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ISSN 0250 3638

MEASURE-THEORETIC PROPERTIES OF THE MAXIMAL ORTHOGONAL TOPOLOGY

by

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PREPRINT SERIES IN MATHEMATICS

No. 2/1984

BUCUREȘTI

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by

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January 1984

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MEASURE-THEORETIC PROPERTIES OF THE MAXIMAL

ORTHOGONAL TOPOLOGY

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Silviu TELEMAN

In the first part of this paper a new topology is introduced on the pure states set $P(A)$ of an arbitrary C^*-algebra A, and its measure-theoretic properties are studied.

In the second part of the paper we introduce and study the canonical irreducible disintegrations of the representations of C^*-algebras. The main result of the paper is the fact that any operator belonging to the Baire C^*-algebra $\mathcal{B}_0(\pi(A))$ over the range $\pi(A)$ of any (cyclic) representation $\pi : A \to \mathcal{L}(H)$ is decomposable with respect to any canonical irreducible disintegration of π (Proposition 2.1).

As an easy application we also prove the intuitively sensible fact that abelian projections are fields of one-dimensional projections (§ 2.IV).

§0. Introduction

Let E be a Hausdorff locally convex topological real vector space and $K \subset E$ a (non empty) compact convex subset. On the set ex K of the extreme points of K, besides the topology induced by that of K, several other topologies have been introduced and studied (see [1], Ch.II, §6; [5] ; [6], Ch.II; [7]; [8] ; [16], [17], [18]).

In [16], [17] and [18], we have begun the study of the measure-theoretic properties of the Choquet topology, for an arbitrary K, and of the orthogonal topology on the pure states set $P(A)$ of an arbitrary C^*-algebra A.

Namely, if C denotes the Choquet topology on ex K, then denote by $\mathcal{B}_0(K)$, respectively by $\mathcal{B}_0(\text{ex } K;\ C)$, the σ-algebra of the Baire measurable subsets of K, and the σ-algebra of the Baire measurable subsets of ex K, with respect to the Choquet topology; i.e., the σ- algebra of subsets of ex K, generated by the closed G_δ-subsets of ex K in the corresponding topology. By $\mathcal{B}(K)$, respectively $\mathcal{B}(\text{ex } K;C)$, we shall denote the σ-algebras of the Borel measurable subsets of K, and of the Borel measurable subsets of ex K, with respect to the Choquet topology; denote $\widetilde{\mathcal{B}}_0(\text{ex } K)=\{D\cap(\text{ex } K);\ D\in\mathcal{B}_0(K)\}$

If μ is any Choquet maximal Radon probability measure on K, then, by virtue of the Choquet-Bishop-de Leeuw Theorem, by the formula

$$\widetilde{\mu}_0(D\cap(\text{ex } K))=\mu(D), \quad D\in\mathcal{B}_0(K),$$

one correctly defines the boundary measure $\widetilde{\mu}_0$ on $\widetilde{\mathcal{B}}_0(\text{ex } K)$. We then have

(1) $$\mathcal{B}_0(\text{ex } K;C)\subset\widetilde{\mathcal{B}}_0(\text{ex } K)^{\sim}_{\widetilde{\mu}_0}$$

(see [16], Theorem 1.5, and [17], Theorem 3).

If $E_0(A) = \{f\in A^*;\ f\geq 0,\ \|f\|\leq 1\}$ is the set of the quasi-states of a C^*-algebra A, then $K=E_0(A)$ endowed with the $\sigma(A^*;A)$-topology, is a compact convex set. In [17] (see Theorem 6) we have shown that for any maximal orthogonal Radon probability measure μ on $E_0(A)$, such that $\|b(\mu)\|=1$, the measure $\widetilde{\mu}_0$ can be "extended" to a measure $\widetilde{\mu}_1$ defined on $\mathcal{B}(P(A);C)$, with good regularity properties, and so that

(2) $$\widetilde{\mathcal{B}}_0(P(A))\subset\mathcal{B}(P(A);C)^{\sim}_{\widetilde{\mu}_1}.$$

The difference between the extensions (1) and (2) is comparable to that existing on compact spaces, between Baire and Borel probability measures, where the Lebesgue completion is, in general, not sufficient to ensure the passage from a Baire measure to a Borel measure.

In fact, the constructions carried out in [16] and in [17], contain, as a particular case, the theory of Radon measures on compact spaces, which obtains when A is a commutative C^*-algebra with the unit element (see [19], for a detailed discussion of these facts).

In [2] (see, also [3] , for a revised version) C.J.K.Batty extended the theory we have developed in [17], by showing that for any compact convex set K, and any Choquet maximal Radon probability measure μ , the measure $\tilde{\mu}_0$ can be"extended" to $\mathcal{B}$(ex K;C); i.e., to be more precise, to the σ-algebra generated by $\tilde{\mathcal{B}}_0$(ex K) and $\mathcal{B}$(ex K;C). It was then easy to prove that

$$\tilde{\mathcal{B}}_0(\text{ex } K) \subset \mathcal{B}(\text{ex } K;\, C)_{\tilde{\mu}_1}^{\sim} ,$$

where $\tilde{\mu}_1$ is the extended measure (see [17], Theorem 6).

Of course, in order to be able to integrate as many functions as possible, the topology on ex K should be as fine as possible, subject only to the condition that Baire measurable, or, better, if possible, Borel measurable subsets, be measurable with respect to any extended boundary measure.

In [16] we have introduced the orthogonal topology on the pure states set P(A) of an arbitrary C^*-algebra, we have shown that, in general, it is strictly stronger than the Choquet topology (see [16], Theorem 7.4) and we have proved that any Baire measurable subset M P(A), with respect to the orthogonal topology, is measurable with respect to the boundary measure $\tilde{\mu}_0$, corresponding to any maximal Radon probability measure μ on $E_0(A)$, such that $\|b(\mu)\| = 1$ (see [16], Theorem 3.11).

In [18] we have introduced the maximal topology M on the extreme boundary ex K of any compact convex set K, and we have raised the problem as whether this topology is, in general, strictly stronger than the Choquet topology. In [18] we have also shown that any boundary measure $\tilde{\mu}_0$ can be "extended" to $\mathcal{B}(\text{ex } K;M)$, with good regularity properties, and such that

$$\tilde{\mathcal{B}}_0(\text{ex } K) \subset \mathcal{B}(\text{ex } K;M)_{\tilde{\mu}_1^M},$$

where $\tilde{\mu}_1^M$ is the extended boundary measure (see [18], Theorems 13 and 14).

C.J.K.Batty has shown that, in general, the maximal topology is strictly stronger than the Choquet topology (personal communication [4]). With his permission, we reproduce here his examples.

<u>Batty's examples:</u>

1) Let X be any compact Hausdorff space and let $x_0, x_1, x_2 \in X$ be three distinct points; let μ be a Radon probability measure on X, such that $\operatorname{supp}\mu = X$ and $\mu(\{x_0\}) = \mu(\{x_1\}) = \mu(\{x_2\}) = 0$. Let A be the Banach subspace of $C_{\mathbb{R}}(X\times[0,\frac{1}{2}])$, consisting of all functions $f \in C_{\mathbb{R}}(X\times[0,\frac{1}{2}])$, such that

$$\text{(1)} \qquad f(x_0,0) = tf(x_1,t) + (1-t)f(x_2,t), \qquad 0 < t \le \tfrac{1}{2},$$

and

$$\text{(2)} \qquad f(x_1,0) + f(x_2,0) = 4\int_{X\times[0,\frac{1}{2}]} f(x,t)\,d\mu(x)\,dt.$$

Let K be the state space of A; i.e.,

$$K = \{L \in A^*;\ \|L\| = L(1) = 1\},$$

endowed with the $\sigma(A^*;A)$-topology.

A separates the points of $X\times[0,\frac{1}{2}]$, with the exception of the pairs $(x_0,0)$ and $(x_2,0)$, which are identified. The Choquet boundary is

$$\operatorname{ex} K = (X\times[0,\tfrac{1}{2}])\setminus\{(x_0,0)\},$$

whereas the Choquet maximal Radon probability measures ν on K, representing $\varepsilon_{(x_0,0)}$, are supported by $\{x_1,x_2\}\times[0,\frac{1}{2}]$, and are given by

$$f(x_0,0)=\int_{[0,\frac{1}{2}]}\left(\frac{t}{1-t}f(x_1,t)+f(x_2,t)\right)d\nu_0(t), \qquad f\in A,$$

where ν_0 is any Borel positive measure on $[0,\frac{1}{2}]$, such that

$$\int_{[0,\frac{1}{2}]}\frac{d\nu_0(t)}{1-t} = 1.$$

If $F\subset K$ is any compact extremal subset, such that $\varepsilon_{(x_0,0)}\in F$, then

$$(3) \qquad \lambda\varepsilon_{(x_1,t)} + (1-\lambda)\varepsilon_{(x_2,t)}\in F,$$

for any $t\in(0,\frac{1}{2}]$ and any $\lambda\in[0,1]$. Since F is compact, from (3) we infer that

$$\tfrac{1}{2}(\varepsilon_{(x_1,0)} + \varepsilon_{(x_2,0)})\in F,$$

and, therefore, from (2), we infer that

$$X\times[0,\tfrac{1}{2}]=(\operatorname{supp}\mu)\times[0,\tfrac{1}{2}]\subset F.$$

It follows that the C-closed subsets of ex K are ex K itself, and the subsets of ex K, which are compact in the original topology.

On the other hand, any compact subset $F\subset X\times[0,\frac{1}{2}]$, containing $\{x_1,x_2\}\times[0,\frac{1}{2}]$ if $(x_o,0)\in F$, is maximally extremal; it follows that any subset of ex K, which is closed in the original topology and contains the set $\{x_1,x_2\}\times[0,\frac{1}{2}]$, is M-closed. Hence, the Choquet topology is strictly weaker than the maximal topology in this case.

2) The preceding example can be used in order to show that, in general, the σ-algebra $\widetilde{\mathcal{B}}_1(\text{ex }K;C)$ of subsets of ex K, generated by $\widetilde{\mathcal{B}}_o(\text{ex }K)$ and by $\mathcal{B}(\text{ex }K;C)$, is strictly included in the σ- algebra $\widetilde{\mathcal{M}}_1(\text{ex }K)$ of subsets of ex K, generated by $\widetilde{\mathcal{B}}_o(\text{ex }K)$ and by $\mathcal{B}(\text{ex }K;M)$.

In order to prove this, let us first remark that for any $\widetilde{B}\in\widetilde{\mathcal{B}}_1(\text{ex }K;C)$ there are Baire measurable subsets B,B' of $X\times[0,\frac{1}{2}]$, such that $(x_o,0)\in B$ and $\widetilde{B}\cap B=B'\setminus\{(x_o,0)\}$. By taking into account the characterization of the M-closed subsets of ex K, given in Example 1, one can find M-closed subsets of ex K, that are not in $\widetilde{\mathcal{B}}_1(\text{ex }K;C)$. It follows that

$$\widetilde{\mathcal{B}}_1(\text{ex }K;C)\subsetneq\widetilde{\mathcal{M}}_1(\text{ex }K).$$

Indeed, let us consider the compact space $X=[0,1]^{[0,1]}$, the point $x_o=(x_o(s))_{s\in[0,1]}\in X$, where $x_o(s)=0,\ \forall s\in[0,1]$, and the set M, given by

$$M=\{(x,t)\in X\times[0,\tfrac{1}{2}];\ \ x(s)\leq s,\ \ s\in[0,1]\}.$$

Then $M\setminus\{(x_o,0)\}$ is M-closed in ex $K=X\times[0,\frac{1}{2}]\setminus\{(x_o,0)\}$.

On the other hand, if B and B' are Baire measurable subsets of $X\times[0,\frac{1}{2}]$, there is a countable subset $I\subset[0,1]$, such that

$$(x,t)\in B,\ \ y\in X,\ \ y(s)=x(s),\forall\ s\in I\ \Rightarrow\ (y,t)\in B,$$

$$(x',t)\in B',\quad y'\in X,\quad y'(s)=x'(s),\ \forall s\in I \Rightarrow (y',t)\in B'.$$

If $(x_o,0)\in B$, then $(M\setminus\{(x_o,0)\})\cap B\neq B'\setminus\{(x_o,0)\}$, and, therefore, $M\setminus\{(x_o,0)\}\notin \widetilde{\mathcal{B}}_1(\mathrm{ex}\ K;C)$, although $M\setminus\{(x_o,0)\}$ is M-closed.

In([16], Remark 2, p.151) we have raised the problem to establish whether the boundary measures, corresponding to maximal orthogonal measures on the quasi-states space of any C^*-algebra A, can be extended to the σ-algebra of all Borel measurable subsets of P(A) with respect to the orthogonal topology.

In this paper we shall introduce the <u>maximal orthogonal</u> topology on P(A), which is stronger than the maximal, the orthogonal and the Choquet topologies, and we shall investigate its properties. Thereby, the foregoing problem will be solved positively.

We shall also apply the results obtained here, and in some previous papers, in order to improve an irreducible disintegration theorem for the representations of C^*-algebras, which we proved in ([16], Theorem 4.3).

§1. <u>The maximal orthogonal topology</u>

In this section we shall introduce the <u>maximal orthogonal</u> topology, which we shall denote by Ω.

I. Let A be any C^*-algebra, $E_o(A)=\{f\in A^*;\ \|f\|\leq 1,\ f\geq 0\}$ the quasi-states space of A, endowed with the $\sigma(A^*;A)$-topology, for which $E_o(A)$ becomes a compact convex set, whose subset of extreme points is given by

$$\mathrm{ex}\ E_o(A) = P(A)\cup\{0\}.$$

Let $E(A)=\{f\in A^*;\ \|f\|=1,\ f\geq 0\}$ be the states space of A.

It is obvious that E(A) is a G_δ-subset of $E_o(A)$ and a face of $E_o(A)$.

We shall say that a compact subset $F\subset E_0(A)$ is Ω - <u>extremal</u> if for any maximal orthogonal Radon probability measure μ on $E_0(A)$, such that $b(\mu)\in F\cap E(A)$, we have $\mu(F)=1$.

<u>Proposition 1.1.</u> <u>For any compact Ω -extremal subset $F\subset E_0(A)$ we have</u> $\overline{co}(F)\cap P(A)=ex\overline{co}(F)\cap E(A)=F\cap P(A)$.

<u>Proof.</u> The inclusions

$$F\cap P(A)\subset\overline{co}(F)\cap P(A)\subset ex\overline{co}(F)\cap E(A)$$

are obvious.

By the Milman Converse Theorem, we have $ex\overline{co}(F)\subset F$; let $f\in ex\overline{co}(F)\cap E(A)$, and let μ be any maximal orthogonal Radon probability measure on $E_0(A)$, such that $b(\mu)=f$. Then we have $\mu(F)=1$, and, therefore, $\mu(\overline{co}(F))=1$. It follows, by H.Bauer's Theorem (see [16], Proposition 1.2), that $\mu|\overline{co}(F)=\varepsilon_f$ and, therefore, $\mu=\varepsilon_f$. We infer that $f\in P(A)$. The Proposition is proved.

<u>Proposition 1.2.</u> <u>For any Ω - extremal compact subset $F\subset E_0(A)$, the set</u> $F_0=\overline{F\cap P(A)}$ <u>is Ω - extremal and</u> $F_0\cap P(A)=F\cap P(A)$.

<u>Proof.</u> Let μ be a maximal orthogonal Radon probability measure, such that $b(\mu)\in F_0\cap E(A)$. Then we have $b(\mu)\in F\cap E(A)$ and, therefore, $\mu(F)=\mu(E(A))=1$. We infer that $\mu(\overline{co}(F)\cap E(A))=1$. Since $\overline{co}(F)\cap E(A)$ is a G_δ-subset of $\overline{co}(F)$, there exists an increasing sequence $(D_n)_{n\geq 0}$ of compact Baire measurable subsets of $\overline{co}(F)$, such that $D_n\subset\overline{co}(F)\cap E(A)$ and $\mu(D_n)\uparrow 1$ (see [17], Lemma 1). Let $D\subset E_0(A)$ be any compact Baire measurable subset, such that $D\cap F_0=\emptyset$. We infer that $D\cap F\cap P(A)=\emptyset$, and, therefore, we have $D\cap ex\overline{co}(F)\cap E(A)=\emptyset$, by Proposition 1.1. We infer that $D_n\cap D\cap ex\overline{co}(F)=\emptyset, \forall\, n\geq 0$; since $D_n\cap D$ is a Baire measurable subset of $\overline{co}(F)$, whereas by Henrichs' Theorem (see

[16], Theorem 3.10), $\mu|\overline{co}(F)$ is Choquet maximal on $\overline{co}(F)$, by the Choquet-Bishop-de Leeuw Theorem we infer that $\mu(D_n \cap D)=0$, $\forall n\geq 0$. It follows that $\mu(D)=0$, and this implies that $\mu(F_o)=1$. The Proposition is proved.

Of course, $F_o=\overline{F\cap P(A)}$ is the smallest Ω-extremal compact subset of $E_o(A)$, such that

$$F_o \cap P(A)=F\cap P(A).$$

It is obvious that any finite union, and any intersection, of compact Ω-extremal subsets is a compact Ω-extremal subset of $E_o(A)$. It follows that the set $\mathcal{F}_\Omega$ of all <u>compact</u> Ω <u>-extremal</u> subsets of $E_o(A)$ is the set of all closed subsets of $E_o(A)$ with respect to a topology in $E_o(A)$, whereas $\tilde{\mathcal{F}}_\Omega=\{P(A)\cap F;\ F\in\mathcal{F}_\Omega\}$ is the set of all closed subsets of $P(A)$, with respect to a topology on $P(A)$, which we shall call the <u>maximal orthogonal topology</u> and we shall denote it by Ω.

It is obvious that any compact extremal, or maximally extremal or orthogonally extremal subset of $E_o(A)$ is Ω-extremal; it follows that the maximal orthogonal topology is stronger than the Choquet, the maximal and the orthogonal topologies.

<u>Proposition 1.3.</u> a) <u>If</u> $F\subset E(A)$ <u>is a compact</u> Ω <u>-extremal subset,</u> <u>then</u> $ex\overline{co}(F)=F\cap P(A)$ <u>and the set</u> $F\cap P(A)$ <u>is</u> Ω <u>-quasicompact (and</u> Ω<u>-closed</u>).

b) <u>If</u> $1\in A$, <u>then</u> $P(A)$ <u>is</u> Ω <u>-quasicompact.</u>

c) <u>If</u> $P(A)$ <u>is</u> Ω<u>-quasicompact, then</u> $1\in A$.

<u>Proof.</u> a) From ([16], Theorem 3.7) and from Proposition 1.1 we infer that

$$ex\overline{co}(F)=ex\overline{co}(F)\cap E(A)=F\cap P(A).$$

Let now $(\tilde{F}_\alpha)_{\alpha\in I}$ be a decreasing net of Ω-closed subsets of P(A), such that $\tilde{F}\cap\tilde{F}_\alpha\neq\emptyset$, where $\tilde{F}=F\cap P(A)$. For any $\alpha\in I$ there exists (cf., Proposition 1.2) a smallest compact, Ω -extremal subset $F_\alpha\subset E_0(A)$, such that $F_\alpha\cap P(A)=\tilde{F}_\alpha$; we then have $F_\beta\subset F_\alpha$, for $\alpha\leq\beta$ in I, and, therefore, $E(A)\supset\bigcap_{\alpha\in I}(F\cap F_\alpha)\neq\emptyset$. We infer that $(\bigcap_{\alpha\in I}\tilde{F}_\alpha)\cap\tilde{F}\neq 0$ (see Proposition 1.1 and [16], Theorem 3.7); hence, $\tilde{F}$ is Ω-quasicompact.

b) If $1\in A$, then E(A) is a compact Ω-extremal subset of $E_0(A)$ and, therefore, $P(A)=E(A)\cap P(A)$ is Ω-quasicompact.

c) If P(A) is Ω-quasicompact, then it is quasicompact for the Choquet topology and, therefore, $1\in A$ (see [16], Proposition 3.19). The Proposition is proved.

II. Let μ be any maximal (maximal orthogonal) Radon probability measure, such that $\|b(\mu)\|=1$, and let $F_1\subset E_0(A)$ be the <u>smallest</u> compact Ω -extremal subset, such that $F_1\supset \operatorname{supp}\mu$.

<u>Theorem 1.1</u> a) <u>The set</u> $F_1\cap P(A)$ <u>is the smallest</u> Ω <u>-closed subset of</u> P(A) , <u>whose</u> $\tilde{\mu}_0$ <u>-outer measure is equal to</u> 1. b) <u>For any</u> $F\in\mathcal{F}_\Omega$ <u>we have</u>

$$\tilde{\mu}_0^*(F\cap P(A))=\mu(F).$$

<u>Remark.</u> Here $\tilde{\mu}_0$ denotes the <u>boundary measure</u> corresponding to μ and defined on $\tilde{\mathcal{B}}_0(P(A))=\{D\cap P(A);\ D\in\mathcal{B}_0(E_0(A))\}$ by the formula

$$\tilde{\mu}_0(D\cap P(A))=\mu(D),\qquad D\in\mathcal{B}_0(E_0(A)).$$

<u>Proof.</u> a) Let $D\in\mathcal{B}_0(E_0(A))$ be such that $D\cap P(A)\supset F_1\cap P(A)$. From $F_1\supset\operatorname{supp}\mu$ we infer that $\mu(\overline{co}(F_1))=1$ and, therefore, $\mu|\overline{co}(F_1)$ is

Choquet maximal as a Radon probability measure on $\overline{co}(F_1)$ (see [16], Henrich's Theorem 3.10, for the case that μ is maximal orthogonal.) Since $D_o=(\complement D)\cap\overline{co}(F_1)\in\mathcal{B}_o(\overline{co}(F_1))$, from

$$D_o\cap ex\ \overline{co}(F_1)\cap E(A)=D_o\cap F_1\cap P(A)=\emptyset$$

we infer that $\mu(D_o)=0$ (since $E(A)$ is a G_δ-subset of $E_o(A)$, there exists a $D_1\in\mathcal{B}_o(E_o(A))$, such that $D_1\subset E(A)$ and $\mu(D_1)=1$). It follows that $\mu(\complement D)=0$, and, therefore,

$$\tilde{\mu}_o(D\cap P(A))=\mu(D)=1.$$

We infer that $\tilde{\mu}_o^*(F_1\cap P(A))=1$.

We have still to prove that $F_1\cap P(A)$ is the smallest Ω-closed subset of $P(A)$ having this property. This will follow from the second part of the Theorem.

b) Let now $F\in\mathcal{F}_\Omega$. Then there exists a compact Baire measurable set $B\in\mathcal{B}_o(E_o(A))$, such that

$$F\subset B \text{ and } \mu(B)=\mu(F).$$

It follows that $F\cap P(A)\subset B\cap P(A)$ and

(1) $$\tilde{\mu}_o^*(F\cap P(A))\leq\tilde{\mu}_o(B\cap P(A))=\mu(B)=\mu(F).$$

If $\mu(F)=0$, the required equality is proved. If $\mu(F)>0$, let us define $\nu=\mu(F)^{-1}\chi_F\mu$. Then by ([16], Proposition 3.16, c), the measure ν is maximal (maximal orthogonal) and $\nu(F)=1$. We infer that $\operatorname{supp}\nu\subset F$ and, therefore, from part a) of the Proof, we have

(2) $$\tilde{\nu}_o^*(F\cap P(A))=1.$$

Let now $B\in\mathcal{B}_o(E_o(A))$ be such that $B\cap P(A)\supset F\cap P(A)$. From (2) we infer that $\tilde{\nu}_o(B\cap P(A))=1$, and, therefore, $\nu(B)=1$. By the definition of ν it follows that $\mu(F)=\mu(B\cap F)\leq\mu(B)$; i.e.,

$$\mu(F)\leq\tilde{\mu}_o(B\cap P(A)),$$

and this implies that

(3) $$\mu(F)\leq\tilde{\mu}_o^*(F\cap P(A)).$$

From (1) and (3) the required equality immediately follows.

c) Let now $F\in\mathcal{F}_\Omega$ be such that

$$\tilde{\mu}_o^*(F\cap P(A))=1.$$

Then, from b), we infer that $\mu(F)=1$ and, therefore, $F_1\subset F$; hence

$$F_1\cap P(A)\subset F\cap P(A),$$

and the Theorem is proved.

<u>Theorem 1.2. If μ is any maximal (maximal orthogonal) Radon probability measure on</u> $E_o(A)$, <u>such that</u> $\|b(\mu)\|=1$ <u>then</u> $\mathcal{B}_o(P(A);\Omega)\subset\tilde{\mathcal{B}}_o(P(A))_{\tilde{\mu}_o}$.

<u>Proof.</u> Let $F\in\mathcal{F}_\Omega$ be such that

$$P(A)\setminus F=\bigcup_{n=0}^{\infty}(F_n\cap P(A)),$$

where $(F_n)_{n\geq 0}$ is a sequence of sets $F_n\in\mathcal{F}_\Omega$, $n\geq 0$, which can be assumed to be increasing. From $F\cap F_n\cap P(A)=\emptyset$ and from Proposition 1.1 we infer that

$$\mathrm{ex}\,\overline{\mathrm{co}}(F\cap F_n)\cap E(A)=\emptyset,\qquad n\geq 0.$$

Since the affine upper semicontinuous function $\overline{\mathrm{co}}(F\cap F_n)\ni f\mapsto 1-\|f\|$

is strictly positive on $\mathrm{ex}\,\overline{\mathrm{co}}(F\cap F_n)$, from ([16], Theorem 1.2; [19], Theorem 2) we infer that $\|f\|<1$, for any $f\in\overline{\mathrm{co}}(F\cap F_n)$ and, therefore,

$$F\cap F_n\cap E(A)=\emptyset, \qquad n\geq 0.$$

It follows that $\mu(F\cap F_n)=0$, and, therefore,

$$\mu(F)+\mu(F_n)\leq 1, \qquad n\geq 0.$$

We infer that

$$\tilde{\mu}_o^*(P(A)\setminus F)=\sup\{\tilde{\mu}_o^*(F_n\cap P(A));\ n\geq 0\}=\sup\{\mu(F_n);\ n\geq 0\}\leq 1-\mu(F)=1-\tilde{\mu}_o^*(F\cap P(A)).$$

(see also [13], Ch.I, §1.5, Proposition I.5.2; [16], Proposition 1.12). It follows that $F\cap P(A)$ is $\tilde{\mu}_o$-measurable and, therefore, we have the inclusion

$$\mathcal{B}_o(P(A);\Omega)\subset\tilde{\mathcal{B}}_o(P(A))_{\tilde{\mu}_o}^{\sim}.$$

The Theorem is proved.

Remark. The preceding Theorem shows that the Lebesgue completion of $\tilde{\mathcal{B}}_o(P(A))$ with respect to $\tilde{\mu}_o$ is strong enough to render "measurable" any Ω-Baire measurable subset of $P(A)$. The next step will be to extend the measure $\tilde{\mu}_o$, in order to render "measurable" any Ω-Borel measurable subset of $P(A)$.

III. Let now $\mathcal{B}_\Omega(E_o(A))$ be the σ-algebra of subsets of $E_o(A)$, generated by $\mathcal{F}_\Omega\cup\mathcal{B}_o(E_o(A))$, and let $\tilde{\mathcal{B}}_\Omega(P(A))$ be the σ-algebra of subsets of $P(A)$, generated by $\tilde{\mathcal{F}}_\Omega\cup\tilde{\mathcal{B}}_o(P(A))$, where

$$\widetilde{\mathcal{B}}_o(P(A))=\{D\cap P(A);\ D\in\mathcal{B}_o(E_o(A))\},$$

and

$$\widetilde{\mathcal{F}}_{\Omega}=\{F\cap P(A);\quad F\in\mathcal{F}_{\Omega}\}.$$

is the set of all Ω-closed subsets of P(A). Of course, we have

$$\widetilde{\mathcal{B}}_{\Omega}(P(A))=\{D\cap P(A);\ \ D\in\mathcal{B}_{\Omega}(E_o(A))\},$$

and, also, the following inclusions

$$\widetilde{\mathcal{B}}_o(P(A))\subset\widetilde{\mathcal{B}}_{\Omega}(P(A)),$$

and

$$\mathcal{B}_o(P(A);\Omega)\subset\mathcal{B}(P(A);\Omega)\subset\widetilde{\mathcal{B}}_{\Omega}(P(A)).$$

We shall now prove

<u>Theorem 1.3. For any maximal (maximal orthogonal) Radon probability measure</u> μ <u>on</u> $E_o(A)$, <u>such that</u> $\|b(\mu)\|=1$, <u>the formula</u>

$$\widetilde{\mu}_o^{\Omega}(\widetilde{M})=\sup\{\mu(M);M\in\mathcal{B}_{\Omega}(E_o(A)),\ M\cap P(A)\subset\widetilde{M}\},$$ <u>for any</u> $\widetilde{M}\in\widetilde{\mathcal{B}}_{\Omega}(P(A))$,

<u>extends</u> $\widetilde{\mu}_o$ <u>to a probability measure</u> $\widetilde{\mu}_o^{\Omega}:\widetilde{\mathcal{B}}_{\Omega}(P(A))\to[0,1]$, <u>which is regular in the sense that</u>

a) $\widetilde{\mu}_o^{\Omega}(\widetilde{M})=\sup\{\widetilde{\mu}_o^{\Omega}(\widetilde{F});\ \widetilde{F}\subset\widetilde{M},\ \widetilde{F}=F\cap P(A),\ F\in\mathcal{F}^1_{\Omega}\}$, <u>for any</u> $\widetilde{M}\in\widetilde{\mathcal{B}}_{\Omega}(P(A))$,

<u>where</u> $\mathcal{F}^1_{\Omega}=\{F\in\mathcal{F}_{\Omega};\ F\subset E(A)\}$.
<u>Moreover, we have</u> $\widetilde{\mu}_o^{\Omega}(\widetilde{F})=\widetilde{\mu}_o^{*}(\widetilde{F})$, <u>for any</u> $\widetilde{F}\in\widetilde{\mathcal{F}}_{\Omega}$; i.e.,

b) $\tilde{\mu}_o^{\Omega}(\tilde{F})=\inf\{\tilde{\mu}_o(\tilde{B}):\ \tilde{F}\subset\tilde{B},\ \tilde{B}\in\tilde{\mathcal{B}}_o(P(A))\}$, <u>for any</u> $\tilde{F}\in\tilde{\mathcal{F}}_{\Omega}$.

<u>Proof.</u> The method of proof is an adaptation of that given by C.J.K.Batty for the case of the Choquet topology on arbitrary compact convex sets (see [2], p.10; and, also, [18], proof of Theorem 13). Henrichs' Theorem (see [16], Theorem 3.10) is involved at several stages of the proof, if μ is assumed to be orthogonal.

a) For any $B\subset E_o(A)$ we shall define

$$\mu_o'(B)=\sup\{\mu(F);F\in\mathcal{F}^1_{\Omega}\ ,\ F\subset B\},$$
$$\mu_o''(B)=\sup\{\mu(F);F\in\mathcal{F}^1_{\Omega}\ ,\ F\cap P(A)\subset B\cap P(A)\}.$$

b) We shall first remark that for any $\varepsilon>0$ there exists an $F_\varepsilon\in\mathcal{F}^1_{\Omega}$, such that $\mu(F_\varepsilon)>1-\varepsilon$. Indeed, $E(A)\subset E_o(A)$ is a G_δ-subset, such that $\mu(E(A))=1$. From ([17], Theorem 2) we infer that for any $\varepsilon>0$ there exists a compact extremal (Baire measurable) subset $F_\varepsilon\subset E(A)$, such that $\mu(F_\varepsilon)$ $1-\varepsilon$. It is obvious that $F_\varepsilon\in\mathcal{F}^1_{\Omega}$.

c) We have the following properties

i) $\mu_o'(B)\le\mu_o''(B)$, for any $B\subset E_o(A)$; obvious.

ii) $\mu_o'(F)=\mu(F)$, for any $F\in\mathcal{F}_{\Omega}$. Indeed, for any $F\in\mathcal{F}_{\Omega}$ we have $F\cap F_\varepsilon\in\mathcal{F}^1_{\Omega}$ and $F\cap F_\varepsilon\subset F$. It follows that $\mu_o'(F)\ge\mu(F\cap F_\varepsilon)>\mu(F)-\varepsilon$, for any $\varepsilon>0$, and this implies that $\mu_o'(F)\ge\mu(F)$. On the other hand, for $F_o\in\mathcal{F}^1_{\Omega}$, $F_o\subset F$ we have $\mu(F_o)\le\mu(F)$, and this implies that $\mu_o'(F)\le\mu(F)$.

iii) $\mu_o'(B)=\mu(B)$, for any $B\in\mathcal{B}_o(E_o(A))$. Indeed, this is an immediate consequence of ([17], Theorem 1, Corollary) and of remark b), above, if we take into account the fact that $F_\varepsilon\cap D_o\in\mathcal{F}^1_{\Omega}$, for any compact extremal Baire measurable subset $D_o\subset E_o(A)$.

iv) $\mu_o'(G)=\mu(G)$, for any G_δ-subset $G\subset E_o(A)$. Indeed, this is an immediate consequence of ([17], Theorem 2), if we also take into account remark b), above.

v) $\mu_o'(B)\le\mu(B)$, for any $B\in\mathcal{B}(E_o(A))$. Indeed, any Radon probability measure on a compact space is regular by closed (compact) subsets.

vi) $\mu_o''(B)\leq\mu(B)$, for any $B\in\mathcal{B}_o(E_o(A))$. Indeed, by Theorem 1,b), for any $F\in\mathcal{F}_\Omega^1$ and any $B\in\mathcal{B}_o(E_o(A))$, such that $F\cap P(A)\subset B\cap P(A)$, we have

$$\mu(F)=\tilde{\mu}_o^*(F\cap P(A))\leq\tilde{\mu}_o(B\cap P(A))=\mu(B).$$

vii) $\mu_o'(B_1)+\mu_o'(B_2)\leq\mu_o'(B_1\cup B_2)$ and $\mu_o''(B_1)+\mu_o''(B_2)\leq\mu_o''(B_1\cup B_2)$, for any $B_1,B_2\subset E_o(A)$, such that $B_1\cap B_2\cap P(A)=\emptyset$. Indeed, the first inequality is obvious by the definition of μ_o', if we take into account the fact that $F_1,F_2\in\mathcal{F}_\Omega^1$, $F_1\subset B_1$, $F_2\subset B_2 \Rightarrow F_1\cap F_2\cap P(A)=\emptyset$ and, therefore, by Proposition 1.3 and the Krein-Milman Theorem, we infer that $F_1\cap F_2=\emptyset$. A similar argument works for the second inequality.

d) Let $\mathcal{A}'=\{B\in\mathcal{B}(E_o(A));\ \mu_o'(B)=\mu(B),\mu_o'(\complement B)=\mu(\complement B)\}$. Then $\mathcal{A}'$ is a σ-algebra, such that $\mathcal{F}_\Omega\subset\mathcal{A}'$, by c, ii) and iv); and also $\mathcal{B}_o(E_o(A))\subset\mathcal{A}'$, by c, iii). It follows that $\mathcal{B}_\Omega(E_o(A))\subset\mathcal{A}'$, by the definition of $\mathcal{B}_\Omega(E_o(A))$.

e) Let $\mathcal{A}''=\{B\in\mathcal{B}_\Omega(E_o(A));\mu_o''(B)=\mu(B),\ \mu_o''(\complement B)=\mu(\complement B)\}$. Then $\mathcal{A}''$ is a σ-algebra, such that $\mathcal{B}_o(E_o(A))\subset\mathcal{A}''$, by c,i), c,vii) and d). We obviously have $\mu(F_o)\leq\mu_o''(F_o)$, for any $F_o\in\mathcal{F}_\Omega$(use b)). On the other hand for any $\varepsilon>0$ there exists a $F_1\in\mathcal{F}_\Omega^1$, such that

$$F_1\cap P(A)\subset(\complement F_o)\cap P(A) \text{ and } \mu_o''(\complement F_o)-\varepsilon<\mu(F_1).$$

Since we have that $F_o\cap F_1\in\mathcal{F}_\Omega^1$ and $F_o\cap F_1\cap P(A)=\emptyset$, by Proposition 1.3, a) and the Krein-Milman Theorem, we infer that $F_o\cap F_1=\emptyset$. We have, therefore,

$$1\geq(F_o\cup F_1)=\mu(F_o)+\mu(F_1)>\mu(F_o)+\mu_o''(\complement F_o)-\varepsilon,$$

and this implies that $\mu_o''(\complement F_o)\leq\mu_o(\complement F_o)$, $F_o\in\mathcal{F}_\Omega$.

On the other hand, we have

$$\mu_0'(\complement F_0)\le\mu_0''(\complement F_0)\le\mu(\complement F_0)=\mu_0'(\complement F_0),$$

by taking into account c, i) and d). It follows that we have $\mu_0''(\complement F_0)=\mu(\complement F_0)$. From c, vii) we now infer that

$$1\ge\mu_0''(F_0)+\mu_0''(\complement F_0)\ge\mu(F_0)+\mu(\complement F_0)=1,$$

and, therefore, $\mu_0''(F_0)=\mu(F_0)$, $F_0\in\mathcal{F}_\Omega$. We infer that we have $\mathcal{F}_\Omega\subset\mathcal{A}''$ and, therefore

$$\mathcal{A}''=\mathcal{B}_\Omega(E_0(A));$$

i.e., $\mu_0''(B)=\mu(B)$, $B\in\mathcal{B}_\Omega(E_0(A))$.

f) $B\in\mathcal{B}_\Omega(E_0(A))$ and $B\cap P(A)=\emptyset\Rightarrow\mu(B)=0$. Indeed, this is an immediate consequence of e). It follows that by the formula

$$(*)\qquad \tilde{\mu}_0^\Omega(B\cap P(A))=\mu(B),\qquad B\in\mathcal{B}_\Omega(E_0(A)),$$

we correctly define a probability measure on $\tilde{\mathcal{B}}_\Omega(P(A))$, and we have

$$\tilde{\mu}_0^\Omega(\tilde{B})=\sup\{\mu(F);\ F\in\mathcal{F}_\Omega,\ F\cap P(A)\subset\tilde{B}\},\ \tilde{B}\in\tilde{\mathcal{B}}_\Omega(P(A)).$$

Equality a) in the statement of the Theorem is now an immediate consequence, whereas equality b) follows from Theorem 1.1, b), and from e) and (*). The Theorem is proved.

Remark. Although the definition of the maximal orthogonal topology involves only the use of the maximal orthogonal Radon probability measure, whose barycenters are states of A, by the preceding Theorem any maximal Radon probability measure whose barycenter is a state of A can induce a measure on $\tilde{\mathcal{B}}_\Omega(P(A))$. It is an open problem to establish whether the maximal orthogonal topology on P(A) is, in general, strictly stronger than the maximal topology. The following Theorem shows that, by restricting the extended boundary measure $\tilde{\mu}_o^{\Omega}$ to $\mathcal{B}(P(A);\Omega)$, and, then, by passing to its Lebesgue completion, no information is lost; i.e., extended boundary measures are, essentially, Borel measures.

Theorem 1.4 a) $\tilde{\mathcal{B}}_{\Omega}(P(A))\subset\mathcal{B}(P(A);\Omega)_{\tilde{\mu}_o^{\Omega}}^{\sim}$;
b) $\tilde{\mu}_o^{\Omega}$ is τ-continuous; i.e.,

$$\tilde{\mu}_o^{\Omega}(\bigcap_{\alpha}\tilde{F}_{\alpha})=\inf\{\tilde{\mu}_o^{\Omega}(\tilde{F}_{\alpha});\alpha\in I\},$$

for any decreasing net $(\tilde{F}_{\alpha})_{\alpha\in I}$ of Ω-closed subsets of $P(A)$.

Proof. We obviously have

$$\mathcal{B}(P(A);\Omega)\subset\tilde{\mathcal{B}}_{\Omega}(P(A)).$$

For any $\tilde{B}\in\tilde{\mathcal{B}}_{\Omega}(P(A))$, by Theorem 1.3, a) there exists a F_{σ}-subset $\tilde{F}\subset P(A)$, and a G_{δ}-subset $\tilde{G}\subset P(A)$ (with respect to the maximal orthogonal topology), such that $\tilde{F}\subset\tilde{B}\subset\tilde{G}$ and

$$\tilde{\mu}_o^{\Omega}(\tilde{F})=\tilde{\mu}_o^{\Omega}(\tilde{B})=\tilde{\mu}_o^{\Omega}(\tilde{G}).$$

It follows that $\tilde{B}\in\mathcal{B}(P(A);\Omega)_{\tilde{\mu}_o^{\Omega}}^{\sim}$.

b) If $F_{\alpha}\subset E_o(A)$ is the smallest (compact) Ω- extremal subset of $E_o(A)$, such that $F_{\alpha}\cap P(A)=\tilde{F}_{\alpha}$, $\alpha\in I$, (see Proposition 1.2) we have

$$\mu(\bigcap_{\alpha\in I}F_{\alpha})=\inf\{\mu(F_{\alpha});\alpha\in I\}=\inf\{\tilde{\mu}_o^{\Omega}(\tilde{F}_{\alpha});\alpha\in I\}\geq\tilde{\mu}_o^{\Omega}(\bigcap_{\alpha\in I}\tilde{F}_{\alpha})=\mu(\bigcap_{\alpha\in I}F_{\alpha}),$$

because $\bigcap_{\alpha\in I}F_{\alpha}$ is a (compact) Ω-extremal subset of $E_o(A)$, such that $(\bigcap_{\alpha\in I}F_{\alpha})\cap P(A)=\bigcap_{\alpha\in I}\tilde{F}_{\alpha}$. The Theorem is proved.

Let now $F_1 \subset E_o(A)$ be the smallest (compact) Ω-extremal subset of $E_o(A)$, such that $F_1 \supset \operatorname{supp}\mu$.

Corollary. The set $\widetilde{F}_1 = F_1 \cap P(A)$ is the smallest Ω-closed subset of $P(A)$, such that $\widetilde{\mu}_o^{\Omega}(\widetilde{F}_1)=1$; i.e., $\widetilde{F}_1$ is the Ω-closed support of $\widetilde{\mu}_o^{\Omega}$.

Proof. By the definition of $\widetilde{\mu}_o^{\Omega}$, given by formula (*) in the proof of Theorem 1.3, we have

$$\widetilde{\mu}_o^{\Omega}(F_1)=\widetilde{\mu}_o^{\Omega}(F_1 \cap P(A))=\mu(F_1)=1.$$

If $\widetilde{F}_o \in \widetilde{\mathcal{F}}_{\Omega}$ is such that $\widetilde{\mu}_o^{\Omega}(\widetilde{F}_o)=1$, and if $F_o \in \mathcal{F}_{\Omega}$ is such that $F_o \cap P(A)=\widetilde{F}_o$, then

$$1=\widetilde{\mu}_o^{\Omega}(\widetilde{F}_o)=\mu(F_o);$$

hence, $F_o \supset \operatorname{supp}\mu$ and, therefore, $F_o \supset F_1$. It follows that $\widetilde{F}_o = F_o \cap P(A) \supset \supset F_1 \cap P(A) = \widetilde{F}_1$.

IV. The regularity of the measure $\widetilde{\mu}_o^{\Omega}$ immediately implies the following extension of Lusin's Theorem.

Theorem 1.5. Let $f: P(A) \to \mathbb{R}$ be any $\widetilde{\mu}_o^{\Omega}$- measurable function. Then, for any $\varepsilon>0$, there exists an Ω-closed subset $\widetilde{F} \subset P(A)$, such that $\widetilde{\mu}_o^{\Omega}(\widetilde{F})>1-\varepsilon$ and $f|\widetilde{F}$ is Ω-continuous. Moreover, $\widetilde{F}$ can be assumed to be Ω-quasicompact.

Proof. Given $\varepsilon>0$, we can find an $F_1 \in \mathcal{F}^1_{\Omega}$, such that $\mu(F_1)>1-\frac{\varepsilon}{2}$, and an Ω-closed subset $\widetilde{F}_o \subset P(A)$, such that $\widetilde{\mu}_o^{\Omega}(\widetilde{F}_o)>1-\frac{\varepsilon}{4}$ and $f|\widetilde{F}_o$ be bounded; therefore, there exist $m, M \in \mathbb{R}$, such that $m \leq f(x) \leq M$, $x \in \widetilde{F}_o$.

Let $\varepsilon_i>0$, $i\in\mathbb{N}^*$, be chosen such that $\sum_{i=1}^{\infty}\varepsilon_i=\frac{\varepsilon}{4}$. If $E_{n,k}=f^{-1}((-\infty, m+k\frac{M-m}{n}])\cap\widetilde{F}_0$, $n\in\mathbb{N}^*$, $k\in\mathbb{N}$, by the regularity of $\widetilde{\mu}_0^{\Omega}$ we can find Ω-closed subsets $\widetilde{F}_{n,k}\subset P(A)$, such that $\widetilde{F}_{n,0}\subset E_{n,0}$ and $\widetilde{F}_{n,k}\subset E_{n,k}\setminus E_{n,k-1}$, $1\le k\le n$, and such that for $\widetilde{H}_n=\bigcup_{k=0}^{n}\widetilde{F}_{n,k}$ we have $\widetilde{\mu}_0^{\Omega}(\widetilde{H}_n)>1-\varepsilon_n$. If $f_n:P(A)\to\mathbb{R}$ is defined by

$$f_n=\sum_{k=0}^{n}(m+k\,\frac{M-m}{n})\chi_{\widetilde{F}_{n,k}},$$

then f_n is continuous on $\widetilde{H}_n$ and $\|(f_n-f)\chi_{\widetilde{H}_n}\|_{\infty}\le\frac{1}{n}$. It follows that the set $\widetilde{F}=F_1\cap(\bigcap_{n=1}^{\infty}\widetilde{H}_n)$ meets the requirements in the statement of the Theorem.

Sometimes a probability space (M,Σ,λ) enjoys the property of being <u>perfect</u> (or quasi-compact). This means that for any λ-measurable function $f:M\to\mathbb{R}$ and any $f_*(\lambda)$-measurable subset $S\subset\mathbb{R}$ ($f_*(\lambda)$ is the full direct image of λ through f, defined on the σ-algebra $f_*(\widetilde{\Sigma}_{\lambda})=\{S\subset\mathbb{R};\ f^{-1}(S)\in\widetilde{\Sigma}_{\lambda}\}$ by $f_*(\lambda)(S)=\lambda(f^{-1}(S))$, $S\in f_*(\widetilde{\Sigma}_{\lambda})$), there exists a Borel measurable subset $B\subset S$, such that $f_*(\lambda)(S\setminus B)=0$ (see [9], p.17, [10], Ch.V, §22; [15], [18]).

<u>Theorem 1.6.</u> <u>The measure $\widetilde{\mu}_0^{\Omega}:\mathcal{B}(P(A);\Omega)^{\sim}_{\widetilde{\mu}_0^{\Omega}}\to[0,1]$ is perfect.</u>

<u>Proof</u>. Let $S\subset\mathbb{R}$ be such that $f^{-1}(S)\in\mathcal{B}(P(A);\Omega)^{\sim}_{\widetilde{\mu}_0^{\Omega}}$ where $f:P(A)\to\mathbb{R}$ is any $\widetilde{\mu}_0^{\Omega}$-measurable function, and $S\subset\mathbb{R}$ is any $f_*(\widetilde{\mu}_0^{\Omega})$-measurable subset. For a given $\varepsilon>0$ we can find an Ω-closed Ω-quasi-compact subset $\widetilde{F}\subset P(A)$, such that $\widetilde{F}\subset f^{-1}(S)$ and $\widetilde{\mu}_0^{\Omega}(f^{-1}(S)\setminus\widetilde{F})<\varepsilon$; and, moreover, such that $f|\widetilde{F}$ be Ω-continuous. Then $f(\widetilde{F})\subset S$ is a compact subset of $\mathbb{R}$, such that

$$f_*(\widetilde{\mu}_0^{\Omega})(S\setminus f(\widetilde{F}))<\varepsilon.$$

The Theorem now immediately follows.

V. It is easy to see that any (relatively) C-closed subset $\widetilde{F}\subset P(A)$ is Ω-closed. It follows that we have

$$\mathcal{B}(P(A);C)\subset\mathcal{B}(P(A);\Omega),$$

and, also, since $\widetilde{\mu}_I(\widetilde{F})=\widetilde{\mu}_o^{\Omega}(\widetilde{F})$, it follows that $\widetilde{\mu}_o^{\Omega}$ is an extension of the measure $\widetilde{\mu}_I$, defined as in [2], or [18].

By taking into account the results from [19], we infer that for any semicontinuous bounded affine function $h_o:E_o(A)\rightarrow \mathbb{R}$, the function $h_o|P(A)$ is $\widetilde{\mu}_o^{\Omega}$-measurable, and we have

$$h_o(b(\mu))=\int_{P(A)} h_o(p)d\widetilde{\mu}_o^{\Omega}(p).$$

Let A_{sa}^{**} be the self-adjoint part of A^{**}, and $\widetilde{A}$ the C^*-algebra obtained by adjoining $1\in A^{**}$ to A; let $(\widetilde{A}_{sa})^m$ be the subset of A_{sa}^{**} consisting of all elements of A^{**}, which are limits (with respect to $\sigma(A^{**};A^*)$) of bounded increasing nets of elements in $\widetilde{A}_{sa}$, and denote by $((\widetilde{A}_{sa})^m)^-$ the norm closure of $(\widetilde{A}_{sa})^m$ in A_{sa}^{**}. We can state the following

<u>Theorem 1.7. For any</u> $x\in A_{sa}^{**}$, <u>such that</u> $x|E(A)$ <u>is (lower) semi-continuous, the function</u> $x|P(A)$ <u>is</u> $\widetilde{\mu}_o^{\Omega}$<u>-measurable and we have</u>

$$(*)\qquad x(b(\mu))=\int_{P(A)} x(p)d\widetilde{\mu}_o^{\Omega}(p).$$

<u>Proof.</u> According to ([14], Proposition 3.11.8), the element x belongs to $((\widetilde{A}_{sa})^m)^-$; therefore, it is sufficient to prove the Theorem for elements x belonging to $(\widetilde{A}_{sa})^m$. By taking into account ([14

Proposition 3.11.7), it is sufficient to prove the Theorem for elements x belonging to $(A_{sa})^m$. Now this is a consequence of ([19], Theorem 2), if we take into account the fact that $\tilde{\mu}_o^{\Omega}$ is an extension of $\tilde{\mu}|\mathcal{B}(P(A);C)$.

Q.E.D.

By slightly extending the notion of a <u>universally measurable element</u>, given in ([14], p.104), and that of a <u>strongly universally measurable</u> bounded affine real function, as defined in ([19], §V), we shall say that an element $x \in A_{sa}^{**}$ is <u>strongly universally measurable</u> if for any $f \in E(A)$ and any $\varepsilon > 0$ there exist $y, z \in A_{sa}^{**}$, such that the following conditions hold:

α) $y|E(A)$ is upper semicontinuous;

β) $z|E(A)$ is lower semicontinuous;

and

γ) $y \leq x \leq z$ and $f(z-y) < \varepsilon$.

We have now

<u>Theorem 1.8.</u> <u>For any strongly universally measurable element</u> $x \in A_{sa}^{**}$ <u>the function</u> $x|P(A)$ <u>is $\tilde{\mu}_o^{\Omega}$-measurable and</u>

$$(*) \qquad x(b(\mu)) = \int_{P(A)} x(p)\, d\tilde{\mu}_o^{\Omega}(p),$$

<u>for any maximal (maximal orthogonal) Radon probability measure</u> μ, <u>such that</u> $\|b(\mu)\| = 1$.

<u>Proof.</u> Similar to that of ([19], Theorem 3).

<u>Remark.</u> Somewhat stronger results are obtained if the measurability of the functions involved is stated with respect to the restriction of $\tilde{\mu}_o^{\Omega}$ to $\mathcal{B}(P(A);C)$. Of course, such statements are true, as one can easily infer.

§2. The canonical irreducible disintegrations of the representations of C^*-algebras

For an arbitrary C^*-algebra A we shall maintain the notations introduced in the preceding section.

I. Any $p \in P(A)$ extends to a normal positive state of A^{**}, and its support $e_p \in A^{**}$ is a minimal projection; conversely, for any minimal projection $e \in A^{**}$ one can define a pure normal state $p \in P(A^{**})$ by the formula

$$eae = p(a)e, \qquad a \in A^{**},$$

and the restriction of p to A is a pure state of A, whose support in A^{**} is equal to e.

The GNS-construction, corresponding to $p \in P(A)$, yields a Hilbert space H_p, a surjective mapping $\theta_p : A \to H_p$, an irreducible representation $\pi_p : A \to \mathcal{L}(H_p)$ and a π_p-cyclic vector $\xi_p^\circ \in H_p$, such that

$$p(a) = (\pi_p(a)\xi_p^\circ \mid \xi_p^\circ) \text{ and } \theta_p(a) = \pi_p(a)\xi_p^\circ, \quad a \in A.$$

On the other hand, it is easy to see that the scalar product on $A^{**}e_p$, given by

$$[ae_p \mid be_p] = p(b^*a), \qquad a, b \in A^{**},$$

endows $A^{**}e_p$ with a Hilbert space structure, whereas the mapping $u_p : H_p \ni \theta_p(a) \mapsto ae_p \in Ae_p \subset A^{**}e_p$ is a unitary operator. It immediately follows that $Ae_p = A^{**}e_p$, and u_p establishes a unitary equivalence of the representation π_p with the (left) regular representation P_p of A in $\mathcal{L}(Ae_p)$, such that

$$u_p \xi_p^\circ = e_p.$$

II. For any state $f_o \in E(A)$ of the arbitrary C^*-algebra A, the GNS-construction, corresponding to f_o, yields a Hilbert space H_{f_o}, a linear mapping $\theta_{f_o}: A \to H_{f_o}$, a representation $\pi_{f_o}: A \to \mathcal{L}(H_{f_o})$ and a π_{f_o}-cyclic vector $\xi^o_{f_o} \in H_{f_o}$, such that

$$f_o(a) = (\pi_{f_o}(a)\xi^o_{f_o} \mid \xi^o_{f_o}) \text{ and } \theta_{f_o}(a) = \pi_{f_o}(a)\xi^o_{f_o}, \quad a \in A.$$

If A is commutative and has the unit element, then, by the Gelfand-Naimark Theorem, A is isomorphic with the C^*-algebra $C(\mathfrak{M}(A))$ of all continuous complex functions on the maximal spectrum $\mathfrak{M}(A)$ of A, which is a compact space.

The formula

$$\mu(\tilde{a}) = f_o(a), \qquad a \in A,$$

where $\tilde{a} \in C(\mathfrak{M}(A))$ corresponds to $a \in A$ by the Gelfand-Naimark isomorphism, determines a Radon probability measure μ on $\mathfrak{M}(A)$, to which one can associate the Hilbert space $L^2(\mathfrak{M}(A), \mu)$. By the Gelfand-Naimark Representation Theorem (see [12], Ch. IV, §17), the mapping $\theta_{f_o}(a) \mapsto \tilde{a} \mapsto [\tilde{a}]$ extends to a unitary operator

$$u: H_{f_o} \to L^2(\mathfrak{M}(A), \mu),$$

which establishes a unitary equivalence of π_{f_o} with $T: A \to \mathcal{L}(L^2(\mathfrak{M}(A), \mu))$; where $[\varphi]$ denotes the class of $\varphi \in \mathcal{L}^2(\mathfrak{M}(A), \mu)$ in $L^2(\mathfrak{M}(A), \mu)$, whereas $T(a)$ is the "multiplication operator" of $[\varphi]$ by $\tilde{a}$:

$$T(a)[\varphi] = [\tilde{a}\varphi], \quad a \in A, \varphi \in \mathcal{L}^2(\mathfrak{M}(A), \mu).$$

Let us now make the following remarks:

1. The pure states of A are in bijection with the evaluation mappings

$$A \ni a \to \tilde{a}(m), \qquad m \in \mathfrak{M}(A);$$

2. If we denote by $\mathcal{L}^{\infty}_{\mathbb{R}}(\mathfrak{M}(A), \mathcal{B}_o(\mathfrak{M}(A)))$ the ordered real Banach space (endowed with the sup-norm) of all bounded Baire measurable real functions on $\mathfrak{M}(A)$, then the same space is also the smallest set of real functions on $\mathfrak{M}(A)$, which contains $C(\mathfrak{M}(A);\mathbb{R})$, and is closed with respect to the taking the point-wise limits of bounded monotone sequences;

3. If we denote by $\mathcal{L}^{\infty}(\mathfrak{M}(A), \mathcal{B}(\mathfrak{M}(A)))$ the space of all Borel measurable bounded complex functions on $\mathfrak{M}(A)$ then any $\varphi \in \mathcal{L}^{\infty}(\mathfrak{M}(A), \mathcal{B}(\mathfrak{M}(A)))$ is μ-equivalent to a bounded Baire measurable complex function $\psi \in \mathcal{L}^{\infty}(\mathfrak{M}(A), \mathcal{B}_o(\mathfrak{M}(A)))$;

4. The representation T can be extended to a representation T^{∞} of $\mathcal{L}^{\infty}(\mathfrak{M}(A), \mathcal{B}_o(\mathfrak{M}(A)))$ into $\mathcal{L}(L^2(\mathfrak{M}(A), \mu))$, which is uniquely determined by the condition:

$$\varphi_n \uparrow \varphi \quad \text{in } \mathcal{L}^{\infty}_{\mathbb{R}}(\mathfrak{M}(A), \mathcal{B}_o(\mathfrak{M}(A))) \Rightarrow T^{\infty}(\varphi_n) \uparrow T^{\infty}(\varphi) \text{ in } \mathcal{L}(L^2(\mathfrak{M}(A), \mu).$$

5. The inclusion $C(\mathfrak{M}(A)) \subset \mathcal{L}^{\infty}(\mathfrak{M}(A), \mathcal{B}_o(\mathfrak{M}(A)))$ has a general abstract analogue. Namely, let A be an arbitrary C^*-algebra and let $\mathcal{U}(A) \subset A^{**}_{sa}$ be the set of all universally measurable elements in the self-adjoint part A^{**}_{sa} of A^{**} (see [14], §4.3.11). If A is canonically embedded in A^{**}, then we can consider the smallest subset $\mathcal{B}_o(A_{sa}) \subset A^{**}$ which contains A_{sa} and is closed with respect to the taking the limits of bounded monotone sequences. Then $\mathcal{B}_o(A_{sa})$ is the self-adjoint part of a C^*-algebra $\mathcal{B}_o(A)$, and $\mathcal{B}_o(A_{sa}) \subset \mathcal{U}(A)$ (see [14], Theorem 4.5.4 and Corollary 4.5.13). In contrast to ([14], §4.5.14), we shall say that $\mathcal{B}_o(A)$ is the C^*-algebra of the Baire operators over A.

III. We shall now present the main features of what we shall call <u>the canonical irreducible disintegration</u> of a given (cyclic) representation $\pi : A \to \mathcal{L}(H)$. Once π is given, the canonical irreducible disintegration of π still depends on the choice of an arbitrary maximal abelian von Neumann subalgebra $\mathcal{C} \subset \pi(A)'$.

For any representation $\pi : A \to \mathcal{L}(H)$ let $\tilde{\pi} : A^{**} \to \mathcal{L}(A)$ be its normal extension and let $\hat{\pi} = \tilde{\pi} | \mathcal{B}_0(A)$.

For any $p \in P(A)$ we shall define the mapping

$$\tau_p : A^{**} \to Ae_p = A^{**}e_p$$

by $\tau_p(a) = ae_p$, $a \in A^{**}$; and also the mapping

$$\tau : A^{**} \to \prod_{p \in P(A)} (Ae_p)$$

by $\tau(a) = (\tau_p(a))_{p \in P(A)}$, $a \in A^{**}$.

We shall denote $\Gamma = \tau(A)$, $\Gamma_0 = \tau(\mathcal{B}_0(A))$; we have

$$\Gamma \subset \Gamma_0 \subset \prod_{p \in P(A)} (Ae_p).$$

The elements of Γ will be called the <u>canonical basic vector fields</u>, whereas the elements of Γ_0 will be called the <u>Baire canonical basic vector fields</u>; of course, both Γ and Γ_0 are vector subspaces of the direct product $\prod_{p \in P(A)} (Ae_p)$.

Let us now assume that π is cyclic, and let $\xi_0 \in H$, $\|\xi_0\| = 1$, be a π-cyclic vector. Let $f_0 \in E(A)$ be defined by $f_0(a) = (\pi(a)\xi_0 | \xi_0)$, $a \in A$, and denote by $\tilde{f}_0$ its canonical extension to A^{**}. The GNS-construction corresponding to f_0 can be identified with (H, π, ξ_0) by the unitary isomorphism

$$u_0 : A/L_{f_0} \to \pi(A)\xi_0, \text{ given by } u_0(a + L_{f_0}) = \pi(a)\xi_0, \quad a \in A,$$

which is then extended by continuity to $u:H_{f_o} \longrightarrow H$.

Let $\mathcal{C} \subset \pi(A)'$ be a maximal abelian von Neumann subalgebra, and let μ_o be the corresponding maximal orthogonal measure, such that $b(\mu_o)=f_o$ and $\mathcal{C}_{\mu_o} = \mathcal{C}$ (see [16], Theorem 3.3).

Endow $P(A)$ with the topology Ω. Then, according to Theorem 1.3, we can associate to μ_o a probability measure $\tilde{\mu}_o^{\Omega}:\mathcal{B}(P(A);\Omega)\to[0,1]$ such that the mapping $P(A)\ni p\mapsto p(a)$ is $\tilde{\mu}_o^{\Omega}$-measurable, for any $a\in\mathcal{B}_o(A)$, and

$$f_o(a)=\int_{P(A)} p(a)d\tilde{\mu}_o^{\Omega}(p), \qquad a\in\mathcal{B}_o(A).$$

Accordingly, we can define on Γ_o the scalar product

$$(1) \quad [\tau(a_1)|\tau(a_2)]=\int_{P(A)} p(a_2^*a_1)d\tilde{\mu}_o^{\Omega}(p), \; a_1,a_2\in\mathcal{B}_o(A).$$

We can also define correctly a mapping

$$V_o:\Gamma_o\to H$$

by $V_o(\tau(a))=\pi(a)\xi_o$, $a\in\mathcal{B}_o(A)$. The correctness of the definition follows from the fact that for $a\in\mathcal{B}_o(A)$, if $p(a)=0$, for any $p\in P(A)$, then $a=0$.

From (1) we immediately infer that V_o is an isometry from Γ_o, endowed with the semi-norm corresponding to the scalar product (1), into the Hilbert space H. Since ξ_o is π-cyclic, the range of V_o is dense in H. Let V be the restriction of V_o to Γ; of course, V is an isometry of Γ into H, whose range is dense in H.

We can now apply the theory developped in ([16], §4). We shall consider the Lebesgue completion

$$\mathcal{A}=\mathcal{B}(P(A);\Omega)^{\sim}_{\tilde{\mu}_o^{\Omega}}$$

and the corresponding extended measure, for which we shall keep for the same notation $\tilde{\mu}_o^{\Omega}$. Then, condition (*) from ([16], p.154) is satisfied and we can consider the completion $\bar{\Gamma}_o^2(\tilde{\mu}_o^{\Omega})$, consisting of all strongly square integrable vector fields, which are "generated" by Γ_o. The same construction, applied to Γ, yields the completion $\Gamma^2(\tilde{\mu}_o^{\Omega})$, consisting of all strongly square integrable vector fields, which are generated by $\Gamma \subset \Gamma_o$. Of course, we have

$$\Gamma^2(\tilde{\mu}_o^{\Omega}) \subset \Gamma_o^2(\tilde{\mu}_o^{\Omega}) ,$$

whereas V_o and V can be extended by continuity to $\Gamma_o^2(\tilde{\mu}_o^{\Omega})$, respectively $\Gamma^2(\tilde{\mu}_o^{\Omega})$, as unitary isomorphisms onto H. We infer that we have

(2) $$\Gamma^2(\tilde{\mu}_o^{\Omega}) = \Gamma_o^2(\tilde{\mu}_o^{\Omega}) .$$

Let us denote by $W:\Gamma^2(\tilde{\mu}_o^{\Omega}) \to H$ the unique unitary extension of V, and by $\tilde{\Gamma}^2(\tilde{\mu}_o^{\Omega})$ the (separated) Hilbert space corresponding to the pre-Hilbert space $\Gamma^2(\tilde{\mu}_o^{\Omega})$. Let $\tilde{W}:\tilde{\Gamma}^2(\tilde{\mu}_o^{\Omega}) \to H$ be the unitary isomorphism of Hilbert spaces, obtained by factoring W through the canonical mapping $Q:\Gamma^2(\tilde{\mu}_o^{\Omega}) \to \tilde{\Gamma}^2(\tilde{\mu}_o^{\Omega})$; i.e., we have $\tilde{W}Q=W$.

Remarks 1. It is customary to denote the Hilbert space $\tilde{\Gamma}^2(\tilde{\mu}_o^{\Omega})$, whose elements are equivalence classes modulo $\tilde{\mu}_o^{\Omega}$ of strongly square integrable vector fields, by $\int_{P(A)}^{\oplus} Ae_p \, d\tilde{\mu}_o^{\Omega}(p)$. No confusion should arise if the symbol Γ is omitted since, in this case, the construction of the field

$$\left\{ (Ae_p)_{p \in P(A)}, \Gamma^2(\tilde{\mu}_o^{\Omega}) \right\}$$

of Hilbert spaces is canonical.

2. The vector fields $\xi \in \Gamma^2(\tilde{\mu}_o^{\Omega})$ are functions

$$\xi : P(A) \to \bar{A}^{**},$$

such that $\xi(p) \in Ae_p = A^{**}e_p \subset A^{**}$, $p \in P(A)$.

3. Since the measure μ_o is orthogonal, the vector space $\Gamma^2(\tilde{\mu}_o^{\Omega})$ is an $\mathcal{L}^{\infty}(P(A);\mathcal{B}(P(A);\Omega)^{\sim}_{\tilde{\mu}_o^{\Omega}})$-module. It follows that

$$\left\{(Ae_p)_{p\in P(A)}, \mathcal{B}(P(A);\Omega)^{\sim}_{\tilde{\mu}_o^{\Omega}}, \tilde{\mu}_o^{\Omega}, \Gamma^2(\tilde{\mu}_o^{\Omega})\right\}$$

is an integrable field of Hilbert spaces, in the sense of W.Wils (see [16], Proposition 4.4, Theorem 4.1 and Theorem 4.2).

4. For any $\varphi \in \mathcal{L}^{\infty}(P(A);\mathcal{B}(P(A);\Omega)^{\sim}_{\tilde{\mu}_o^{\Omega}})$ we can consider the linear operator $T_{\varphi}:\Gamma^2(\tilde{\mu}_o^{\Omega}) \to \Gamma^2(\tilde{\mu}_o^{\Omega})$, given by

$$(T_{\varphi}\xi)(p) = \varphi(p)\xi(p), \quad p\in P(A), \xi\in\Gamma^2(\tilde{\mu}_o^{\Omega}),$$

which factors through Q as an operator

$$\tilde{T}_{\varphi}:\tilde{\Gamma}^2(\tilde{\mu}_o^{\Omega}) \to \tilde{\Gamma}^2(\tilde{\mu}_o^{\Omega}),$$

which belongs to $\mathcal{L}(\tilde{\Gamma}^2(\tilde{\mu}_o^{\Omega}))$ and depends only on the class $[\varphi]$ of φ in $L^2(P(A);\mathcal{B}(P(A);\Omega)^{\sim}_{\tilde{\mu}_o^{\Omega}}, \tilde{\mu}_o^{\Omega})$. Therefore, we can denote $\tilde{T}_{\varphi} = T_{[\varphi]}$.

5. The mapping $T:L^{\infty}(P(A),\mathcal{B}(P(A),\Omega)^{\sim}_{\tilde{\mu}_o^{\Omega}}, \tilde{\mu}_o^{\Omega}) \to \mathcal{L}(\tilde{\Gamma}^2(\tilde{\mu}_o^{\Omega}))$ given by $[\varphi] \mapsto T_{[\varphi]}$, is an injective $*$-homomorphism of the W^*-algebra $L^{\infty}(P(A),\mathcal{B}(P(A);\Omega)^{\sim}_{\tilde{\mu}_o^{\Omega}}, \tilde{\mu}_o^{\Omega})$ into $\mathcal{L}(\tilde{\Gamma}^2(\tilde{\mu}_o^{\Omega}))$. This follows immediately from the fact that

$$(*) \qquad (e_p)_{p\in P(A)} \in \Gamma^2(\tilde{\mu}_o^{\Omega}),$$

which can be proved as in ([16], Proposition 4.5).

6. More generally, we can consider a field $(a_p)_{p\in P(A)}$, of operators $a_p\in\mathcal{L}(Ae_p)$, $p\in P(A)$, such that there exists a constant M, having the property

$$\|a_p\|\leq M, \qquad \tilde{\mu}_o^{\omega}\text{-a.e. on } P(A).$$

We shall say that $(a_p)_{p\in P(A)}$ is an <u>integrable field of operators</u> if

$$(**) \qquad (\xi_p)_{p\in P(A)}\in\Gamma^2(\tilde{\mu}_o^{\omega}) \Rightarrow (a_p\xi_p)_{p\in P(A)}\in\Gamma^2(\tilde{\mu}_o^{\omega}).$$

It is obvious that any integrable field of operators determines a linear operator $a:\Gamma^2(\tilde{\mu}_o^{\omega})\to\Gamma^2(\tilde{\mu}_o^{\omega})$, and there exists a uniquely determined operator $\tilde{a}\in\mathcal{L}(\tilde{\Gamma}^2(\tilde{\mu}_o^{\omega}))$, such that $\tilde{a}Q=Qa$.

It is customary to call the operator of the form $\tilde{W}\tilde{a}\tilde{W}^{-1}(\in\mathcal{L}(H))$, corresponding to integrable fields of operators, <u>decomposable operators</u>, and to denote

$$\tilde{a} = \int_{P(A)}^{\oplus} a_p d\tilde{\mu}_o^{\omega}(p).$$

The operators of the form $\tilde{W}T_{[\varphi]}\tilde{W}^{-1}$ are decomposable, and they correspond to the integrable fields of operators of the form $(\varphi(p)1_p)_{p\in P(A)}$ where 1_p is the identity operator in Ae_p, $p\in P(A)$. Such operators are called <u>diagonalizable operators.</u> For any $a\in\mathcal{B}_0(A)$ we can consider the field of left regular representations $(\rho_p(a))_{p\in P(A)}$. It it obvious from equality (2) above that any such field of operators is integrable. If we denote

$$\tilde{\rho}(a)=\int_{P(A)}^{\oplus}\rho_p(a)d\tilde{\mu}_o^{\omega}(p), \qquad a\in\mathcal{B}_0(A),$$

then $\tilde{W}\tilde{\rho}(a)\tilde{W}^{-1}=\tilde{\pi}(a)$, $a\in\mathcal{B}_0(A)$. We have, therefore, Proposition 2.1. Any Baire operator in $\mathcal{L}(H)$ over $\pi(A)$ is decomposable.

Proof. The Baire operators in $\mathcal{L}(H)$ over $\pi(A)$ are, by definition, those operators in $\mathcal{L}(H)$ whose real and imaginary parts belong to the smallest vector subspace of $\mathcal{L}(H)_{sa}$, which is closed under the taking of limits of monotone bounded sequences, and contains $\pi(A)_{sa}$ (see [14], Theorem 4.5.4). The set $\mathcal{B}_0(\pi(A))$ of the Baire operators over $\pi(A)$ is a C^*-algebra and $\mathcal{B}_0(\pi(A))=\tilde{\pi}(\mathcal{B}_0(A))$ (see [14], Theorem 4.5.9). The Proposition now immediately follows from the equality just preceding it.

Remark. This result extends to the possibly non-separable case the well-known method of reducing a von Neumann algebra $\mathcal{a}$ acting on a separable Hilbert space: one chooses a separable weakly dense C^*-subalgebra $\mathcal{a}_0 \subset \mathcal{a}$, with whose help an integrable field of separable Hilbert spaces is constructed. Subsequently it is proved that any operator in $\mathcal{a}$ is decomposable. But, in this case, $\mathcal{a}=\mathcal{B}_0(\mathcal{a}_0)$ (see [14], Corollary 2.8.8).

It is obvious that for any strongly square integrable vector field $\xi=(\xi_p)_{p\in P(A)}\in\Gamma^2(\tilde{\mu}_0^{\Omega})$ and any $a\in\mathcal{B}_0(A)$ we have

$$\|\hat{\pi}(a)W\xi\|^2=\int_{P(A)}\|\rho_p(a)\xi_p\|^2 d\tilde{\mu}_0^{\Omega}(p).$$

In particular, we have

$$\|\hat{\pi}(a)\xi_0\|^2=\int_{P(A)}\|ae_p\|^2 d\tilde{\mu}_0^{\Omega}(p),$$

for any $a\in\mathcal{B}_0(A)$.

Proposition 2.2. The algebra of the diagonalizable operators coincides with $\mathcal{C}$.

Proof. The mapping

$$S:L^{\infty}(P(A),\mathcal{B}(P(A);\Omega)^{\sim}_{\tilde{\mu}_0^{\Omega}},\tilde{\mu}_0^{\Omega}) \to \mathcal{L}(H)$$

given by $S([\varphi])=\widetilde{W}T_{[\varphi]}\widetilde{W}^{-1}$, obviously is a $*$-homomorphism. From the equality

$$\|S([\varphi])\xi_0\|^2=\int_{P(A)}|\varphi(p)|^2 d\tilde{\mu}_0^{\Omega}(p),$$

we infer that S is injective.

It is now obvious that the range $\mathcal{R}(S)$ is an abelian $*$-subalgebra of $\mathcal{L}(H)$, and also that

$$\mathcal{C}\subset\mathcal{R}(S)\subset\pi(A)'.$$

Since $\mathcal{C}$ is maximal abelian in $\pi(A)'$, we infer that $\mathcal{C}=\mathcal{R}(S)$ (for a more detailed proof, see [16], Theorem 4.3).

The preceding theory extends the irreducible disintegration theory we have developed in [16].

The new feature shown by this extension is the decomposability of the Baire operators, as well as the topological properties of the measures with whose help the disintegration can be carried out.

As in [16], we can apply these results to the central (factor) disintegration (reduction) of the identical representation of any von Neumann algebra, as well as to the irreducible disintegration of the unitary representations of the locally compact groups (see [16], §5 and §6).

IV. As an example, let us consider the field of operators $(e_p)_{p\in P(A)}$, where $e_p\in\mathcal{L}(Ae_p)$ acts by multiplication to the left: $ae_p\mapsto e_pae_p=p(a)e_p$. Since $p\mapsto p(a)$ is bounded and $\tilde{\mu}_0^{\Omega}$-measurable, we infer that $(p(a)e_p)_{p\in P(A)}\in\Gamma^2(\tilde{\mu}_0^{\Omega})$, and therefore, the field of operators $(e_p)_{p\in P(A)}$ is integrable. If we denote

$$\tilde{e} = \int_{P(A)}^{\oplus} e_p d\tilde{\mu}_o^{\Omega}(p),$$

then $\tilde{W}\tilde{e}\tilde{W}^{-1}$ is the projection on $\overline{\mathcal{E}\xi_o}$; hence, it is an abelian projection in $\mathcal{C}'$ and, more precisely, with the notation from ([16], Theorem 3.4) we have

$$\tilde{W}\tilde{e}\tilde{W}^{-1} = e_{\mu_o} ,$$

and this shows that the projection e_{μ_o} is decomposable. Hence, abelian projections are shown to be fields of one-dimensional projections, an intuitively sensible fact.

<u>Remark.</u> In ([16], Proposition 5.6) we have proved that any $\hat{\alpha}$-measurable function φ: supp $\check{\alpha} \to \mathbb{R}$ coincides $\hat{\alpha}$ - a.e. with a (unique) continuous function. With the help of the regularity property of $\hat{\alpha}$, which was proved in ([17], Theorem 6), one can show that any such function φ <u>is</u> continuous on an open dense subset of supp $\check{\alpha}$. This result is well known for (compact; i.e., quasi-compact and Hausdorff) hyperstonean spaces, where the Hausdorff separation property is usually used in the proof. In our case, where the space supp $\check{\alpha}$ generally fails to be Hausdorff, the proof can use the regularity of the measure $\hat{\alpha}$ instead.

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