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by

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by Andrei IORDAN

1. INTRODUCTION. Let $D\subset\mathbb{C}^n$ be a bounded pseudoconvex domain with smooth boundary. We denote by $A^\infty(D)$ the set of holomorphic functions in D which have a C^∞ extension to $\overline{D}$. A compact set $E\subset\partial D$ is a peak set for $A^\infty(D)$ if there exists $f\in A^\infty(D)$ such that $f=0$ on E and $\mathrm{Re}f>0$ on $\overline{D}\setminus E$. Such a function will be called a strong support function for E.

If D is strictly pseudoconvex, Chaumat and Chollet proved in [1] that each closed subset of a peak set for $A^\infty(D)$ is a peak set for $A^\infty(D)$. The result was extended by A.V.Noell in [3] to bounded pseudoconvex domains in $\mathbb{C}^2$ of finite type. An example of pseudoconvex domain (not of finite type) where this property fails to be true is also given in [3].

Here, we prove that the result is also true for peak sets for $A^\infty(D)$, where D is a bounded pseudoconvex domain in $\mathbb{C}^n$ which is strictly pseudoconvex except at a finite number of points. We know by [4] that such domains have the Mergelyan property.

The methods used in the paper are based on those used by Chaumat and Chollet to prove the assertion in the strictly pseudoconvex case.

We recall here two propositions of [1] which we shall use in the paper.

Proposition 1([1]). Let D be a bounded strictly pseudoconvex domain with smooth boundary, K a peak set for $A^\infty(D)$, $f\in A^\infty(D)$ a strong support function for K and $p\in K$. Then there exists a neighborhood ω of p, a C^∞ totally real n-dimensional submanifold M of $\partial D\cap\omega$ and $c>0$ such that $K\cap\omega\subset M$ and $\mathrm{Re}f(z)\geq cd(z,M)^2$ for each $z\in\overline{D}\cap\omega$.

The proof of proposition 1 depends only on the strong pseudoconvexity at p, so the theorem is true for points of strong pseudoconvexity in peak sets in weakly pseudoconvex domains. This remark was used by A.V.Noell in [3].

Here, we shall give a generalization of proposition 1 for weakly pseudoconvex domains (proposition 3). The estimation for Ref(z) depends on the number of the zero-eigenvalues of the Levi form and it is in a certain sense the best possible.

Proposition 2 ([1]). Let $D \subset C^n$ be a bounded pseudoconvex domain with smooth boundary, E a compact subset of ∂D, Ω a neighborhood of E in C^n and ρ a continuous function on Ω which vanishes on E. We suppose that there exists $G \in C^{\infty}(\Omega \cap \bar{D})$ such that:

a) $\{z \in \bar{D} \cap \Omega \mid G(z)=0\}=E$

b) for each $\alpha \in N^n$, $k \in N$, there exists $C_{\alpha k}>0$ such that $|D^{\alpha}\bar{\partial}(G(z))| \leq C_{\alpha k}\rho(z)^k$ for each $z \in \bar{D} \cap \Omega$.

c) There exists $c>0$ such that $\operatorname{Re}G(z) \geq c\rho(z)$ for each $z \in \bar{D} \cap \Omega$.

Then E is a peak set for $A^{\infty}(D)$.

I wish to thank dr.G.Cussi for his advices and his continuous encouragements.

2. Local properties of strong support functions in weakly pseudoconvex domains.

Proposition 3. Let $D \subset C^n$ be a pseudoconvex domain with smooth boundary, $E \subset \partial D$ a peak set for $A^{\infty}(D)$, f a strong support function for E and $p \in E$ such that the Levi form has q zero-eigenvalues at p. Then there exists a neighbourhood ω of p, an n+q-dimensional generic submanifold M of $\partial D \cap \omega$ and $c>0$ such that $E \cap \omega \subset M$ and $\operatorname{Re} f(z) \geq c\,d(z,M)^2$ for each $z \in \bar{D} \cap \omega$.

Proof

We must present a complete proof of this result, though most of the arguments are similar to those of the proof of proposition 1.

By making a complex-linear change of coordinates in C^n we may suppose that p is the origin and in the neighborhood U_1 of the origin D is given by $D\cap U_1=\{(z',w)\in U_1 | \rho(z',w)<0\}$ where $z'=(z_1,\dots,z_{n-1})$, $z_j=x_j+iy_j$, $w=u+iv$ and $\rho(z',w)=u+R_1(z')+R_2(z',w)$, where $R_1(z')$ is a second order omogenuous polynomial in z', $\bar{z}'$, and $R_2(z',w)=O(|z'|\,|w|+|w|^2+|z'|^3)$.

Because (0,0) is a local minimum for Ref, by the Hopf lemma we obtain that $\frac{\partial \mathrm{Re}f}{\partial u}(0,0)<0$, $\frac{\partial \mathrm{Re}f}{\partial v}(0,0)=0$, $\frac{\partial \mathrm{Re}f}{\partial x_j}(0,0)=\frac{\partial \mathrm{Re}f}{\partial y_j}(0,0)=0$, $1\leq j\leq n-1$.

It follows that in a neighborhood U_2 of the origin, $U_2\subset U_1$, we have $\mathrm{Re}f(z',w)=\frac{\partial \mathrm{Re}f}{\partial u}(0,0)u+K_1(z',w)+K_2(z',w)$ where $K_1(z',w)$ is a second order pluriarmonic polynomial in $z',\bar{z}',w,\bar{w}$ and $K_2(z',w)=O((|z'|+|w|)^3)$.

From the Cauchy-Riemann equations at the origin we obtain that $\frac{\partial \mathrm{Im}f}{\partial v}(0,0)<0$, $\frac{\partial \mathrm{Im}f}{\partial u}(0,0)=0$, $\frac{\partial \mathrm{Im}f}{\partial x_j}(0,0)=\frac{\partial \mathrm{Im}f}{\partial y_j}(0,0)=0$, $j=1,\dots,n-1$.

Because $\frac{\partial(\rho,\mathrm{Im}f)}{\partial(w,\bar{w})}(0,0)=\frac{i}{2}\frac{\partial \mathrm{Re}f}{\partial u}(0,0)\neq 0$ it follows that the set $\Sigma=\{(z',w)\,|\,\rho(z',w)=0,\ \mathrm{Im}f(z',w)=0\}$ is in a neighborhood U_3 of the origin, $U_3\subset U_2$, an 2n-2-dimensional C^∞-submanifold of the boundary which contains $E\cap U_3$.

So, there exists a C^∞-function $h=h(z')$ defined in a neighborhood V_1 of $0\in C^{n-1}$ such that $\Sigma=\{(z',w)\,|\,w=h(z')\}$.

We have $\rho(z',h(z'))=0=\mathrm{Re}\ h(z')+R_1(z')+R_2(z',h(z'))$ and because the first order derivatives of h vanish at the origin we obtain that $\mathrm{Re}\ h(z')=-R_1(z')+O(|z'|^3)$.

We define $\theta(z')=\mathrm{Re}\ f(z',h(z'))=\frac{\partial \mathrm{Re}f}{\partial u}(0,0)\mathrm{Re}\ h(z')+K_1(z',h(z'))+K_2(z',h(z'))=-\frac{\partial \mathrm{Re}f}{\partial u}(0,0)R_1(z')+K_1(z',0)+O(|z'|^3)$.

The complex tangent space of ∂D at (0,0) is $\{(z',w)\,|\,w=0\}$, hence the complex Hessian of θ has n-q-1 strictly positive eigenvalues and q zero-eigenvalues at 0.

Because f is a strong support function for E we have $\theta(z')\geq 0$ and $\theta(z')=0$ if and only if $(z',h(z'))\in E$.

We denote by $Z=\{z\in V_1 \mid \theta(z')=0\}$.

From [2] it follows that there exists a complex-linear change of coordinates in C^{n-1} such that in the new coordinates (which we shall denote also $z'=(z'_1,\dots,z'_{n-1})$) we have:

$$\theta(z')=\sum_{j=1}^{n-q-1}((1+\lambda_j)x_j^2+\sum_{j=1}^{n-q-1}(1-\lambda_j)y_j^2+\sum_{i=1}^{n-q-1}\sum_{j=n-q}^{n-1} a_{ij}x_ix_j +$$

$$+b_{ij}y_iy_j+c_{ij}x_iy_j+d_{ij}x_jy_j)+\sum_{i,j=n-q}^{n-1}(\alpha_{ij}x_ix_j+$$

$$+\beta_{ij}y_iy_j+\gamma_{ij}x_iy_j)+O(|z|^3)\quad,\ \lambda_j\geq 0 \qquad (1)$$

The set $N=\{z'\in V_1 \mid \frac{\partial\theta}{\partial x_j}(z')=0,\ 1\leq j\leq n-q-1\}$ is in a neighborhood $V_2\subset V_1$ of $0\in C^{n-1}$ an $n+q-1$-dimensional generic submanifold of C^{n-1} which contains $Z\cap V_2$ [2].

We denote by $\tau(z)=J(\operatorname{grad}\rho(z))$ where J represents the complex structure on $C^n=R^{2n}$. Because $T_0(\Sigma)=\{(z,w) \mid w=0\}$, it follows that τ is transversal to Σ at $(0,0)$, hence there exists a neighborhood $U_4\subset U_3$ such that τ is transversal to Σ on U_4.

Therefore there exists a C^∞-diffeomorphism φ defined on $O_\varepsilon=\{(z',t) \mid z'\in V_2,\ t\in(-\varepsilon,\varepsilon)\}$ with values in ∂D such that

$$\varphi(z',0)=(z',h(z'))\ \text{and}\ \frac{\partial\varphi}{\partial t}(z',0)=\tau(z',h(z')), \qquad (2)$$

$$\text{Because } Z\cap V_2\subset N \text{ we have } E\cap U_4\subset\varphi(Z\times\{0\})\subset\varphi(N\times\{0\}) \qquad (3)$$

We denote by $\phi(z',t)=\operatorname{Re} f(\varphi(z',t))$ and by

$$\tilde N=\{(z',t)\in O_\varepsilon \mid r_j(z',t)=0,\ 1\leq j\leq n-q-1\}\ \text{where}\ r_j(z',t)=\frac{\partial\phi}{\partial x_j}(z',t).$$

Let us denote by $\{e_1,\dots,e_n\}$ the standard basis in $C^n=R^{2n}$ and by H the orthogonal complement (in respect to the inner product in R^{2n}) of S_o, where S_o is the subspace of $R^{2n-2}\times R$ generated by

$\{e_1,\dots,e_{n-q-1}\}$.

Because $r_j(z',0)=\frac{\partial\phi}{\partial x_j}(z',0)=\frac{\partial\theta}{\partial x_j}(z')$ (4), from (1) we conclude that $(\text{grad } r_j)(0,0)=2(1+\lambda_j)e_j+h_j$ (5) where $h_j\in H$.

It follows that $\frac{\partial(r_1,\dots,r_{n-q-1})}{\partial(x_1 y_1,\dots,x_{n-1},y_{n-1},t)}(0)$ has maximal rank $n-q-1$ and $\tilde{N}$ is the neighborhood of the origin a C^∞-submanifold of O_ε of dimension $n+q$.

From (1) and (4) we obtain that the restriction to S_o of the Hessian of ϕ at the origin is strictly positive, so there exists $c_o>0$ such that $\phi(d+x)\geqslant c_o\|x\|^2$ (6) for each $d\in\tilde{N}$ and $x\in S_o$.

We shall prove now that $S_o\oplus T_{(0,0)}(\tilde{N})=R^{2n-2}\times R$ (7)

We have $\dim S_o=n-q-1$ and $\dim T_{(0,0)}(\tilde{N})=n+q$, hence, to prove (7) it is sufficient to prove that $T_{(0,0)}(\tilde{N})\cap S_o=\{0\}$.

Let $x=\mu_1 e_1+\dots+\mu_{n-q-1}e_{n-q-1}\in T_{(0,0)}(\tilde{N})=[\text{grad } r_1(0,0),\dots$ $\dots,\text{grad } r_{n-q-1}(0,0)]^\perp$.

It follows that $x\perp \text{grad } r_j(0,0)$ for $j=1,\dots,n-q-1$ or, by (5) $\langle\sum_{i=1}^{n-q-1}\mu_i e_i,\ 2(1+\lambda_j)e_j+h_j\rangle=0$ for $j=1,\dots,n-q-1$, where $\langle,\rangle$ is the inner product in R^{2n}. We obtain $2(1+\lambda_j)\mu_j=0$ for $j=1,\dots,n-q-1$, hence $x=0$ and (7) is proved.

We consider now the map $\Psi:(U\cap\tilde{N})\times(V\cap S_o)\longrightarrow O_\varepsilon$, $\Psi(d,x)=d+x$ where U and V are suitable neighborhoods of the origin. From (7), for U, V and O_ε sufficiently small we obtain that Ψ is a C^∞-diffeomorphism.

We denote $M=\varphi(\tilde{N})$ which is an $n+q$-dimensional submanifold of ∂D. From (3) and (4) we obtain that $E\cap U\subset M$.

Because $\varphi\circ\Psi$ is a C^∞-diffeomorphism from $(U\cap\tilde{N})\times(V\cap S_o)$ on $\varphi(O_\varepsilon)$ we obtain that there exists $\gamma>0$ such that

$\|x\|\geqslant\gamma\|\varphi(x+d)-\varphi(d)\|\geqslant\gamma\, d(\varphi(x+d),M)$ and from (6) it follows that $\text{Re}f(z)\geqslant\gamma^2 c_o d(z,M)^2$ (8) for each $z\in\partial D\cap\varphi(O_\varepsilon)$.

Because $\langle \operatorname{grad} \operatorname{Re} f(0), -\operatorname{grad} \rho(0) \rangle = 1$, there exists $\gamma_2 > 0$ such that $\langle \operatorname{grad} \operatorname{Re} f(z), -\operatorname{grad} \rho(z) \rangle \geq \gamma_2$ (9) for each $z \in \varphi(O_\varepsilon)$, if O_ε is small enough.

Using (8), (9) and Taylor developments in the normal directions to ∂D we obtain (as in [1]) that there exists a neighborhood ω of the origin and $c>0$ such that $\operatorname{Re} f(z) \geq c\, d(z,M)^2$ for each $z \in \bar{D} \cap \omega$.

To finish the proof of proposition 3, we have to prove that M is generic in the neighborhood of the origin.

Let us define $\tilde{\varphi}: O_\varepsilon \times R \longrightarrow C^n$,

$$\tilde{\varphi}(z',t,t') = (\varphi_1(z',t), \ldots, \varphi_{2n-2}(z',t), \varphi_{2n-1}(z',t)+t', \varphi_{2n}(z',t)).$$

We have $\tilde{\varphi}(z',t,0) = \varphi(z',t)$ and $\frac{\partial \tilde{\varphi}_j}{\partial x_k} = \frac{\partial \varphi_j}{\partial x_k}, \frac{\partial \tilde{\varphi}_j}{\partial y_k} = \frac{\partial \varphi_j}{\partial y_k}$,

$1 \leq j \leq 2n$, $1 \leq k \leq n-1$, $\frac{\partial \tilde{\varphi}_j}{\partial t} = \frac{\partial \varphi_j}{\partial t}$, $1 \leq j \leq 2n$,

$\frac{\partial \tilde{\varphi}_j}{\partial t'} = 0$ $j=1,\ldots,2n-2$ and $j=2n$, $\frac{\partial \tilde{\varphi}_{2n-1}}{\partial t'} = 1$.

From (2) we obtain that

$$\frac{\partial(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{2n})}{\partial(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, t, t')}(0) = \begin{bmatrix} 1 & \cdots\cdots\cdots & 0 & 0 \\ \cdots\cdots & \cdots\cdots & & \\ 0 & \cdots\cdots 1 & 0 & 0 \\ 0 & \cdots\cdots 0 & 0 & 1 \\ 0 & \cdots\cdots 0 & 1 & 0 \end{bmatrix}$$

and it follows that $\tilde{\varphi}$ is a C^∞-diffeomorphism in the neighborhood of the origin.

Because $\tilde{\varphi}(O_\varepsilon \times \{0\}) = \varphi(O_\varepsilon)$ we obtain that $\varphi(O_\varepsilon) \cap U = \tilde{\varphi}(O_\varepsilon \times \{0\}) \cap U = \{z \in U \mid \rho(z) = 0\}$ for U sufficiently small.

Because $M = \varphi(\tilde{N}) = \tilde{\varphi}(\tilde{N} \times \{0\})$ we have $z \in M \cap U$ if and only if $\tilde{\varphi}^{-1}(z) = (z',t,0)$ with $(z',t) \in \tilde{N}$.

From the definition of $\tilde{N}$ we obtain that

$$M\cap U=\{z\in U\mid \rho(z)=0,\ r_1(pr(\tilde{\varphi}^{-1}(z))=\dots=r_{n-q-1}(pr(\tilde{\varphi}^{-1}(z)))=0\},$$ where

$pr:R^{2n}\longrightarrow R^{2n-1}$ is the projection on the first $2n-1$ variables.
Let us denote $s_j(z)=r_j(pr(\tilde{\varphi}^{-1}(z)))$, $j=1,\dots,n-q-1$.

We have $$\left[\frac{\partial(\tilde{\varphi}_1,\dots,\tilde{\varphi}_{2n})}{\partial(x_1,y_1,\dots,x_{n-1}y_{n-1},t,t')}(0)\right]^{-1}=$$

$$=\frac{\partial(\tilde{\varphi}_1,\dots,\tilde{\varphi}_{2n})}{\partial(x_1,y_1,\dots,x_{n-1}y_{n-1},t,t')}(0)$$

and it follows that $\frac{\partial s_j}{\partial x_k}(0)=\frac{\partial r_j}{\partial x_k}(0)$, $\frac{\partial s_j}{\partial y_k}(0)=\frac{\partial r_j}{\partial y_k}(0)$

for $1\leq k\leq n-1$, $1\leq j\leq n-q-1$, $\frac{\partial s_j}{\partial x_n}(0)=0$, $\frac{\partial s_j}{\partial y_n}(0)=\frac{\partial r_j}{\partial t}(0)$, $1\leq j\leq n-q-1$.

From (5) we obtain that $\frac{\partial(\rho,s_1,\dots,s_{n-q-1})}{\partial(x_1,y_1,\dots,x_n,y_n)}(0)$ and
$\frac{\partial(\rho,s_1,\dots,s_{n-q-1})}{\partial(z_1,\dots,z_n)}(0)$ have maximal rank, hence M is an $n+q$-dimensional generic submanifold of C^n in the neighborhood of the origin.

<u>Remark 1.</u> Let D be a pseudoconvex domain with smooth boundary, E a peak set for $A^\infty(D)$ and $p\in E$ such that the Levi form has q zero-eigenvalues at p. We know from [2] that E is locally contained in the neighborhood of p in an $n+q-1$-dimensional submanifold M of ∂D such that $\dim_C TC_z(M)\leq q$ for each $z\in M$, where $TC_z(M)$ is the maximal complex subspace of the tangent space of M at z.

It follows that the generic manifold M obtained in proposition 3 such that $\mathrm{Re}f(z)\geq c d(z,M)^2$ for each $z\in\bar{D}\cap\omega$ is generally the lowest dimensional generic submanifold which satisfies such an estimation.

3. <u>Peak sets in pseudoconvex domain with isolated degeneracies</u>

<u>Theorem 1.</u> Let D be a bounded domain in C^n with smooth

boundary which is strictly pseudoconvex except at a finite number of points. Let $K\subset\partial D$ be a peak set for $A^{\infty}(D)$ and E a closed subset of K. Then E is a peak set for $A^{\infty}(D)$.

Proof.

We denote by $w(\partial D)$ the set of weakly pseudoconvex boundary points of D. Let $f\in A^{\infty}(D)$ be a strong support function for K.

Let $P\in w(\partial D)\cap E$ be an isolated point of K and let us suppose that $E\setminus\{P\}$ is a peak set for $A^{\infty}(D)$ with strong support function $g\in A^{\infty}(D)$. Then E is a peak set for $A^{\infty}(D)$.

Indeed, we choose Ω_1 a neighborhood of $E\setminus\{P\}$ and Ω_2 a neighborhood of P such that $\Omega_1\cap\Omega_2=\emptyset$, $\Omega_2\cap(K\setminus\{P\})=\emptyset$. We consider $\Omega=\Omega_1\cup\Omega_2$, $\rho=0$ and

$$G(z)=\begin{cases} g(z) & \text{if } z\in\Omega_1\cap\overline{D} \\ f(z) & \text{if } z\in\Omega_1\cap\overline{D}\ , \end{cases}$$

and by proposition 2 we obtain that E is a peak set for $A^{\infty}(D)$.

Hence, we can remove from E the isolated points of $K\cap w(\partial D)$.

Let $P\in w(\partial D)\cap K$ and $\{\Omega_n\}_{n\geq 1}$ a fundamental neighborhood system of P in C^n, such that $\Omega_{n+1}\subset\subset\Omega_n$ and $\Omega_1\cap w(\partial D)=\{P\}$. We denote by $U_n=\Omega_{n+1}\setminus\Omega_{n+3}$, $V_n=\Omega_n\setminus\Omega_{n+4}$, $U_n\subset\subset V_n$. Because for each $n\geq 1$, $K\cap\overline{U}_n\subset\partial D\setminus w(\partial D)$, there exist (by proposition 3) finite coverings $\omega_{jn}\subset\subset\omega'_{jn}\subset\subset V_n$, $1\leq j\leq r_n$ of $K\cap\overline{U}_n$, M_{jn} totally real n-dimensional submanifolds of ω'_{jn} and $0<c_{jn}<1$ such that $K\cap\omega'_{jn}\subset M_{jn}$ and $\operatorname{Re} f(z)\geq c_{jn}d(z,M_{jn})^2$ for each $z\in\overline{D}\cap\omega'_{jn}$.

Let $\chi'_{jn}\in C_0^{\infty}(C^n)$, $0\leq\chi'_{jn}\leq 1$, $\operatorname{sup}\chi'_{jn}\subset\subset\omega'_{jn}$, $\chi'_{jn}=1$ on ω_{jn} and we define $\rho(z)=\sum_{n=1}^{\infty}\left(\sum_{j=1}^{r_n}\frac{c_{jn}}{2^j}\chi'_{jn}(z)d(z,M_{jn})^2\right)$. Because $\bigcap_{i=m}^{m+5}V_i=\emptyset$ (1) for each $m\geq 1$

and $\operatorname{supp}\chi'_{jn}\subset\subset\omega'_{jn}\subset\subset V_n$, it follows that $\rho\in C_0^{\infty}(C^n)$.

If $z\in K\cap\operatorname{supp}\chi'_{jn}$, then $z\in K\cap\omega'_{jn}\subset M_{jn}$ and we obtain that ρ vanishes on K.

We denote $\varphi_n = \sum_{j=1}^{r_n} \frac{1}{2^j}\chi'_{jn}$. We have $0 \le \varphi_n \le 1$ and from (1) it follows that $\sum_{n=1}^{\infty}\varphi_n \le 4$.

Hence, if $z \in \bar{D}$ we have $4\,\mathrm{Re} f(z) \geq \sum_{n=1}^{\infty}\varphi_n(z)\mathrm{Re} f(z) =$

$$= \sum_{n=1}^{\infty}\sum_{j=1}^{r_n} \frac{1}{2^j}\chi'_{jn}(z)\mathrm{Re} f(z) \geq \sum_{n=1}^{\infty}\sum_{j=1}^{r_n} \frac{c_{jn}}{2^j}\chi'_{jn}(z)d(z,M_{jn})^2 = \rho(z) \quad (2)$$

Let D_1 be a compact neighborhood of D which contains ω'_{jn} for each n and j and $s \in C_0^\infty(C^n)$, $s \geq 0$ such that $E = \{z \in D_1 | s(z) = 0\}$. Let $\{\chi_{jn}\}$ a partition of unity subordinate to the covering $\bigcup_{n=1}^{\infty}\bigcup_{j=1}^{r_n}\omega_{jn}$ and we denote by $s_{jn} = \chi_{jn}s$. Then, from [5] we obtain that there exist $\tilde{s}_{jn} \in C_0^\infty(C^n)$ such that $\mathrm{supp}\,\tilde{s}_{jn} \subset\subset \omega_{jn}$, $\tilde{s}_{jn} = s_{jn}$ on M_{jn}, $\bar{\partial}\tilde{s}_{jn}$ vanishes to infinite order on M_{jn} and $\mathrm{Re}\,\tilde{s}_{jn} \geq$

$\geq -d_{jn}d(z,M_{jn})^2$ with $d_{jn} > 0$.

Because $\bar{\partial}\tilde{s}_{jn}$ vanishes to infinite order on M_{jn} and $\chi'_{jn} = 1$ on $\mathrm{supp}\,\tilde{s}_{jn}$ it follows that for each $\alpha \in N^n$ and $k \in N$, there exist $C_{\alpha k jn} > 0$ such that

$$|D^\alpha \bar{\partial}\tilde{s}_{jn}(z)| \le C_{\alpha k jn}\left(\frac{c_{jn}}{2^j}\right)^k \left(\chi_{jn}^{'k+1}(z)d(z,M_{jn})^{2k}\right).$$

Let $\varphi_{\alpha k}(z) = \sum_{n=0}^{\infty}\sum_{j=1}^{r_n} C_{\alpha k jn}\chi'_{jn}(z)$. From (1) we obtain that $\varphi_{\alpha k} \in C_0^\infty(C^n)$ and we denote by $C_{\alpha k} = \sup \varphi_{\alpha k}(z)$.

We choose $0 < \alpha_n < 1$, $\alpha_n \le \min_{1 \le j \le r_n}\left\{\frac{c_{jn}}{2^j}\right\}$, $\alpha_n \le \min_{1 \le j \le r_n}\left\{\frac{1}{d_{jn}}\right\}$ and we define $\tilde{s} = \sum_{n=1}^{\infty}\alpha_n^2 \sum_{j=1}^{r_n}\tilde{s}_{jn}$.

We have $|D^\alpha\bar{\partial}\tilde{s}(z)| \le \sum_{n=1}^{\infty}\alpha_n^2 |D^\alpha\bar{\partial}\tilde{s}_{jn}(z)| \le$

$$\le \sum_{n=1}^{\infty}\sum_{j=1}^{r_n} C_{\alpha k jn}\left(\frac{c_{jn}}{2^j}\right)^k \chi_{jn}^{'k+1}(z)\, d(z,M_{jn})^{2k} \le$$

$$\leq \Big(\sum_{n=1}^{\infty}\sum_{j=1}^{r_n} C_{\alpha k j n}\,\chi'_{jn}(z)\Big)\Big(\sum_{n=1}^{\infty}\sum_{j=1}^{r_n}\Big(\frac{c_{jn}}{2^j}\Big)^k \chi'^k_{jn}(z)\,d(z,M_{jn})^{2k}\Big)\leq$$

$$\leq C_{\alpha k}\,\rho^k(z) \qquad (3)$$

for each $z\in C^n$ and

$$\operatorname{Re}\tilde{s}(z)=\sum_{n=1}^{\infty}\alpha_n^2\sum_{j=1}^{r_n}\operatorname{Re}\tilde{s}_{jn}(z)\geqslant-\sum_{n=1}^{\infty}\alpha_n^2\sum_{j=1}^{r_n}d_{jn}\chi'_{jn}(z)\,d(z,M_{jn})^2\geqslant$$

$$\geqslant-\sum_{n=1}^{\infty}\alpha_n\sum_{j=1}^{r_n}\chi'_{jn}(z)\,d(z,M_{jn})^2\geqslant-\sum_{n=1}^{\infty}\sum_{j=1}^{r_n}\frac{c_{jn}}{2^j}\chi'_{jn}(z)\,d(z,M_{jn})^2$$

$$=-\rho(z) \qquad (4)$$

for each $z\in D$.

Let $w(\partial D)=\{P_1,\dots,P_s\}$ and for each P_i, $i=1,\dots,s$ we repeat the same procedure. Therefore there exists neighborhoods Ω_1^i of P_i, $\omega_{jn}^i\subset\subset\omega_{jn}'^i\subset\subset V_n^i$ finite coverings of $K\cap\bar{U}_n^i$; M_{jn}^i totally real n-dimensional submanifolds of $\omega_{jn}'^i$, continuous functions

$$\rho_i(z)=\sum_{n=1}^{\infty}\sum_{j=1}^{r_n}\frac{c_{jn}^i}{2^j}\chi'^i_{jn}(z)\,d(z,M_{jn}^i)^2$$ which vanish on K and

$\tilde{s}_i\in C_0^\infty(C^n)$ such that (2), (3) and (4) are verified.

Let $K_o=K\setminus\bigcup_{i=1}^{s}\Omega_1^i$. Because $K_o\cap w(\partial D)=\emptyset$, there exist finite coverings $\omega_{jo}\subset\subset\omega'_{jo}$ of K_o, $1\leq j\leq r_o$, $(\bigcup_{j=1}^{r_o}\omega'_{jo})\cap(\bigcup_{i=1}^{s}\Omega_2^i)=\emptyset$, M_{jo} totally real submanifolds of ω'_{jo}, $\chi'_{jo}\in C_0^\infty(C^n)$, $\operatorname{supp}\chi'_{jo}\subset\subset\omega'_{jo}$, $\chi'_{jo}=1$ on ω_{jo} such that if we denote $\rho_o(z)=\sum_{j=1}^{r_o}\chi'_{jo}(z)\,d(z,M_{jo})$

we have

i) $\rho_o=0$ on K (5)

ii) There exists $c_o>0$ such that $\operatorname{Re} f(z)\geqslant c_o\rho(z)$ (6)

iii) If $\{\chi_{jo}\}$ is a C^∞ partition of unity subordinate to the covering

$\overset{r_o}{\underset{j=1}{\cup}} \omega_{jo}$, there exist C^∞-extensions $\tilde{s}_{jo}$ of $s_{jo} = \chi_{jo} s$ and we denote $\tilde{s}_o = \sum_{j=1}^{r_o} \tilde{s}_{jo}$. Then, for each $\alpha \in N^n$, $k \in N$ and $z \in C^n$ we have

$$|D^\alpha \bar{\partial} \tilde{s}_o(z)| \leq C^0_{\alpha k} \rho(z)^k \tag{7}$$

and for each $z \in \bar{D}$, Re $\tilde{s}_o \geq -d_o \rho(z)$ (8).

Let $E \cap w(\partial D) = \{P_1, \ldots, P_t\}$ and $\Omega = (\bigcup_{i=1}^{t} \Omega_1^i) \cup (\bigcup_{j=1}^{r_o} \omega_{jo}) \supset E$.
We denote $\rho = \sum_{i=0}^{t} \rho_i$, $\tilde{s} = \sum_{i=0}^{t} \tilde{s}_i$, $G = f + \delta \tilde{s}$ with $\delta > 0$.

Then:

a) $\rho = 0$ on K, hence $\rho = 0$ on E

b) If $z \in C^n$, $\sum_{i=0}^{t} \rho_i(z) \leq c_o \mathrm{Re} f(z) + \sum_{i=1}^{t} 4 \mathrm{Re} f(z)$,

hence Re $f(z) \geq \frac{1}{c_o + 4t} \rho(z)$.

c) If $z \in C^n$, $\alpha \in N^n$, $k \in N$ from (3) and (7) we obtain that there exist $C_{\alpha k} > 0$ such that $|D^\alpha \bar{\partial} \tilde{s}(z)| \leq C_{\alpha k} \rho^k(z)$.

d) If $z \in \bar{D}$, from (4) and (8) we obtain that

$$\mathrm{Re}\ \tilde{s}(z) \geq -d_o \rho_o(z) - \sum_{i=1}^{t} \rho_i(z) \geq -d \rho(z) \quad \text{where } d > \max(d_o, 1).$$

If $\delta > 0$ is small enough $(\delta < \frac{1}{d(c_o + 4t)})$ from a) and d) we obtain that Re $G(z) \geq c \rho(z)$ for each $z \in \bar{D}$.

Let $z \in \Omega$, $G(z) = 0$. Then $\rho(z) = 0$ and we have the possibilities:

i) $z \in A = \bigcup_{j=0}^{r_o} (\mathrm{supp}\, \chi_{jo}) \cup (\bigcup_{i=0}^{t} \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{r_n} \mathrm{supp}\, \chi^i_{jn})$

Then $\tilde{s}(z) = \sum_{i=0}^{t} \tilde{s}_i(z) = \sum_{j=0}^{r_o} \chi_{jo}(z) s(z) + \sum_{i=1}^{t} \sum_{n=1}^{\infty} (\alpha^i_n)^2 \sum_{j=1}^{r_n} \chi^i_{jn} s(z) =$

$= s(z) (\sum_{j=0}^{r_o} \chi_{jo} + \sum_{i=1}^{t} \sum_{n=1}^{\infty} (\alpha^i_n)^2 \sum_{j=1}^{r_n} \chi^i_{jn})$,

so $s(z)=0$ and $z\in E$.

ii) If $z\notin A$ then $\tilde{s}(z)=0$ and $G(z)=f(z)$. Because A is a covering of $K\setminus w(\partial D)$, it follows that $\mathrm{Re}f(z)=0$ only if $z\in K\cap w(\partial D)\cap\Omega=\{p_1,\ldots$ $\ldots, p_{t2}\}=E\cap w(\partial D)$.

Hence $G(z)=0$ on Ω if and only if $z\in E$ and by proposition 2. E is a peak set for $A^{\infty}(D)$.

Remark 2. For the proof of theorem 2 it suffices the local version of proposition 1.

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