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by

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ON COMPOSITION SERIES OF A MODULE WITH RESPECT TO A SET OF GABRIEL TOPOLOGIES

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Composition series for modules with respect to a Gabriel topology (or equivalently, with respect to a hereditary torsion theory) were introduced by Goldman [1] in 1975, but only for torsion-free modules, and have been further studied, among others, by Beachy, Golan, etc.

A series of recent papers of Boyle and Feller [1], [2], [3] consider the semicritical socle series as well as the Krull critical composition series of a module over a right noetherian ring.

Teply [1] announced in 1982 the definition of the notion of composition series of a module for sets of torsion theories, which works in particular for the chain of torsion theories for Krull dimension over a right noetherian ring.

The aim of this paper is to place all these notions in a latticial setting. This allows to obtain simplified proofs in a slight generalized context of some previous results and to enlighten the machinery involved in their proofs.

Section 0 presents the basic notions and facts on Loewy series of a lattice, semi-artinian lattices, semi-atomic lattices, as well as two simple but useful lemmas which will be exploited in the sequel.

The next section show essentially that there exists an identity between the composition series of a right R-module M with respect to a Gabriel topology F on R, the composition series of the lattice $C_F(M)$ of all F-saturated submodules of M, and the composition series of the image of M in a suitable quotient category of the category Mod-R, from which can be easily deduced the invariance of the length of such composition series as well as the invariance of the injective hulls of the factors of the series.

In section 2 is presented the definition of the concept of composition series of a module with respect to a set of Gabriel topologies, and is proved in a very simple manner, by using the two lemmas from section 0, a result due to Teply [1] , [2], concerning the invariance of the type of such a composition series.

0. PRELIMINARIES

Throughout this section L will denote a modular lattice with a least element 0 and with a greatest element 1. The notation and terminology will follow Stenström [1].

0.1. Lemma. Let $(x_i)_{0\leq i\leq n}$ be a finite family of elements of L such that

$$0 = x_0 < x_1 < \ldots < x_n = 1.$$

If $y \in L$, $y \neq 0$, then there exists j, $1 \leq j \leq n$, and $z \in]x_{j-1}, x_j]$, $u \in]0,y]$ such that the intervals $[0,u]$ and $[x_{j-1}, z]$ are similar.

Proof. Denote by j the least integer $\leq n$ such that $y \wedge x_j \neq 0$. Clearly $j > 0$; then $x_{j-1} < (y\wedge x_j) \vee x_{j-1}$, for otherwise $y \wedge x_j \leq x_{j-1}$, and so $y \wedge x_j = (y \wedge x_j) \wedge x_{j-1} = y \wedge x_{j-1} = 0$, a contradiction. Denote $z = (y \wedge x_j) \vee x_{j-1}$. Then $z \in]x_{j-1}, x_j]$ and

$$[x_{j-1}, z] \simeq [x_{j-1}\wedge(y\wedge x_j), y\wedge x_j] = [y \wedge x_{j-1}, y\wedge x_j] = [0,u],$$

where $u = y \wedge x_j$. ∎

Throughout the remainder of this section L will be supposed to be an upper continuous and modular lattice.

We shall recall now some definitions and properties. An atom of L is a nonzero element $a \in L$ such that whenever $b \in L$ and $b < a$, then $b = 0$, i.e., the interval $[0,a]$ has exactly two elements, 0 and a. If $x, y \in L$ and $x < y$, then the interval $[x,y]$ is said to be simple if y is an atom in the sublattice $[x,y]$ of L. The lattice L is called

<u>semi-atomic</u> if 1 is a join of atoms, and L is called <u>semi-artinian</u> if for every $x \in L$, $x \neq 1$, the sublattice $[x,1]$ of L contains an atom. If L is a semi-atomic lattice, then L is complemented, and for every $x,y \in L$ with $x \leq y$ the interval $[x,y]$ of L is a semi-atomic lattice.

The <u>(ascending) Loewy series</u> of L

$$s_0(L) < s_1(L) < \ldots < s_{\lambda(L)}(L) \qquad (*)$$

is defined inductively as follows: $s_0(L) = 0$, $s_1(L)$ is the <u>socle</u> So(L) of L (i.e., the join of all atoms of L), and if the elements $s_\beta(L)$ of L have been defined for all ordinals $\beta < \alpha$, then $s_\alpha(L) = \bigvee_{\beta<\alpha} s_\beta(L)$ if α is a limit ordinal, and $s_\alpha(L) = So([s_\gamma(L),1])$ if $\alpha = \gamma + 1$; $\lambda(L)$ is the least ordinal λ such that $s_\lambda(L) = s_{\lambda+1}(L)$, and is called the <u>Loewy length</u> of L. The intervals $[s_\alpha(L), s_{\alpha+1}(L)]$ are called the <u>factors</u> of the series $(*)$, and they are for each ordinal α, $\alpha < \lambda(L)$ semi-atomic lattices. The lattice L is semi-artinian if and only if $s_{\lambda(L)}(L) = 1$. Note also that if x,y and z are elements of L such that $x < y < z$, then $[x,z]$ is a semi-artinian lattice if and only if $[x,y]$ and $[y,z]$ are both semi-artinian.

For all these summarized facts concerning Loewy series of a lattice the reader is referred to Năstăsescu [1].

<u>0.2. Lemma.</u> Let $(x_i)_{0\leq i\leq n}$ be a finite family of elements of L $(n \geq 1)$, such that

$$0 = x_0 < x_1 < \ldots < x_n$$

and $[x_{i-1}, x_i]$ are simple intervals for each i, $1 \leq i \leq n$.

If the interval $[x_n, 1]$ contains no atom, then $\lambda(L) \leq n$ and $x_n = s_{\lambda(L)}(L)$.

Proof. We shall prove by induction that $x_i \leq s_i(L)$ for each i, $0 \leq i \leq n$. Trivially this holds if $i = 0$ and $i = 1$, so assume it holds for $i < n$ and prove it for $i+1$. By modularity, one has

$$[s_i(L), x_{i+1} \vee s_i(L)] \simeq [x_i \vee s_i(L), (x_i \vee s_i(L)) \vee x_{i+1}] \simeq$$
$$\simeq [(x_i \vee s_i(L)) \wedge x_{i+1}, x_{i+1}] = [x_i \vee (s_i(L) \wedge x_{i+1}), x_{i+1}] \subseteq [x_i, x_{i+1}].$$

But $[x_i, x_{i+1}]$ is a simple interval, so $[s_i(L), x_{i+1} \vee s_i(L)]$ is a simple interval or is reduced to a single element, and therefore $x_{i+1} \vee s_i(L) \leq s_{i+1}(L)$. In particular $x_{i+1} \leq s_{i+1}(L)$.

Let now $k \geq 1$ be the least natural number such that $x_n \leq s_k(L)$ and $x_n \nleq s_{k-1}(L)$. Since each interval $[s_j(L), s_{j+1}(L)]$ is a semi-atomic lattice, hence a semi-artinian lattice, it follows that $[0, s_k(L)]$ is a semi-artinian lattice. Hence $x_n = s_k(L)$, for otherwise, the interval $[x_n, s_k(L)]$ would contain an atom, a contradiction. By the same argument, $s_k(L) = s_{k+1}(L)$; so $\lambda(L) = k$, and consequently $x_n = s_{\lambda(L)}(L)$. ∎

1. COMPOSITION SERIES OF A MODULE WITH RESPECT TO A GABRIEL TOPOLOGY

Let L be a modular lattice with elements 0 and 1, $0 \neq 1$. Recall that a (Jordan-Hölder) composition series of L is a chain

$$0 = a_o < a_1 < \ldots < a_n = 1$$

such that each interval $[a_{i-1}, a_i]$, $1 \leq i \leq n$ is a simple interval. The lattice L is said to be <u>of finite length</u> if L has a composition series; in this case, a well-known result asserts that any two composition series of L are equivalent (see e.g. Stenström [1]). Note that a modular lattice with 0 and 1 is of finite length if and only if it is both artinian and noetherian.

Let now R be an associative, unitary and nonzero ring, and Mod-R the category of unitary right R-modules. If M is a right R-module then $\mathcal{L}(M)$ will denote the lattice of all submodules of M.

The set of all right Gabriel topologies on R will be denoted by <u>Gab</u>(R). If $F \in$ <u>Gab</u>(R), then $(\mathcal{T}_F, \mathcal{F}_F)$ will denote the corresponding hereditary torsion theory on Mod-R, and t_F the torsion radical associated to $(\mathcal{T}_F, \mathcal{F}_F)$. If $M \in$ Mod-R, we shall use the following notation

$$C_F(M) = \{N \in \mathcal{L}(M) \mid M/N \in \mathcal{F}_F\}.$$

For each $P \in \mathcal{L}(M)$ $\widetilde{P}$ will denote the F-saturation of P in M, i.e., $\widetilde{P}/P = t_F(M/P)$. Thus $P \in C_F(M)$ iff $P = \widetilde{P}$, i.e., P is F-saturated. If $(N_i)_{i \in I}$ is a family of elements of $C_F(M)$, then $\bigvee_{i \in I} N_i = (\sum_{i \in I} N_i)^{\sim}$ and $\bigwedge_{i \in I} N_i = \bigcap_{i \in I} N_i$ are elements of $C_F(M)$. Moreover, $C_F(M)$ is an upper continuous and modular lattice with respect to the partial ordering given by "$\subseteq$" (inclusion) and with respect to the operations "$\vee$" and "$\wedge$", having the least element $t_F(M)$ and the greatest element M.

Recall that $M \in \text{Mod-}R$ is said to be F-cocritical if $M \neq 0$, $M \in \mathcal{F}_F$ and $M/M' \in \mathcal{T}_F$ for every nonzero submodule M' of M, or equivalently, if $M \neq 0$ and $C_F(M) = \{0, M\}$, or equivalently, if $M \in \mathcal{F}_F$ and $T_F(M)$ is a simple object in the quotient category $\text{Mod-}R/\mathcal{T}_F$, where

$$T_F : \text{Mod-}R \longrightarrow \text{Mod-}R/\mathcal{T}_F$$

is the canonical functor.

An F-composition series of $M \in \text{Mod-}R$ is a chain

$$t_F(M) = M_o < M_1 < \ldots < M_n = M$$

of submodules of M such that M_i/M_{i-1} is an F-cocritical module for each i, $0 < i \leq n$.

1.1. Proposition. Let $M \in \text{Mod-}R$ and

$$M_o < M_1 < \ldots < M_n = M \qquad (*)$$

be a chain of submodules of M. Then $(*)$ is an F-composition series of M if and only if $(*)$ is a composition series of the lattice $C_F(M)$.

Proof. Suppose that $(*)$ is an F-composition series of M. The exact sequence of R-modules

$$0 \longrightarrow M_{n-1}/M_{n-2} \longrightarrow M/M_{n-2} \longrightarrow M/M_{n-1} \longrightarrow 0$$

with $M_{n-1}/M_{n-2} \in \mathcal{F}_F$ and $M/M_{n-1} \in \mathcal{F}_F$ yields $M/M_{n-2} \in \mathcal{F}_F$. Then, from the exact sequence of R-modules

$$0 \longrightarrow M_{n-2}/M_{n-3} \longrightarrow M/M_{n-3} \longrightarrow M/M_{n-2} \longrightarrow 0$$

with $M_{n-2}/M_{n-3} \in \mathcal{F}_F$ and $M/M_{n-2} \in \mathcal{F}_F$ one gets $M/M_{n-3} \in \mathcal{F}_F$, and so on, $M/M_i \in \mathcal{F}_F$ for all i, $0 \leq i < n$, i.e., $M_i \in C_F(M)$.

Let now $X \in C_F(M)$ with $M_{i-1} \leq X \leq M_i$. Then $M/X \in$

$\in \mathcal{F}_F$, hence $M_i/X \in \mathcal{F}_F$. But $(M_i/M_{i-1})/(X/M_{i-1}) \simeq M_i/X \in \mathcal{T}_F$ because M_i/M_{i-1} is F-cocritical, hence $M_i/X \in \mathcal{T}_F \cap \mathcal{F}_F = \{0\}$, i.e., $X = M_i$, and so $[M_{i-1}, M_i]$ is a simple interval in $C_F(M)$, i.e., (*) is a composition series of the lattice $C_F(M)$.

Conversely, suppose that (*) is a composition series of the lattice $C_F(M)$. Then necessarily $M_o = t_F(M)$. We have to prove that M_i/M_{i-1} is F-cocritical for each i, $0 < i \leqslant n$. First of all $M_i/M_{i-1} \in \mathcal{F}_F$ because $M/M_{i-1} \in \mathcal{F}_F$. Let $Y/M_{i-1} \in C_F(M_i/M_{i-1})$ with $Y \neq M_{i-1}$. Then $(M_i/M_{i-1})/(Y/M_{i-1}) \simeq$ $\simeq M_i/Y \in \mathcal{F}_F$. The exact sequence of R-modules

$$0 \rightarrow M_i/Y \rightarrow M/Y \rightarrow M/M_i \rightarrow 0.$$

with $M_i/Y \in \mathcal{F}_F$ and $M/M_i \in \mathcal{F}_F$ yields $M/Y \in \mathcal{F}_F$, i.e., $Y \in$ $\in C_F(M)$. But $M_{i-1} < Y \leqslant M_i$, hence $Y = M_i$ because $[M_{i-1}, M_i]$ is a simple interval in $C_F(M)$. Consequently M_i/M_{i-1} is an F-cocritical module for each i, $0 < i \leqslant n$. ∎

<u>1.2. Corollary</u>. The following assertions are equivalent for a right R-module M:

(1) M has an F-composition series.

(2) $C_F(M)$ is a lattice of finite length.

(3) $T_F(M)$ is an object of finite length in Mod-R/$\mathcal{T}_F$.

Moreover, if

$$t_F(M) = M_o < M_1 < \ldots < M_n = M$$

is a composition series of the lattice $C_F(M)$, then the injective hulls $E_R(M_i/M_{i-1})$ of the modules M_i/M_{i-1} are unique up to order and isomorphism.

Proof. Since the lattice $C_F(M)$ is isomorphic to the lattice $\mathcal{S}(T_F(M))$ of all subobjects of the object $T_F(M)$ (see e.g. Albu and Năstăsescu [1]), it follows immediately the equivalence of the assertions (1), (2) and (3).

Applying the exact functor T_F to the chain

$$t_F(M) = M_o < M_1 < \dots < M_n = M$$

one obtains a composition series

$$0 = T_F(M_o) \subset T_F(M_1) \subset \dots \subset T_F(M_n) = T_F(M)$$

of $T_F(M)$ in Mod-$R/\mathcal{T}_F$. By the Jordan-Hölder theorem we have only to prove that if X_1 and X_2 are two F-cocritical R-modules with $T_F(X_1) \simeq T_F(X_2)$, then $E_R(X_1) \simeq E_R(X_2)$. Since $T_F(X_1) \simeq T_F(X_2)$ it follows that the injective hull $E(T_F(X_1))$ of $T_F(X_1)$ is isomorphic to the injective hull $E(T_F(X_2))$ of $T_F(X_2)$. But $X_1, X_2 \in \mathcal{F}_F$, hence

$$E(T_F(X_1)) \simeq T_F(E_R(X_1)) \quad \text{and} \quad E(T_F(X_2)) \simeq T_F(E_R(X_2))$$

(see Gabriel [1]), and so $S_F T_F(E_R(X_1)) \simeq S_F T_F(E_R(X_2))$, where S_F : Mod-$R/\mathcal{T}_F \longrightarrow$ Mod-R is the right adjoint of the functor T_F. Since $E_R(X_1)$ and $E_R(X_2)$ are injective modules, hence F-closed, it follows that

$$E_R(X_1) \simeq S_F T_F(E_R(X_1)) \simeq S_F T_F(E_R(X_2)) \simeq E_R(X_2). \blacksquare$$

2. RELATIVE $\mathfrak{X}$-COMPOSITION SERIES OF A MODULE

Definition (Teply [1],[2]). Let $\mathfrak{X}$ be a nonempty set of (right) Gabriel topologies on R. A right R-module M is said to have an $\mathfrak{X}$-composition series if there exists a chain of submodules of M

$$M_o < M_1 < \dots < M_n = M \qquad (*)$$

such that

(1) Each M_i/M_{i-1} is F_i-cocritical for some $F_i \in \mathcal{X}$.

(2) If M_i/M_{i-1} contains an F-cocritical submodule for some $F \in \mathcal{X}$, then $F = F_i$.

(3) $F_1 \leq F_2 \leq \dots \leq F_n$.

(4) $M_o = t_{F_1}(M)$. ∎

Note that the original definition of Teply requires $M_o = 0$, i.e., M is F_1-torsion-free. If $\mathcal{X}$ has only one element F one obtains the definition of an F-composition series.

If M has an $\mathcal{X}$-composition series $(*)$, then by renumbering the Gabriel topologies F_i one gets for M a so called $\mathcal{X}$-composition series of type

$$(F_1, n_1; F_2, n_2; \dots ; F_k, n_k),$$

where $k \geq 1$, $n_1, n_2, \dots , n_k$ are natural numbers ≥ 1, and $F_1, F_2, \dots , F_k \in \mathcal{X}$ such that

(i) $F_1 < F_2 < \dots < F_k$.

(ii) The first n_1 factors of the series $(*)$ are F_1-cocritical, the next n_2 factors are F_2-cocritical, and so on, the last n_k factors of $(*)$ are F_k-cocritical.

<u>2.1. Lemma.</u> Let $F \in \underline{\text{Gab}}(R)$ and $X \in \text{Mod-}R$. Then $C_F(X) = \{X\}$ if and only if $X \in \mathcal{T}_F$.

<u>Proof.</u> If $C_F(X) = \{X\}$, then $t_F(X) = X$ because $t_F(X) \in C_F(X)$, i.e., $X \in \mathcal{T}_F$. Conversely, if $X \in \mathcal{T}_F$ and $Y \in$

$\in C_F(X)$, then $X/Y \in \mathcal{T}_F \cap \mathcal{F}_F = \{0\}$, hence $Y = X$. ∎

2.2. Proposition. Let $X \in$ Mod-R and $F_1, F_2 \in \underline{\text{Gab}}(R)$ such that $F_1 \leq F_2$. If X is F_1-cocritical, then either X is F_2-cocritical or $X \in \mathcal{T}_{F_2}$.

Proof. Since $F_1 \leq F_2$, it follows that $\mathcal{F}_{F_2} \subseteq \mathcal{F}_{F_1}$, hence $C_{F_2}(X) \subseteq C_{F_1}(X) = \{0, X\}$, and so $C_{F_2}(X) = \{0, X\}$ or $C_{F_2}(X) = \{X\}$, i.e., X is F_2-cocritical or $X \in \mathcal{T}_{F_2}$ by 2.1. ∎

2.3. Theorem. Let $M \in$ Mod-R and $\emptyset \neq \mathcal{X} \subseteq \underline{\text{Gab}}(R)$. If M has an $\mathcal{X}$-composition series $M_o < M_1 < \ldots < M_n = M$ of type $(F_1, n_1; F_2, n_2; \ldots; F_k, n_k)$, then any other $\mathcal{X}$-composition series of M beginning with M_o is of the same type, and the injective hulls of the factors M_i/M_{i-1} are unique up to order and isomorphism. Moreover,

$$t_{F_{j+1}}(M) = M_{n_1+n_2+\ldots+n_j}$$

for each j, $1 \leq j \leq k-1$.

Proof. Let $M_o = N_o < N_1 < \ldots < N_m = M$ be another $\mathcal{X}$-composition series of M of type $(G_1, m_1; G_2, m_2; \ldots; G_s, m_s)$. We have to prove that $s = k$, $G_i = F_i$ and $m_i = n_i$ for each i, $1 \leq i \leq k$.

First of all, $M/M_{n-1} \in \mathcal{F}_{F_k} \subseteq \mathcal{F}_{F_1}$, and, as in the proof of 1.1 one deduces that $M/M_i \in \mathcal{F}_{F_1}$ for all i, i.e., the given chain of submodules of M is a chain in the lattice $C_{F_1}(M)$. The same is true also for the other above considered $\mathcal{X}$-composition series of M. Note that by the same argument, if $k \geqslant 2$, then $M_{n_1} \in C_{F_2}(M)$.

By 0.1, applied to the interval $[M_o, M]$ considered in the lattice $\mathcal{L}(M)$, one deduces that there exists j, $1 \leqslant j \leqslant n$, $Z \in \mathcal{L}(M)$, $M_{j-1} < Z \leqslant M_j$, and $U \in \mathcal{L}(N_1)$, $N_o < U$ such that $U/N_o \simeq$ $\simeq Z/M_{j-1}$. But U/N_o is G_1-cocritical, hence $G_1 = F_p \geqslant F_1$, where M_j/M_{j-1} is F_p-cocritical for some $p \geqslant 1$. By symmetry, $F_1 \geqslant G_1$, and so $F_1 = G_1$.

If $M_{n_1} \neq M$, then applying again 0.1 to the interval $[M_{n_1}, M]$ considered in the lattice $\mathcal{L}(M)$, one deduces that $[M_{n_1}, M]$, considered now as an interval in the lattice $C_{F_1}(M)$, contains no atom, for otherwise the R-module M_j/M_{j-1} would contain a submodule which is F_1-cocritical for some $j > n_1$, a contradiction. By 0.2, $M_{n_1} = s_\lambda(C_{F_1}(M)) = s_\lambda(C_{G_1}(M)) = N_{m_1}$, where $\lambda = \lambda(C_{F_1}(M)) = \lambda(C_{G_1}(M))$. The same is also true if $M_{n_1} = M$.

By 1.2, $n_1 = m_1$ and $E_R(M_i/M_{i-1})$ are unique up to order and isomorphism for all i, $i \leqslant n_1$. Applying now again 0.1, 0.2 and 1.2 to the chains

$$M_{n_1} < M_{n_1+1} < \dots < M_n = M$$
$$M_{n_1} = N_{n_1} < N_{n_1+1} < \dots < N_m = M$$

of elements of $C_{F_2}(M)$ one gets $n_2 = m_2$, etc.

We shall prove the last assertion of the theorem only for $j = 1$. By 2.2, $M_1/M_o \in \mathcal{T}_{F_2}$; on the other hand, $M_o \in \mathcal{T}_{F_1} \subseteq \mathcal{T}_{F_2}$, hence $M_1 \in \mathcal{T}_{F_2}$, and so on, $M_i \in \mathcal{T}_{F_2}$ for all $i \leqslant n_1$. It follows that $M_{n_1} = t_{F_2}(M_{n_1}) \leqslant t_{F_2}(M)$. But we have seen that $M_{n_1} \in C_{F_2}(M)$, hence $t_{F_2}(M) \leqslant M_{n_1}$, and thus $t_{F_2}(M) = M_{n_1}$. ∎

<u>2.4. Remark.</u> It is easy to see that the notion of an <u>F-semicritical module</u>, as it appears in Teply [2], can be refor-

mulated in a latticial setting as follows: an R-module M is F-semicritical iff $M \in \mathcal{F}_F$ and $C_F(M)$ is a semi-atomic lattice of finite length. Note also that the F-semicritical socle series of M is exactly the Loewy series of the lattice $C_F(M)$. ∎

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