ldots,n), \tag{3.31}$$

and hence, by [8], theorem 3.1, the functions Φ_π° $(\pi\in\Pi_n)$ occurring in (3.20) above, are precisely the extreme points of $\partial f(0)$. Let us note that, in the above proof of theorem 3.3, we have obtained (3.20) without using extreme point methods.

The representation (3.21), of an arbitrary submodular function $f:C_n\to R$, involves $|\Pi_n|=n!$ affine functions $\Phi_\pi^{\circ}+f(0)$. Now we shall improve this, as follows:

Theorem 3.4. There exists a subset Ω_n of Π_n, of cardinality

$$|\Omega_n| = \binom{n}{\left[\frac{n}{2}\right]}, \tag{3.32}$$

such that

$$C_n = \bigcup_{\pi\in\Omega_n} \mathcal{E}(B_\pi), \tag{3.33}$$

and hence, for each submodular function $f:C_n \to R$ we have

$$f(x) = f(0) + \max_{\pi\in\Omega_n} \phi^0_\pi(x) \qquad (x\in C_n). \tag{3.34}$$

Proof. Let us first observe that the permutations $\pi\in\Pi_n$ are in one-to-one correspondence with the "maximal" chains

$$S_1 \subset S_2 \subset \ldots \subset S_n = \{1,\ldots,n\}, \tag{3.35}$$

where $|S_i| = i$ $(i=1,\ldots,n)$. Indeed, to each $\pi\in\Pi_n$ there corresponds the chain (3.35) with

$$S_k = \{\pi(1),\ldots,\pi(k)\} \qquad (k=1,\ldots,n) \tag{3.36}$$

and conversely, for each chain (3.35) with $|S_i| = i$ $(i=1,\ldots,n)$, there is a unique permutation $\pi\in\Pi_n$ such that (3.36) holds, namely, $\{\pi(1)\} = S_1, \{\pi(k)\} = S_k \setminus S_{k-1}$ $(k=2,\ldots,n)$. We claim that the minimum of the number of maximal chains (3.35) such that each $S\subseteq\{1,\ldots,n\}$ belongs to one of these maximal chains, is $\binom{n}{\left[\frac{n}{2}\right]}$. Indeed, by a theorem of Dilworth [4], the above minimum is equal to the maximum of the cardinalities of antichains in $2^{\{1,\ldots,n\}}$, and the latter is known to be $\binom{n}{\left[\frac{n}{2}\right]}$ (see e.g. [3], p.335, part e); one can also give a direct proof, by showing that in $2^{\{1,\ldots,n\}}$ the elements at the level $\left[\frac{n}{2}\right]$, i.e., the subsets $T\subseteq\{1,\ldots,n\}$ with $|T| = \left[\frac{n}{2}\right]$, constitute an antichain of maximum cardinality). This proves the claim.

Now let $\Omega_n \subset \Pi_n$ be the image, by the above correspondence between maximal chains (3.35) and permutations $\pi\in\Pi_n$, of a minimal family of maximal chains as in the above claim, so (3.32) holds. Then, since each $x\in C_n\setminus\{0\}$ is of the form $x = \sum_{i\in S} e_i$, where $\emptyset \neq S \subseteq \{1,\ldots,n\}$, for each such x there exists a permutation $\pi\in\Omega_n$ such that $S=\{\pi(1),\ldots,\pi(|S|)\}$, whence $x=\sum_{i=1}^{|S|} e_{\pi(i)} \in \mathcal{V}(B_{\pi})$. This proves (3.33). Finally, by (3.33), for each $x \in C_n \setminus \{0\}$ there exists $\pi \in \Omega_n$ such that $x \in \mathcal{V}(B_{\pi}) \setminus \{0\} = \{\sum_{i=1}^{k} e_{\pi(i)}\}_{k=1}^{n}$, whence, by (2.24) and (3.21), we obtain

$$f(x)-f(0)=\phi_{\pi}^{0}(x) \leq \max_{\pi' \in \Omega_n} \phi_{\pi'}^{0}(x) \leq \max_{\pi' \in \Pi_n} \phi_{\pi'}^{0}(x)=f(x)-f(0),$$

which yields (3.34).

Example 3.1. The v-triangulation (2.3) of co C_3 contains $3!=6$ simplices B_{π} and, by thoerem 3.4, we can represent any submodular $f: C_3 \to R$ in the form (3.34), using only $\binom{3}{[\frac{3}{2}]}=3$ of these simplices, say $\Omega_3=\{B_1, B_2, B_3\}$, such that $C_3=\bigcup_{i=1}^{3} \mathcal{V}(B_i)$; for example, one can take $B_1=\mathrm{co}\,\{0, e_1, e_1+e_2, \sum_{i=1}^{3} e_i\}$, $B_2=\mathrm{co}\,\{0, e_2, e_2+e_3, \sum_{i=1}^{3} e_4\}$ and $B_3=$ $=\mathrm{co}\,\{0, e_3, e_1+e_3, \sum_{i=1}^{3} e_i\}$.

Remark 3.4. To each function $f: C_n \to R$ and each v-triangulation $\mathcal{D}=\{D_i\}_{i=1}^{p}$ of co C_n one can associate a polyhedral concave function $\tilde{f}^{\mathcal{D}}: C_n \to R$, replacing max by min in (3.1), and one can prove for it the corresponding results, with the obvious changes (thus, in the result for $\tilde{f}^{\mathcal{D}}$ corresponding to theorem 3.1, the inequalities are reversed, and in the results corresponding to theorems 3.3, 3.4, submodular functions are replaced by supermodular ones and max is replaced by min, etc.).

Now we shall show that tight convex extensions associated to v-triangulations of co C_n, which are different from the standard one of (2.3), are also of interest. For simplicity, let us consider the case $n=2$. In this case, there is only one "non-standard" v-triangulation of co C_2, namely, $\mathcal{D}=\{D_1, D_2\}$, where

$$D_1=\mathrm{co}\,\{0, e_1, e_2\}, \quad D_2=\mathrm{co}\,\{e_1, e_2, e_1+e_2\}. \tag{3.37}$$

The unique affine functions $\Psi_1, \Psi_2: R^2 \to R$, which interpolate a function $f: C_2 \to R$ at the vertices of D_1 and D_2 respectively, i.e., which satisfy

$$\Psi_1(e_i)=f(e_i)=\Psi_2(e_i) \qquad (i=1,2), \tag{3.38}$$

$$\Psi_1(0)=f(0),\ \Psi_2(e_1+e_2)=f(e_1+e_2)\ , \tag{3.39}$$

are

$$\Psi_1(\sum_{i=1}^{2}\xi_i e_i)=\sum_{i=1}^{2}\xi_i(f(e_i)-f(0))+f(0) \qquad (\sum_{i=1}^{2}\xi_i e_i\in R^2), \tag{3.40}$$

$$\Psi_2(\sum_{i=1}^{2}\xi_i e_i)=(f(e_1+e_2)-f(e_2))\xi_1+(f(e_1+e_2)-f(e_1))\xi_2+$$

$$+f(e_1)+f(e_2)-f(e_1+e_2) \qquad (\sum_{i=1}^{2}\xi_i e_i\in R^2). \tag{3.41}$$

Now we shall show that <u>the supermodular functions $f:C_2\to R$ correspond to the v-triangulation (3.37) of co C_2 and to convex functions</u>, in a similar manner as the submodular ones correspond, by theorem 3.3, to the standard v-triangulation (2.3) of co C_2 and to convex functions.

<u>Proposition 3.1. For the v-triangulation (3.37) of co C_2 and for any $f:C_2\to R$, the following statements are equivalent:</u>

1°. f <u>is supermodular.</u>

2°. <u>The tight extension $\bar{f}^{\mathcal{D}}$ is convex.</u>

3°-6°= <u>the statements 1°-4° of theorem 3.2, respectively.</u>

<u>Proof.</u> $1^\circ\Rightarrow 6^\circ$. If 1° holds, then, by (3.40) and (3.41),

$$\Psi_1(e_1+e_2)=f(e_1)+f(e_2)-f(0)\leqslant f(e_1+e_2),$$

$$\Psi_2(0)=f(e_1)+f(e_2)-f(e_1+e_2)\leqslant f(0),$$

which, together with (3.38), (3.39), yield (3.13).

The equivalences $2^\circ\Leftrightarrow\ldots\Leftrightarrow 6^\circ$ (actually, we need here only the implications $6^\circ\Rightarrow\ldots\Rightarrow 2^\circ$) hold by theorem 3.2.

$2^\circ\Rightarrow 1^\circ$. If 2° holds, then, by (3.38), (2.9) for $x=\frac{e_1+e_2}{2}\in D_1$, and $\bar{f}^{\mathcal{D}}|_{C_2}=f$, we obtain

$$\frac{1}{2}\{f(e_1)+f(e_2)\}=\Psi_1(\frac{e_1+e_2}{2})=\bar{f}^{\mathcal{D}}(\frac{e_1+e_2}{2})\leqslant$$

$$\leqslant\frac{1}{2}\{\bar{f}^{\mathcal{D}}(0)+\bar{f}^{\mathcal{D}}(e_1+e_2)\}=\frac{1}{2}\{f(0)+f(e_1+e_2)\}\ .$$

By remark 3.4, to proposition 3.1 there corresponds the following result on submodularity and concavity (where Ψ_1 , Ψ_2 are as above):

Proposition 3.2. Under the assumptions of proposition 3.1, the following statements are equivalent:

1° f is submodular.

2° $\overline{f}^{\mathcal{D}}$ is concave.

3° $\overline{f}^{\mathcal{D}}$ is polyhedral concave.

4° We have

$$\overline{f}^{\mathcal{D}}(x)=\min(\Psi_1(x),\Psi_2(x)) \qquad (x\in \text{co } C_2). \qquad (3.42)$$

5° We have

$$f(x)=\min(\Psi_1(x),\Psi_2(x)) \qquad (x\in C_2). \qquad (3.43)$$

6° We have

$$\Psi_i(x)\geq f(x) \qquad (x\in C_2,\ i=1,2). \qquad (3.44)$$

Indeed, $f:C_2\to R$ is a submodular function if and only if $-f:$ $:C_2\to R$ is supermodular and, since $-\Psi_1$, $-\Psi_2$ and $-f$ satisfy the equalities corresponding to (3.38), (3.39) (i.e., $-\Psi_1(e_i)=-f(e_i)$, etc.), formula (3.13) for $-f$ means that $-\Psi_i(x)\leq -f(x)$ $(x\in C_2, i=1,2)$ which is equivalent to (3.44). Similarly, for $\tilde{\tilde{f}}^{\mathcal{D}}$ of remark 3.4, we have

$$\tilde{\tilde{f}}^{\mathcal{D}}=\min(\Psi_1,\Psi_2)=-\max(-\Psi_1,-\Psi_2)=-\widetilde{(-f)}^{\mathcal{D}}, \qquad (3.45)$$

so $-f=\widetilde{(-f)}^{\mathcal{D}}$ holds if and only if $f=\tilde{\tilde{f}}^{\mathcal{D}}$.

Remark 3.4. a) Geometrically, we see that each $f:C_2\to R$ has exactly two tight extensions, $\overline{f}^{\mathcal{B}}=\hat{f}|_{\text{co } C_2}$ and $\overline{f}^{\mathcal{D}}:\text{co } C_2\to R$, and their graphs are dyhedral surfaces in R^3, consisting of two triangles; namely, $\overline{f}^{\mathcal{B}}(\overline{f}^{\mathcal{D}})$ interpolates f at the vertices of the triangles B_1, B_2 (respectively, D_1, D_2), so the two triangles of Graph $\overline{f}^{\mathcal{B}}$(Graph $\overline{f}^{\mathcal{D}}$) have the side $[(0,f(0)),\ (e_1+e_2,\ f(e_1+e_2))]$ (respectively, $[(e_1,f(e_1)),\ (e_2,f(e_2))]$) in common. If f is submodular, then $\overline{f}^{\mathcal{B}}$ (respectively, $\overline{f}^{\mathcal{D}}$) is convex (respectively, concave) and, by (3.20) (respectively, (3.42)), $\overline{f}^{\mathcal{B}}(\overline{f}^{\mathcal{D}})$ extends tightly f "from below" (respectively, "from above"), and the converse statements are also valid. This explains, in part, ~~why~~ submodular functions "have some aspects similat to" (or, "behave like") convex functions and to concave functions (see also §4).

b) In contrast with the case n=2, we conjecture that for any v-triangulation $\mathcal{D}=\{D_i\}_{i=1}^{p}$ of co C_3, there exists a supermodular function $f:C_3\rightarrow R$, such that if $\Psi_i:R^n\rightarrow R$ is the unique affine function which extends tightly f to D_i (i=1,...,p), then we do not have $f(x)=\max_{1\leq i\leq p}\Psi_i(x)$ $(x\in C_3)$; or, equivalently, there exist $i_o\leq p$ and $x_o\in C_3$ such that $f(x_o)<\Psi_{i_o}(x_o)$. This is indeed true e.g. for the standard v-triangulation (2.3) of co C_3, for the "minimal" v-triangulation of co C_3, consisting of 5 simplices, given in [11], namely, $D_1'=$co $\{0,e_1,e_2,e_3\}$, $D_2'=$co $\{e_1,e_2,e_3,e_1+e_2+e_3\}$ and $D_3',D_4',D_5'=$co $\{e_i, e_j, e_i+e_j, \sum_{k=1}^{3} e_k\}$ (i,j=1,2,3; i≠j), and for the v-triangulation $\{D_i\}_{i=1}^{6}$ of co C_3, given in remak 4.3 a) below.

4. POLYHEDRAL CONVEX (CONCAVE) EXTENSIONS

Let us recall that if E is a set and W is a family of functions $w:E\rightarrow R$, a function $f:E\rightarrow R$ is said to be W-convex [5], if there exists a subfamily W_f of W such that

$$f=\sup_{w\in W_f} w, \qquad (4.1)$$

or, equivalently (see e.g. [19], proposition 3.3), if

$$f=\sup_{\substack{w\in W\\ w\leq f}} w. \qquad (4.2)$$

Definition 4.1. For E and W as above, we shall say that a function $f:E\rightarrow R$ is finitely W-convex, or, briefly, FW-convex, if there exists a finite subfamily W_f of W, such that (4.1) holds (with sup replaced by max).

For $E=C_n$, theorems 3.2-3.4 suggest to consider the $F\mathcal{M}_n$-convex functions $f:C_n\rightarrow R$, where $\mathcal{M}_n$ is the family of all modular functions $w:C_n\rightarrow R$. Also, we shall say that $f:C_n\rightarrow R$ is $F\mathcal{M}_n$-concave, if -f is

$F\mathcal{M}_n$-convex.

Remark 4.1. a) Since (by §2) $w:C_n\to R$ is modular if and only if it can be extended to an affine function $\Psi=\Phi+c:R^n\to R$, a function $f:C_n\to R$ is $F\mathcal{M}_n$-convex if and only if there exists a finite subfamily $W_f=\{w_1,\dots,w_r\}$ of $\mathcal{M}_n$ such that

$$f=\max_{1\leqslant i\leqslant r} w_i=\max_{1\leqslant i\leqslant r}(\varphi_i+c_i), \tag{4.3}$$

where $\varphi_i\in\mathcal{M}_n$, $\varphi_i(0)=0$, $c_i=w_i(0)\in R$; thus, $f:C_n\to R$ is $F\mathcal{M}_n$-convex if and only if it admits a polyhedral convex extension $\bar{f}:R^n\to R$.

b) By theorem 3.4, every submodular function $f:C_n\to R$ is $F\mathcal{M}_n$-convex and we have (4.3) with $r\leqslant\binom{n}{\left[\frac{n}{2}\right]}$ (in particular, $r\leqslant 2$ for $n=2$ and $r\leqslant 3$ for $n=3$) and with $c_1=\dots=c_r=f(0)$. On the other hand, by a theorem of Rosenmüller and Weidner [14], every convex game (i.e., every supermodular function $f:C_n\to R$ with $f(0)=0$, $f\geqslant 0$) is $F\mathcal{M}_n$-convex, satisfying (4.3) with $\varphi_i\geqslant 0$, $c_i\leqslant 0$ $(i=1,\dots,r)$ and $r\leqslant\leqslant 2^n$ (where 2^n is not minimal [14]). Hence, every submodular function $f:C_n\to R$ with $f(0)=0$, $f\leqslant 0$, is $F\mathcal{M}_n$-concave, satisfying $f(x)=$
$=\min_{1\leqslant i\leqslant n}\{\Phi_i(x)+c_i\}$ $(x\in C_n)$, where $\Phi_i\in(R^n)^*$, $\Phi_i\leqslant 0$, $c_i\geqslant 0$ $(i=1,\dots,r)$; thus, such a submodular function is simultaneously $F\mathcal{M}_n$-convex and $F\mathcal{M}_n$-concave (but, in general, not modular).

For $n=2$, since each $f:C_2\to R$ is either submodular or supermodular, from propositions 3.1 and 3.2 it follows that each $f:C_2\to R$ is $F\mathcal{M}_2$-convex and $F\mathcal{M}_2$-concave. Although for $n\geqslant 3$ there are functions $f:C_3\to R$, which are neither submodular, nor supermodular, we shall prove now, for an arbitrary n, the following result:

Theorem 4.1. Every function $f:C_n\to R$ is $F\mathcal{M}_n$-convex and $F\mathcal{M}_n$-concave.

Proof. Let $C_n=\{x_1,\dots,x_{2^n}\}$. Then, for each i there exist $\Phi_i\in\in(R^n)^*$ and $c_i\in R$ such that

$$\Phi_i(x_i)=c_i\ ,\quad \Phi_i(x_j)<c_i \qquad (x_j\in C_n\setminus\{x_i\}), \tag{4.4}$$

since each x_j is an exposed point of co C_n; one can see this also directly, by taking, for example,

$$\Phi_S(x)=\sum_{i\in S}\xi_i-\sum_{i\in C_n\setminus S}\xi_i \qquad (x=\{\xi_i\}\in R^n,\ S\subseteq\{1,\dots,n\}), \quad (4.5)$$

$$c_S=|S| \qquad (S\subseteq\{1,\dots,n\}). \quad (4.6)$$

Let α_i, $\beta_i\in R$ $(i=1,\dots,2^n)$ be any numbers such that

$$0<\alpha_i<\min_{x_j\in C_n\setminus\{x_i\}}\frac{f(x_j)-f(x_i)}{\Phi_i(x_j)-c_i} \qquad (i=1,\dots,2^n), \quad (4.7)$$

$$\beta_i=f(x_i)-\alpha_i c_i \qquad (i=1,\dots,2^n), \quad (4.8)$$

and let

$$\Psi_i=\alpha_i\Phi_i+\beta_i \qquad (i=1,\dots,2^n). \quad (4.9)$$

$$\overline{f}(x)=\max_{1\leq i\leq 2^n}\Psi_i(x) \qquad (x\in R^n). \quad (4.10)$$

Then, each $\Psi_i:R^n\to R$ is affine, so $\overline{f}:R^n\to R$ is polyhedral convex and, by (4.4), (4.7), (4.8) we obtain, for each $i\in\{1,\dots,2^n\}$,

$$\Psi_i(x_i)=\alpha_i\Phi_i(x_i)+\beta_i=\alpha_i c_i+\beta_i=f(x_i), \quad (4.11)$$

$$\Psi_i(x_k)=\alpha_i\Phi_i(x_k)-\alpha_i c_i+\alpha_i c_i+\beta_i<\{f(x_k)-f(x_i)\}+$$
$$+f(x_i)=f(x_k) \qquad (x_k\in C_n\setminus\{x_i\}), \quad (4.12)$$

whence

$$\overline{f}(x_k)=\max_{1\leq i\leq 2^n}\Psi_i(x_k)=f(x_k) \qquad (k=1,\dots,2^n), \quad (4.13)$$

which proves that f is $F\mathcal{M}_n$-convex. Thus, since $f:C_n\to R$ was arbitrary, $-f$ is also $F\mathcal{M}_n$-convex, so f is $F\mathcal{M}_n$-concave.

Remark 4.2. a) By (4.11), (4.12), the non-vertical hyperplane

$$\{(x,d)\in R^{n+1}\,|\,(\Psi_i,-1)(x,d)=0\}=\{(x,\Psi_i(x))\,|\,x\in R^n\}=\text{Graph }\Psi_i \quad (4.14)$$

contains the point $(x_i,f(x_i))$, and the "upper half-space"

$$\{(x,d)\in R^{n+1}\,|\,(\Psi_i,-1)(x,d)\leq 0\}=\{(x,d)\,|\,\Psi_i(x)\leq d\}=\text{Epi }\Psi_i \quad (4.15)$$

contains Epi f. Thus, the above proof amounts to showing that for the 2^n points $(x_k,f(x_k))\in R^{n+1}$ and for any fixed one of them, there is an "upper half-space" in R^{n+1} containing all 2^n points, with its boundary containing only the fixed one; or, equivalently, each $(x_k, f(x_k))$ is an exposed point of co (Epi f). Note also that the hyperplane (4.14) supports the epigraph of $\overline{f} = \max_{1 \leq i \leq 2^n} \Psi_i$ at $(x_i, \overline{f}(x_i)) = (x_i, f(x_i))$, by (4.11) and since $\Psi_i \leq \overline{f}$. If $c_i > 0$ for all i with $x_i \neq 0$ (e.g., in the case (4.6)), then the functions Ψ_i of (4.9) satisfy, besides $\Psi_i(0) \leq f(0)$, also

$$\Psi_i(0) = \beta_i = f(x_i) - \alpha_i c_i < f(x_i) = \Psi_i(x_i) \qquad (x_i \in C_n \setminus \{0\}). \qquad (4.16)$$

b) Theorem 4.1 shows that, in contrast with the "continuous case" (where each simultaneously convex and concave function is necessarily affine), every function $f: C_n \to R$ admits an extension to a polyhedral convex function $\overline{f}_1 : R^n \to R$ and an extension to a polyhedral concave function $\overline{f}_2 : R^n \to R$; in particular, if $f: C_n \to R$ admits a simultaneously convex and concave (i.e., an affine) extension $\overline{f}: R^n \to R$, then f is modular.

c) We shall show elsewhere that W-convexity of functions f on Z^n and of subsets G of Z^n (in the sense of [6]) are useful in discrete optimization and might be "the appropriate convexity concepts in discrete structures" (the problem of finding such concepts seems to be still open; see e.g. [1], p.10). An additional advantage is that there exist already a number of results on general W-convexity (see e.g. [6], [5], [19] and the references therein).

From (4.13) and (4.10) we obtain, for any $f: C_n \to R$, the representation $f = \max_{1 \leq i \leq 2^n} \Psi_i|_{C_n}$, involving 2^n affine functions Ψ_i. By remark 4.1 b), for convex games $f: C_n \to R$ the number 2^n is not minimal [14]. Moreover, let us give now the following result, which, in particular, improves the minimal number given in [14] for convex games $f: C_3 \to R$.

Proposition 4.1. For any supermodular function $f: C_3 \to R$, we have the representation

$$f(x) = \max(\Psi_1(x), \Psi_2(x)) \qquad (x \in C_3), \qquad (4.17)$$

where $\Psi_1, \Psi_2 : R^3 \to R$ are the uniquely determined affine functions which interpolate f at the vertices of the simplices

$$D_1=\mathrm{co}\,\{0,e_1,e_2,e_3\},\quad D_2=\mathrm{co}\{e_1+e_2,e_1+e_3,e_2+e_3,\sum_{i=1}^{3}e_i\} \tag{4.18}$$

respectively, i.e., which satisfy

$$\Psi_1(0)=f(0),\quad \Psi_1(e_i)=f(e_i) \qquad (i=1,2,3), \tag{4.19}$$

$$\Psi_2(\sum_{k=1}^{3}e_k)=f(\sum_{k=1}^{3}e_k),\quad \Psi_2(e_i+e_j)=f(e_i+e_j) \quad (i,j=1,2,3;\ i\neq j). \tag{4.20}$$

Proof. Clearly,

$$\Psi_1(\sum_{i=1}^{3}\xi_i e_i)=\sum_{i=1}^{3}\xi_i(f(e_i)-f(0))+f(0) \qquad (\sum_{i=1}^{3}\xi_i e_i\in R^3), \tag{4.21}$$

$$\Psi_2(\sum_{i=1}^{3}\xi_i e_i)=\sum_{i=1}^{3}\{f(\sum_{j=1}^{3}e_j)-f(\sum_{j\neq i}e_j)\}\xi_i+\sum_{i=1}^{3}f(\sum_{j\neq i}e_j)-2f(\sum_{j=1}^{3}e_j)$$

$$(\sum_{i=1}^{3}\xi_i e_i\in R^3). \tag{4.22}$$

Thus, by (4.21), (4.22) and supermodularity, we have

$$\Psi_1(e_i+e_j)=f(e_i)+f(e_j)-f(0)\leqslant f(e_i+e_j) \qquad (i,j=1,2,3;\ i\neq j),$$

$$\Psi_1(\sum_{i=1}^{3}e_i)=\sum_{i=1}^{3}f(e_i)-2f(0)\leqslant f(e_1+e_2)+f(e_3)-f(0)\leqslant f(\sum_{i=1}^{3}e_i),$$

$$\Psi_2(0)=\{f(e_1+e_2)+f(e_1+e_3)-f(\sum_{i=1}^{3}e_i)\}+\{f(e_2+e_3)-f(\sum_{i=1}^{3}e_i)\}\leqslant$$

$$\leqslant f(e_1)+\{f(e_3)-f(e_1+e_3)\}\leqslant f(0),$$

$$\Psi_2(e_i)=f(\sum_{j=1}^{3}e_j)-f(\sum_{j\neq i}e_j)+\Psi_2(0)\leqslant f(\sum_{j=1}^{3}e_j)-f(\sum_{j\neq i}e_j)+f(0)\leqslant f(e_i)$$

$$(i=1,2,3)$$

which, together with (4.19) and (4.20), proves (4.17).

Remark 4.3. a) $\mathrm{co}\ C_3\neq D_1\cup D_2$, and $\{D_1,D_2\}$ is only a proper subfamily of a v-triangulation of co C_3 (e.g., one can take, in addition, $D_3=\mathrm{co}\ \{e_1+e_2,e_1,e_3,e_1+e_3\}$, $D_4=\mathrm{co}\ \{e_1+e_2,e_2,e_3,e_2+e_3\}$, $D_5=\mathrm{co}\ \{e_1+e_2,e_1+e_3,e_2+e_3,e_3\}$ and $D_6=\mathrm{co}\ \{e_1+e_2,e_1,e_2,e_3\}$), but $C_3=\mathcal{E}(D_1)\cup\mathcal{E}(D_2)$.

b) For an arbitrary n, if $f:C_n\rightarrow R$ is supermodular, then for the unique affine function $\Psi_1:R^n\rightarrow R$ which interpolates f at the vertices of the n-simplex

$$\tilde{D}_1=\text{co }\{0,e_k\}_{k=1}^n \tag{4.23}$$

we have $\Psi_1\leq f$. Indeed, if $\Psi_1(0)=f(0)$, $\Psi_1(e_i)=f(e_i)$ $(i=1,\ldots,n)$ and if $\Psi_1(\sum_{k\in S} e_k)\leq f(\sum_{k\in S} e_k)$ for all $S\subseteq\{1,\ldots,n\}$ with $|S|\leq s-1$, where $s\leq n$, then for any $S\subseteq\{1,\ldots,n\}$ with $|S|=s$ and any $k_o\in S$ we have

$$\Psi_1(\sum_{k\in S} e_k)=\Psi_1(\sum_{k\in S\setminus\{k_o\}} e_k)+\Psi_1(e_{k_o})-\Psi_1(0)\leq$$

$$\leq f(\sum_{k\in S\setminus\{k_o\}} e_k)+f(e_{k_o})-f(0)\leq f(\sum_{k\in S} e_k).$$

c) If f is a convex game, then, since the unique affine function $\Psi_2:R^n\to R$ which interpolates f at the vertices of the n-simplex

$$\tilde{D}_2=\text{co }\left\{\sum_{i\neq k} e_i\ ,\ \sum_{i=1}^n e_i\right\}_{k=1}^n , \tag{4.24}$$

is just the last function constructed in the proof of the Rosenmüller-Weidner theorem mentioned above, we have, by [14], $\Psi_2\leq f$; propositions 3.2 and 4.1 show that, if n=2 or n=3, then the same conclusion holds for any supermodular f.

5. APPLICATIONS TO COMBINATORIAL OPTIMIZATION

For a function $f:C_n\to R$, we shall consider the combinatorial optimization problems

$$(P_{min})\qquad \min f(C_n)=\min_{x\in C_n} f(x)=\min_{S\subseteq\{1,\ldots,n\}} f(\sum_{i\in S} e_i), \tag{5.1}$$

$$(P_{max})\qquad \max f(C_n)=\max_{x\in C_n} f(x)=\max_{S\subseteq\{1,\ldots,n\}} f(\sum_{i\in S} e_i). \tag{5.2}$$

Lovász has observed ([10], lemma 4.3) that for any function $f:C_n\to R$ with $f(0)=0$ and with Lovász extension $\hat{f}$, we have

$$\min f(C_n)=\min \hat{f}(\text{co } C_n), \tag{5.3}$$

and that, if f is submodular, whence $\hat{f}$ is convex, then, by (5.3), we can study (P_{min}) by applying known results of convex minimization theory to the problem

$(\hat{P}_{min})$ $\quad \min \hat{f}(\mathrm{co}\, C_n)$. (5.4)

Generalizing this method, in the present section we shall consider, for a function $f:C_n \to R$, various extensions $\overline{f}:\mathrm{co}\, C_n \to R$, and we shall study problems (P_{min}), (P_{max}) with the aid of the optimization problems

$(\overline{P}_{min})$ $\quad \min \overline{f}(\mathrm{co}\, C_n)$, (5.5)

$(\overline{P}_{max})$ $\quad \max \overline{f}(\mathrm{co}\, C_n)$, (5.6)

when the minima in (5.1) and (5.5) are equal, or the maxima in (5.2) and (5.6) are equal, or both. Of course, if $\overline{f}$ is any convex extension of f (see theorem 4.1), then (5.6) is attained at some $x_0 \in \mathcal{E}(\mathrm{co}\, C_n)=C_n$, and hence the maxima in (5.2) and (5.6) coincide; however, we shall not use here this remark.

<u>Theorem 5.1</u>. <u>For any tight extension</u> $\overline{f}$ <u>of a function</u> $f:C_n \to R$, <u>we have</u>

$$\min f(C_n)=\min \overline{f}(\mathrm{co}\, C_n), \tag{5.7}$$

$$\max f(C_n)=\max \overline{f}(\mathrm{co}\, C_n). \tag{5.8}$$

<u>Proof</u>. It will be sufficient to consider (P_{min}). Since $\overline{f}|_{C_n}=f$, we have the inequality $\geq$ in (5.7). For the opposite inequality, let $\overline{f}=\overline{f}^{\mathcal{D}}$, $\mathcal{D}=\{D_i\}_{i=1}^{p}$ and $\{\Psi_i\}_{i=1}^{p}$ be as in definition 2.2. Then, by $\mathrm{co}\, C_n=\bigcup_{i=1}^{p} D_i$ and (2.9), there exists $i_0 \leq p$ such that

$$\min \overline{f}(\mathrm{co}\, C_n)=\min \overline{f}(D_{i_0})=\min \Psi_{i_0}(D_{i_0}) . \tag{5.9}$$

But, the affine function Ψ_{i_0} attains its minimum on D_{i_0} at some $x_k^{i_0} \in \mathcal{E}(D_{i_0})$. Then, by condition (ji) of definition 2.1, we have $x_k^{i_0} \in C_n$, whence, by (2.10) and (5.9),

$$\min f(C_n) \leq f(x_k^{i_0})=\Psi_{i_0}(x_k^{i_0})=\min \Psi_{i_0}(D_{i_0})=\min \overline{f}(\mathrm{co}\, C_n),$$

which, together with the inequality $\geq$ observed above, yields (5.7).

<u>Remark 5.1</u>. From theorem 5.1 it follows that if $f:C_n \to R$ admits a convex tight extension $\overline{f}:\mathrm{co}\, C_n \to R$ (e.g., by theorem 3.3, this is

the case when f is submodular), then one can study (P_{min}) with the following method: By theorem 3.2, implication $1^{\circ} \Rightarrow 3^{\circ}$, $\overline{f}$ can be extended to a continuous convex function $\tilde{f}:R^n \rightarrow R$, and then, by theorem 5.1, we have

$$\min f(C_n)=\min \overline{f}(\mathrm{co}\ C_n)=\min \tilde{f}(\mathrm{co}\ C_n); \tag{5.10}$$

hence, from results on the problem of minimizing the continuous convex function $\tilde{f}:R^n \rightarrow R$ on the convex set co C_n, one can obtain results on problem (P_{min}). Let us note that the "economical" (in particular, the "minimal") representations of a function $f:C_n \rightarrow R$ as $f= \max_{1\leq i\leq p} \Psi_i|_{C_n}$, with $\Psi_i:R^n \rightarrow R$ affine (e.g., (3.34) for a submodular function f, or (4.17) for a supermodular function $f:C_3 \rightarrow R$) are not appropriate for such a study of (P_{min}), since for them the polyhedral convex extension $\overline{f}= \max_{1\leq i\leq p} \Psi_i|_{\mathrm{co}\ C_n}$ of f is not tight and (5.7), (5.10) need not hold.

Using the method described in remark 5.1, let us prove

Theorem 5.2. *If $f:C_n \rightarrow R$ admits a convex tight extension $\overline{f}$: :co $C_n \rightarrow R$ (in particular, if f is submodular), then*

$$\min f(C_n)= \max_{\Phi\in(R^n)^*} \min_{x\in C_n} \{f(x)-\Phi(x)+\min \Phi(C_n)\}. \tag{5.11}$$

Proof. By theorem 3.2, implication $1^{\circ} \Rightarrow 3^{\circ}$, we can extend $\overline{f}$ to a polyhedral convex continuous function $\tilde{f}:R^n \rightarrow R$. Hence, by (5.10) and [16], theorem 2.1, and since $-\Phi(x)+\inf \Phi(\mathrm{co}\ C_n)\leq 0$ $(x\in \mathrm{co}\ C_n)$,

$$\min f(C_n)= \max_{\Phi\in(R^n)^*} \inf_{x\in R^n} \{\tilde{f}(x)-\Phi(x)+\inf \Phi(\mathrm{co}\ C_n)\}=$$

$$= \max_{\Phi\in(R^n)^*} \inf_{x\in \mathrm{co}\ C_n} \{\overline{f}(x)-\Phi(x)+\inf \Phi(\mathrm{co}\ C_n)\}. \tag{5.12}$$

But, since $C_n=\mathcal{E}(\mathrm{co}\ C_n)$, for each $\Phi\in(R^n)^*$ we have $\inf \Phi(\mathrm{co}\ C_n)= =\min \Phi(C_n)$. Furthermore, since $\overline{f}$ is a convex tight extension of f, $\overline{f}-\Phi$ is a convex tight extension of $f-\Phi|_{C_n}$, and hence, by (5.7) applied to $f-\Phi|_{C_n}$, the right hand sides of (5.12) and (5.11) coincide.

Remark 5.2. Just as [16], theorem 2.1, used above, is obtained in [16] from the well-known duality theorem of Fenchel-Rockafellar ([12], theorem 1), theorem 5.2 can be also deduced from the recent Fenchel-Rockafellar-type theorem of [7], theorem 3.3 (identifying the functions $\Phi:\{1,\dots,n\}\to R$, used in [7], with functions $\Phi\in(R^n)^*$, as in remark 3.3 a) above).

Similarly, for problem (P_{max}), let us prove

Theorem 5.3. If $f:C_n\to R$ admits a convex tight extension $\overline{f}$: $:\mathrm{co}\, C_n\to R$ (in particular, if f is submodular), then

$$\max f(C_n)=\max_{\Phi\in(R^n)^*}\ \min_{x\in C_n}\{f(x)-\Phi(x)+\max \Phi(C_n)\}. \tag{5.13}$$

Proof. Extend $\overline{f}$, as in the above proof, to a continuous convex function $\tilde{f}:R^n\to R$. Then, by (5.8), $\tilde{f}|_{\mathrm{co}\, C_n}=\overline{f}$ and [18], theorem 2.1, we obtain (considering $x_{\Phi,\varepsilon}\in\mathrm{co}\, C_n$ with $-\Phi(x_{\Phi,\varepsilon})+\sup \Phi(\mathrm{co}\, C_n)<\varepsilon$)

$$\max f(C_n)=\sup_{\Phi\in(R^n)^*}\ \inf_{x\in R^n}\{\tilde{f}(x)-\Phi(x)+\sup \Phi(\mathrm{co}\, C_n)\}=$$

$$=\sup_{\Phi\in(R^n)^*}\ \inf_{x\in \mathrm{co}\, C_n}\{\overline{f}(x)-\Phi(x)+\sup \Phi(\mathrm{co}\, C_n)\}. \tag{5.14}$$

But, similarly to the proof of theorem 5.2, for each $\Phi\in(R^n)^*$ we have $\sup \Phi(\mathrm{co}\, C_n)=\max \Phi(C_n)$ and $\overline{f}-\Phi$ is a convex tight extension of $f-\Phi|_{C_n}$, whence, by (5.7) applied to $f-\Phi|_{C_n}$ and by (5.14), we obtain

$$\max f(C_n)=\sup_{\Phi\in(R^n)^*}\ \min_{x\in C_n}\{f(x)-\Phi(x)+\max \Phi(C_n)\}, \tag{5.15}$$

so it remains to show that the supremum is attained. Let $x_o\in C_n$ be such that $f(x_o)=\max f(C_n)$ and, as in the proof of theorem 4.1, choose $\Phi_o\in(R^n)^*$ satisfying

$$\Phi_o(x)<\max \Phi_o(C_n)=\Phi_o(x_o) \qquad (x\in C_n\setminus\{x_o\}). \tag{5.16}$$

Then, for each $x\in C_n\setminus\{x_o\}$ there exists $\lambda_x>0$ such that

$$f(x_o)-f(x)\leq\lambda\{\max \Phi_o(C_n)-\Phi_o(x)\} \qquad (\lambda\geq\lambda_x). \tag{5.17}$$

Let $\lambda_o=\max_{x\in C_n\setminus\{x_o\}}\lambda_x$. Then, by $\max f(C_n)=f(x_o)$, (5.17), $\lambda_o>0$ and $\Phi_o(x_o)=\max \Phi_o(C_n)$, we obtain

$$\max f(C_n)=\min_{x\in C_n} \{f(x)-\lambda\Phi_o(x)+\max(\lambda\Phi_o)(C_n)\} \quad (\lambda\geq\lambda_o), \tag{5.18}$$

so the sup in (5.15) is attained for all $\lambda\Phi_o$ $(\lambda\geq\lambda_o)$.

<u>Theorem 5.4. Under the assumptions of theorem 5.3, we have</u>

$$\max f(C_n)= \max_{\Phi\in(R^n)^*} \min_{\substack{x\in C_n \\ \Phi(x)=\max\Phi(C_n)}} f(x). \tag{5.19}$$

<u>Proof</u>. Similarly to the above proof of theorem 5.3, using now [17], theorem 2.1, we obtain (5.19) with $\max\limits_{\Phi\in(R^n)^*}$ replaced by sup, so it remains to show that this supremum is attained. But, since for each $\Phi\in(R^n)^*$, $F_\Phi=\{x\in C_n | \Phi(x)=\max\Phi(C_n)\}$ is a subset of C_n, the function $\mu:(R^n)^*\to R$ defined by

$$\mu(\Phi)=\min f(F_\Phi) \quad (\Phi\in(R^n)^*) \tag{5.20}$$

assumes at most 2^{2^n} distinct values on $(R^n)^*$, and hence sup $\mu((R^n)^*)$ is attained.

<u>Remark 5.3</u>. a) One can also give the following direct proof of theorem 5.4: The inequality $\geq$ in (5.19) is obvious. Furthermore, let $x_o\in C_n$ be such that $f(x_o)=\max f(C_n)$ and, as in the proof of theorem 4.1, choose $\Phi_o\in(R^n)^*$ satisfying (5.16). Then $\{x\in C_n | \Phi_o(x)=\max\Phi_o(C_n)\}=\{x_o\}$, whence

$$\max f(C_n)=f(x_o)= \min_{\substack{x\in C_n \\ \Phi_o(x)=\max\Phi_o(C_n)}} f(x).$$

b) Theorems 5.3 and 5.4 may be regarded as "duality theorems" for the NP-hard problem (P_{max}) of (5.2). Indeed, although these results are not of max-min type, they involve the natural "dual variables" $\Phi\in(R^n)^*$.

<u>6. APPENDIX: ON THE "GREEDY SOLUTIONS" OF A LINEAR MAXIMIZATION PROBLEM</u>

By remark 3.3 a), formula (3.28), the linear optimization problem considered in [10], p.247, can be formulated as the problem of maximizing, for a given $x\in R^n_+$, the linear function $h_x(\Phi)=\Phi(x)$

3°. For each $x\in R^n_+$, every greedy solution Φ_π is an optimal solution of problem (6.1).

Proof. By (3.24) with $M=C_n$, $g=f$ and $x_o=0$, a function $\Phi\in(R^n)^*$ is a feasible solution of (6.1) (i.e., $\Phi\in\partial f(0)$) if and only if $\Phi(x)\leqslant f(x)$ $(x\in C_n)$. Hence, by the above interpretation of greedy solutions, the equivalence $1^\circ\Leftrightarrow 2^\circ$ follows from theorem 3.3, equivalence $1^\circ\Leftrightarrow 6^\circ$.

$1^\circ\Rightarrow 3^\circ$. Assume that 1° holds and let $x\in R^n_+$ be arbitrary. Choose $\pi\in\Pi_n$ such that $x\in K_\pi$, and let $\Phi_\pi\in(R^n)^*$ be the greedy solution of (6.1) interpolating f at the vertices of B_π. Then, by the implication $1^\circ\Rightarrow 2^\circ$ above, $\Phi_\pi\in\partial f(0)$. Furthermore, since $x\in K_\pi$, for any $f:C_n\to R$ we have, by theorem 2.2, $\Phi_\pi(x)=\hat f(x)$. Hence, by 1° and (3.27) we obtain

$$\Phi_\pi(x)=\hat f(x)=\max_{\Phi\in\partial f(0)}\Phi(x),$$

so Φ_π is an optimal solution of problem (6.1).

Finally, the implication $3^\circ\Rightarrow 2^\circ$ is obvious.

Remark 6.1. One does not need all $x\in R^n_+$ in order to exhaust all "greedy functions" Φ_π $(\pi\in\Pi_n)$. In fact, already n! elements $x\in R^n_+$ are sufficient to this end, e.g., one can take, for each $\pi\in\Pi_n$, one element $x_\pi\in\mathrm{Int}\,K_\pi$. Then, one can replace, in 2° and 3° above, "for each $x\in R^n_+$", by: for each x_π $(\pi\in\Pi_n)$. Moreover, in general, a proper subset of these x_π's will suffice, as shown by the following example: Let f be the rank function of the matroid $M=(\{1,2,3\},\mathcal{J})$, in which the family of independent sets is $\mathcal{J}=\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\}\}$ (for these concepts, see e.g. [20]), transformed, by (1.3), into a function $f:C_3\to R$ (so $f(0)=0$). Then there are only two distinct greedy functions (since three permutations $\pi\in\Pi_3$ yield the same greedy function $\Phi_1(x)=\xi_1+\xi_2$ and the other three permutations yield $\Phi_2(x)=\xi_1+\xi_3$, for $x=\{\xi_1,\xi_2,\xi_3\}\in R^3$). We shall return to related problems in a subsequent paper.

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