INSTITUTUL
DE
MATEMATICĂ

INSTITUTUL NAȚIONAL
PENTRU CREAȚIE
ȘTIINȚIFICĂ ȘI TEHNICĂ

ISSN 0250 3638

TWO PROBLEMS OF OPERATORIAL EXTRAPOLATIONS

by

Radu GADIDOV

PREPRINT SERIES IN MATHEMATICS
No. 31/1984

BUCUREȘTI

TWO PROBLEMS OF OPERATORIAL EXTRAPOLATIONS

by

Radu GADIDOV*)

May 1984

*) The National Institute for Scientific and Tehnical Creation, Department of Mathematics, Bd. Pacii 220, 79622 Bucharest, Romania

Two problems of operatorial extrapolations

by R. Gadidov

INTRODUCTION

Let H , H_1 , H_2 be (complex) Hilbert spaces , $H = H_1 \vee H_2$, $T_i \in \mathcal{L}(H_i)$ i=1,2 contractions , $U_i \in \mathcal{L}(K_i)$ i=1,2 their minimal isometric dilations and let $A = P^H_{H_2|H_1}$ (*) .

In the present paper we shall deal with the problems below :

PROBLEM A : the existence of a contraction $T \in \mathcal{L}(H)$ such that :

$$T_{|H_1} = T_1 \ , \ T^*_{|H_2} = T^*_2$$

and the parametrization of the set of these contractions ,

PROBLEM B : the existence of an $\mathcal{L}(H)$ - valued function $\{F_n\}_{n \in Z}$ of pozitive type , such that :

$$F_o = 1_H \ , \ F_{n|H_1} = T_1^n \ , \ F^*_{n|H_2} = T_2^{*n} \quad , \quad n \geq 1$$

and the parametrization of the set of these functions .

Let us note that if one of the two problems has solutions ,then the other has too. (If $\{F_n\}_{n \in Z}$ is a solution of Problem B , then $T = F_1$ is a solution of Problem A and conversely , if T is a solution of Problem A , then the $\mathcal{L}(H)$ - valued function $\{F_n\}_{n \in Z}$ defined by : $F_n = T^n$, $n \geq o$ and $F_n = T^{*|n|}$, $n \leq o$ is a solution of Problem B).

Based on the celebrated Nagy-Foiaş lifting theorem it will be proved that Problem B has solution iff $A\,T_1 = T_2\,A$.

Using the techniques in [1] we obtain a necessary and sufficient

---

(*) If G is a closed linear subspace of H , P^H_G denotes the orthogonal projection of H onto G .

condition for a contractive intertwining dilation of a contraction which intertwines two other contractions, be an isometry.

Though this condition seems nothing but a reformulation, it may be effectively used in some particular cases. As an exemplification, we shall prove the existence of isometric interwining dilations of Hankel operators which are strict contractions, in slightly more general conditions than those imposed by Adamian-Arov-Kreĭn in [2].

1.

The main result of this section is the following:

THEOREM 1.1. Problem B has solutions if and only if $A T_1 = T_2 A$.

In this case, if $\mathcal{F}$ is the set of the solutions of Problem B and $CID(A) = \{ B \in \mathcal{L}(K_1, K_2) : B$ contraction, $B U_1 = U_2 B$, $P_{H_2}^{K_2} = A P_{H_1}^{K_1} \}$ then there exists a bijection between $CID(A)$ and $\mathcal{F}$.

Proof Let us suppose that $A T_1 = T_2 A$ and let $B \in CID(A)$.

Since the operator:

$$H_B = \begin{pmatrix} 1 & B^* \\ B & 1 \end{pmatrix} : \begin{matrix} K_1 \\ \oplus \\ K_2 \end{matrix} \longrightarrow \begin{matrix} K_1 \\ \oplus \\ K_2 \end{matrix}$$ is positive, one may define a new inner product $\langle \cdot , \cdot \rangle_K$ on $K_1 \oplus K_2$ by:

$$\langle k , k' \rangle_K = \langle H_B k , k' \rangle \qquad k , k' \in K_1 \oplus K_2$$

This new inner product is only positive semidefinite in general and gives rise, via the usual process of factorization and completion, to a new Hilbert space K, such that K_1, K_2 are naturally embedded in K, $K = K_1 \vee K_2$ and $B = P^K_{K_2}|K_1$.

K will be called the renormed Hilbert space of $K_1 \oplus K_2$ by B.

Since for every h_1, $h_1' \in H_1$, h_2, $h_2' \in H_2$,

$$\langle h_1 + h_2 , h_1' + h_2' \rangle_H^{(*)} = \langle h_1 , h_1' \rangle_{H_1} + \langle h_2 , A h_1' \rangle_{H_2} + \langle h_2 , h_2' \rangle_{H_2} +$$

(*) If G is a Hilbert space, $\langle \cdot , \cdot \rangle_G$ denotes as usual the inner product of G.

$+ \langle Ah_1 , h'_2 \rangle_{H_2} = \langle h_1 , h'_1 \rangle_{H_1} + \langle Bh_1 , h'_2 \rangle_{K_2} + \langle h_2 , Bh'_1 \rangle_{K_2} +$

$+ \langle h_2 , h'_2 \rangle_{K_2} = \langle h_1 + B^* h_2 , h'_1 \rangle_{K_1} + \langle h_2 + Bh_1 , h'_2 \rangle_{K_2} =$

$= \left\langle \begin{pmatrix} 1 & B^* \\ B & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} , \begin{pmatrix} h'_1 \\ h'_2 \end{pmatrix} \right\rangle = \langle h_1 + h_2 , h'_1 + h'_2 \rangle_K$, K is naturally embedded in K .

Let U be the isometry defined by :

$$U(k_1 + k_2) = U_1 k_1 + U_2 k_2 \qquad k_1 \in K_1 , k_2 \in K_2 .$$

Since $U_i \in \mathcal{L}(K_i)$ $i=1,2$ are the minimal isometric dilations of the contractions T_i $i=1,2$, it is obvious that $K = \bigvee_{n \geq 0} U^n H$.

The isometry U gives rise to an $\mathcal{L}(H)$ - valued function $\{F_n\}_{n \in Z}$ of positive type , defined by :

$$F_n = P^K_H U^n|_H , n \geq 0 , F_n = P^K_H U^{*|n|}|_H , n \leq 0 .$$

Let us show that $F_n|_{H_1} = T_1^n$, $F^*_n|_{H_2} = T_2^{*n}$, $n \geq 0$.

Consider $h_1, h'_1 \in H_1$, h_2 , $h'_2 \in H_2$, $n \geq 0$.

Then :

$\langle F_n(h_1 + h_2) , h'_1 + h'_2 \rangle_H = \langle U^n(h_1 + h_2) , h'_1 + h'_2 \rangle_K =$

$= \langle U_1^n h_1 + U_2^n h_2 , h'_1 + h'_2 \rangle_K = \langle U_1^n h_1 + B^* U_2^n h_2 , h'_1 \rangle_{K_1} +$

$+ \langle BU_1^n h_1 + U_2^n h_2 , h'_2 \rangle_{K_2} = \langle T_1^n h_1 , h'_1 \rangle_{H_1} + \langle U_2^n h_2 , Bh'_1 \rangle_{K_2} +$

$+ \langle AT_1^n h_1 + T_2^n h_2 , h'_2 \rangle_{H_2} = \langle T_1^n h_1 + T_2^n h_2 , h'_1 \rangle_H + \langle T_1^n h_1 + T_2^n h_2 , h'_2 \rangle_H +$

$+ \langle (U_2^n - T_2^n) h_2 , Bh'_1 \rangle_{K_2} = \langle T_1^n h_1 + T_2^n h_2 , h'_1 + h'_2 \rangle_H +$

$+ \langle (U_2^n - T_2^n) h_2 , Bh'_1 \rangle_{K_2}$

Thus :

$(1.1)_n$ $\langle F_n(h_1 + h_2), h'_1 + h'_2 \rangle_H = \langle T_1^n h_1 + T_2^n h_2 , h'_1 + h'_2 \rangle_H +$

$+ \langle (U_2^n - T_2^n) h_2 , Bh'_1 \rangle_{K_2}$, which proves that F_n is uniquely defined by B and if $h_2 = 0$, then :

$\langle F_n h_1 , h'_1 + h'_2 \rangle_H = \langle T_1^n h_1 , h'_1 + h'_2 \rangle_H$, hence $F_n|_{H_1} = T_1^n$, $n \geq 0$.

Similarly, it can be proved that $F^{*}_{n|H_2} = T_2^{*n} \quad n \geq 0$.

Conversely, let $\{F_n\}_{n\in Z}$ be a solution of Problem B.

Since :

$$\langle AT_1h_1 , h_2\rangle_{H_2} = \langle T_1h_1 , h_2\rangle_H = \langle F_1h_1 , h_2\rangle_H = \langle h_1 , F_1^{*}h_1\rangle_H =$$

$$= \langle h_1 , T_2^{*}h_2\rangle_{H_2} = \langle Ah_1 , T_2^{*}h_2\rangle_{H_2} = \langle T_2Ah_1 , h_2\rangle_{H_2} ,$$ for every

$h_1 \in H_1$ and $h_2 \in H_2$, it follows that $A T_1 = T_2 A$.

Let $U \in \mathcal{L}(K)$ be the minimal isometric dilation of the $\mathcal{L}(H)$ - valued function $\{F_n\}_{n\in Z}$ of positive type and $K_i = \bigvee_{n\geq 0} U^n H_i \quad i=1,2.$

It is obvious that $K_i \quad i=1,2$ are invariant subspaces for U, hence $U_i = U_{|K_i} \quad i=1,2$ are isometries and $K_i = \bigvee_{n\geq 0} U_i^n K_i \quad i=1,2$.

Since :

$$\langle U_i^n h_i , h_i'\rangle_{K_i} = \langle U^n h_i , h_i'\rangle_K = \begin{cases} \langle F_n h_1 , h_1'\rangle_H & i=1 \\ \langle h_2 , F_n^{*}h_2'\rangle_H & i=2 \end{cases} =$$

$$= \begin{cases} \langle T_1^n h_1 , h_1'\rangle_{H_1} & i=1 \\ \langle h_2 , T_2^{*}h_2'\rangle_{H_2} & i=2 \end{cases} = \langle T_i^n h_i , h_i'\rangle_{H_i} \quad h_i , h_i' \in H_i \quad i=1,2 ,$$

$U_i \in \mathcal{L}(K_i)$ are the minimal isometric dilations of $T_i \quad i=1,2$.

Define $B : K_1 \longrightarrow K_2$ by :

$\langle Bk_1 , k_2\rangle = \langle k_1 , k_2\rangle_K \quad k_1 \in K_1 , \quad k_2 \in K_2$.

We shall prove that $B \in CID(A)$.

Let us prove first that $P^{K_2}_{H_2} B = A P^{K_1}_{H_1}$.

If $h_1 \in H_1$, $h_2 \in H_2$, $n \geq 0$, then :

$$\langle BU_1^n h_1 , h_2\rangle_{K_2} = \langle U_1^n h_1 , h_2\rangle_K = \langle U^n h_1 , h_2\rangle_K = \langle F_n h_1, h_2\rangle_H =$$

$$= \langle T_1^n h_1 , h_2\rangle_H = \langle AT_1^n h_1 , h_2\rangle_{H_2} = \langle AP^{K_1}_{H_1} U_1^n h_1 , h_2\rangle_{K_2}$$ and since

$\bigvee_{n\geq 0} U_1^n H_1 = K_1$ it follows that $P^{K_2}_{H_2} B = AP^{K_1}_{H_1}$.

It remains to prove that $BU_1 = U_2 B$.

Let $h_1 \in H_1$, $h_2 \in H_2$, $n , m \geq 0$.

Then :

$$\langle BU_1U_1^nh_1 , U_2^mh_2\rangle_{K_2} = \langle U_1^{n+1}h_1 , U_2^mh_2\rangle_K = \langle U^{n+1}h_1 , U^mh_2\rangle_k .$$

If $m \geq 1$, then :

$$\langle BU_1U_1^nh_1 , U_2^mh_2\rangle_{K_2} = \langle U^nh_1 , U^{m-1}h_2\rangle_K = \langle U_1^nh_1 , U_2^{m-1}h_2\rangle_K =$$

$$= \langle U_2BU_1^nh_1 , U_2^mh_2\rangle_{K_2} ,$$

and if $m=o$, then :

$$\langle BU_1U_1^nh_1 , h_2\rangle_{K_2} = \langle U^{n+1}h_1 , h_2\rangle_K = \langle F_{n+1}h_1 , h_2\rangle_H =$$

$$= \langle T_1^{n+1}h_1 , h_2\rangle_H = \langle AT_1^{n+1}h_1 , h_2\rangle_{H_2} = \langle T_2AT_1^n h_1 , h_2\rangle_{H_2} =$$

$$= \langle AT_1^n h_1 , T_2^*h_2\rangle_{H_2} = \langle BU_1^nh_1 , T_2^*h_2\rangle_{K_2} = \langle BU_1^nh_1 , U_2^*h_2\rangle_{K_2} =$$

$$= \langle U_2BU_1^nh_1 , h_2\rangle_{K_2}$$

where we used $P_{H_2}^{K_2} B = A P_{H_1}^{K_1}$ and $U_{2|H_2}^* = T_2^*$.

In the sequel we shall suppose that $A T_1 = T_2 A$.

For the parametrization of the set of solutions of Problem A, let us recall some notations from [3].

Choose as minimal isometric dilations of the contractions T_i , $i=1,2$ (see for exemple [7] Ch.I) :

$$K_i = H_i \oplus \mathcal{D}_{T_i} \oplus \dots$$

$$\text{and } U_i = \begin{pmatrix} T_i & o & o & \cdot & \cdot & \cdot \\ D_{T_i} & o & o & \cdot & \cdot & \cdot \\ o & 1_{\mathcal{D}_{T_i}} & o & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \end{pmatrix} \qquad i=1,2$$

Let :

$$\mathcal{F}_A = \overline{\{D_AT_1h_1 \oplus D_{T_1}h_1 : h_1 \in H_1\}} , \quad \mathcal{R}_A = (\mathcal{D}_A \oplus \mathcal{D}_{T_1}) \ominus \mathcal{F}_A ,$$

$$\mathcal{F}^A = \overline{\{D_Ah_1 \oplus D_{T_2}Ah_1 : h_1 \in H_1\}} , \quad \mathcal{R}^A = (\mathcal{D}_A \oplus \mathcal{D}_{T_2}) \ominus \mathcal{F}^A ,$$

$$p_A = P_{\mathcal{F}_A}^{\mathcal{D}_A \oplus \mathcal{D}_{T_1}} , \quad q^A = P_{\mathcal{D}_{T_2}}^{\mathcal{D}_A \oplus \mathcal{D}_{T_2}} , \quad p^A = P_{\mathcal{F}^A}^{\mathcal{D}_A \oplus \mathcal{D}_{T_2}} .$$

Between $\mathcal{F}_A$ and $\mathcal{F}^A$ acts the natural unitary operator :

$\sigma_A : \mathcal{F}_A \longrightarrow \mathcal{F}^A$, $\sigma_A(D_A T_1 h_1 \oplus D_{T_1} h_1) = D_A h_1 \oplus D_{T_2} A h_1$

PROPOSITION 1.2. There exists a bijection between the set of solutions of Problem A and the set of contractions $\Gamma_1 : \mathcal{R}_A \longrightarrow \mathcal{R}^A$

Proof. We have a natural unitary operator F ,

$F : H \longrightarrow H_2 \oplus \mathcal{D}_A$, $F(h_1 + h_2) = (h_2 + Ah_1) \oplus D_A h_1$, $h_i \in H_i$ $i=1,2$

Let $T \in \mathcal{L}(H)$ be a solution of Problem A and $B \in CID(A)$ such that the $\mathcal{L}(H)$ - valued function $\{F_n\}_{n\in Z}$ corresponding to B by Theorem 1.1 satisfies : $F_1 = T$.

Define $T' = FTF^* \in \mathcal{L}(H_2 \oplus \mathcal{D}_A)$.

Since $T^*|_{H_2} = T_2^*$ and $Fh_2 = h_2 \oplus o$, $h_2 \in H_2$ then :

$$T' = \begin{pmatrix} T_2 & o \\ X_1^* & Y^* \end{pmatrix} \qquad (1.2)$$

where $X_1 \in \mathcal{L}(\mathcal{D}_A , H_2)$, $Y \in \mathcal{L}(\mathcal{D}_A)$ are contractions .

Using $(1.1)_1$ and $T|_{H_1} = T_1$ it follows :

$\langle Th_2 , h_1' + h_2' \rangle = \langle T_2 h_2 , h_1' + h_2' \rangle + \langle (U_2 - T_2)h_2 , Bh_1' \rangle$ for every $h_1' \in H_1$, h_2 , $h_2' \in H_2$, then :

$$\left\langle T'\begin{pmatrix} h_2 \\ o \end{pmatrix}, \begin{pmatrix} h_2' + Ah_1' \\ D_A h_1' \end{pmatrix}\right\rangle = \left\langle \begin{pmatrix} T_2 h_2 \\ o \end{pmatrix}, \begin{pmatrix} h_2' + Ah_1' \\ D_A h_1' \end{pmatrix}\right\rangle +$$

$+ \langle (U_2 - T_2)h_2 , Bh_1' \rangle$, and using (1.2) :

$\langle X_1^* h_2 , D_A h_1 \rangle = \langle (U_2 - T_2)h_2 , Bh_1' \rangle$.

From [3] Lemma 2.1 :

$\langle (U_2 - T_2)h_2 , Bh_1' \rangle = \langle D_{T_2} h_2 , q^A(\sigma_A p_A + \Gamma_1(1 - p_A))D_A h_1 \rangle$,

$\Gamma_1 : \mathcal{R}_A \longrightarrow \mathcal{R}^A$ being a contraction .

Define $X = q^A(\sigma_A p_A + \Gamma_1(1 - p_A))|\mathcal{D}_A$. Then :

$\langle X_1^* h_2 , D_A h_1 \rangle = \langle X^* D_{T_2} h_2 , D_A h_1 \rangle$ for every $h_1 \in H_1$, $h_2 \in H_2$,

which implies : $X_1 = D_{T_2} X$.

Similarly , using $(1.1)_1$ and $T^*|_{H_2} = T_2^*$ it follows that :

$Y = (1 - q^A)(\sigma_A p_A + \Gamma_1(1 - p_A))|\mathcal{D}_A$.

Conversely , it is obvious that if $\Gamma_1 : \mathcal{R}_A \longrightarrow \mathcal{R}^A$ is a contraction and T' is defined by (1.2) , with X_1,Y as above , then $T = F^*T'F$ is a solution of Problem A .

REMARK 1.3. Problem A may be seen as a generalization of Theorem 1 , in [6] , presently H_1 and H_2 being no more orthogonal in H .

In [5] it is shown that for fixed contractions $T_i \in \mathcal{L}(H_i)$ i=1,2 acting on the complex Hilbert spaces H_i i=1,2 and $A \in \mathcal{L}(H_1,H_2)$ a contraction which intertwines T_1 and T_2 (i.e. $AT_1 = T_2A$) , there exists a bijection between CID(A) and the set of all A-choice sequences $\{\Gamma_n\}_{n \geq 1}$. (We recall that an A-choice sequence is a sequence of contractions, $\{\Gamma_n\}_{n \geq 1}$ such that : $\Gamma_1 : \mathcal{R}_A \longrightarrow \mathcal{R}^A$ and $\Gamma_n : \mathcal{D}_{\Gamma_{n-1}} \longrightarrow \mathcal{D}_{\Gamma^*_{n-1}}$, $n \geq 2$) .

Using the algorithm in [4] Theorem 4.1.,and Proposition 1.2., one can obtain :

COROLLARY 1.4. With the notations from Theorem 1.1., let $B \in$ CID(A) be given by the A-choice sequence $\{\Gamma_n\}_{n \geq 1}$. Then the corresponding $\mathcal{L}(H)$ - valued function of positive type $\{F_n\}_{n \in Z}$ is a contraction (i.e. $F_n = F_1^n$ $n \geq 0$) if and only if $\Gamma_n = 0$, $n \geq 2$.

2.

In this section the existence of isometric intertwining dilations of Hankel operators which are strict contractions will be studied.

Let us begin with a general remark :

REMARK 2.1. Let $T_i \in \mathcal{L}(H_i)$ i =1,2 be contractions, $U_i \in \mathcal{L}(K_i)$ their minimal isometric dilations , $A \in \mathcal{L}(H_1,H_2)$ a contraction which intertwines T_1 and T_2 and let $B \in$ CID(A) .

Let K be the renormed Hilbert space of $K_1 \oplus K_2$ by B .

Then B is an isometry if and only if $K = K_2$.

Indeed , since $B = P^K_{K_2} | K_1$,it is obvious that B is an isometry if and only if $P^K_{K_2} k_1 = k_1$, $k_1 \in K_1$, therefore $K_1 \subseteq K_2$.

The main result of this section is the following :

THEOREM 2.2. Let $T_1 \in \mathcal{L}(H_1)$ be the unilateral shift of dimension n_1 ($n_1 = \dim \mathcal{D}_{T_1^*}$), $T_2^* \in \mathcal{L}(H_2)$ the unilateral shift of dimension n_2 ($n_2 = \dim \mathcal{D}_{T_2}$) and suppose that $n_1 \leq n_2$.

If $A \in \mathcal{L}(H_1, H_2)$ is a Hankel operator (i.e. $AT_1 = T_2A$) such that $\|A\| < 1$, then there exist isometric intertwining dilations of A.

For the proof of this Theorem we shall need some preliminary results.

Let us first note that in this case $D_{T_1^*}$ is the orthogonal projection of H_1 onto $\ker T_1^*$ and D_{T_2} is the orthogonal projection of H_2 onto $\ker T_2$.

Since $\mathcal{F}_A = D_A T_1 H_1$, then $h_1 \in \mathcal{R}_A (= H_1 \ominus \mathcal{F}_A)$ if and only if $T_1^* D_A h_1 = o$ hence iff $h_1 \in D_A^{-1} \ker T_1^* = D_A^{-1} \mathcal{D}_{T_1^*}$. Thus:

$$\mathcal{R}_A = D_A^{-1} \mathcal{D}_{T_1^*} \quad \text{and} \quad \dim \mathcal{R}_A = n_1$$

Now: $\mathcal{F}^A = \{ D_A h_1 \oplus D_{T_2} A h_1 : h_1 \in H_1 \}$, so $h_1 \oplus d_2 \in \mathcal{R}^A$ if and only if $D_A h_1 + A^* d_2 = o$, hence $h_1 = -D_A^{-1} A^* d_2 = -A^* D_{A^*}^{-1} d_2$. Thus:

$$\mathcal{R}^A = \{ -A^* D_{A^*}^{-1} d_2 \oplus d_2 : d_2 \in \mathcal{D}_{T_2} \} \quad \text{and} \quad \dim \mathcal{R}^A = n_2 \tag{2.1}$$

LEMMA 2.3. Let $\Gamma_1 : \mathcal{R}_A \longrightarrow \mathcal{R}^A$ be an isometry and $T' \in \mathcal{L}(H_2 \oplus H_1)$ defined by:

$$T' = \begin{pmatrix} T_2 & o \\ X^* D_{T_2} & Y^* \end{pmatrix}$$

where $X = q^A(\sigma_A p_A + \Gamma_1(1 - p_A))$, $Y = (1 - q^A)(\sigma_A p_A + \Gamma_1(1 - p_A))$, q^A, p_A, σ_A being as in Section 1.

Then:

i) T' is a $*$ - isometry

ii) $\mathcal{D}_{T'} = \{ q^A d_* \oplus (1 - q^A) d_* : d_* \in \mathcal{D}_{\Gamma_1^*} \}$

Proof i) Let h_i H_i $i=1,2$. Since:

$$T'^* = \begin{pmatrix} T_2^* & X \\ o & Y \end{pmatrix}$$

then :

$$\left\| T'^* \begin{pmatrix} h_2 \\ h_1 \end{pmatrix} \right\|^2 = \|T_2^* h_2 + Xh_1\|^2 + \|Yh_1\|^2 = \|T_2^* h_2\|^2 + \|Xh_1\|^2 +$$

$$+ \|Yh_1\|^2 = \|h_2\|^2 + \|Xh_1 + Yh_1\|^2 = \|h_2\|^2 + \|(\sigma_A p_A + \Gamma_1(1-p_A))h_1\|^2 =$$

$$= \|h_2\|^2 + \|\sigma_A p_A h_1\|^2 + \|\Gamma_1(1-p_A)h_1\|^2 = \|h_2\|^2 + \|p_A h_1\|^2 +$$

$$+ \|(1-p_A)h_1\|^2 = \|h_1\|^2 + \|h_2\|^2$$

ii) Let $h_2 \oplus h_1 \in \mathcal{D}_{T'}$. Since T'^* is an isometry , $T' \begin{pmatrix} h_2 \\ h_1 \end{pmatrix} = o$,

hence $T_2 h_2 = o$ and $X^* D_{T_2} h_2 + Y^* h_1 = o$. Then : $D_{T_2} h_2 = h_2$ and

$$X^* D_{T_2} h_2 + Y^* h_1 = X^* h_2 + Y^* h_1 = (\sigma_A^* p^A + \Gamma_1^*(1-p^A))(h_1 \oplus h_2) = o$$

From the above relation it follows that $\sigma_A^* p^A(h_1 \oplus h_2) = o$ and $\Gamma_1^*(1-p^A)(h_1 \oplus h_2) = o$, therefore , σ_A being unitary, $p^A(h_1 \oplus h_2) = o$

Hence $h_1 \oplus h_2 \in \mathcal{R}^A$ and $\Gamma_1^*(h_1 \oplus h_2) = o$.

Since Γ_1 is an isometry , $d_* = h_1 \oplus h_2 \in \mathcal{D}_{\Gamma_1^*}$ and $q^A d_* = h_2$, $(1-q^A)d_* = h_1$.

Conversely , let $d_* \in \mathcal{D}_{\Gamma_1^*}$. Then :

$$T' \begin{pmatrix} q^A d_* \\ (1-q^A)d_* \end{pmatrix} = \begin{pmatrix} T_2 q^A d_* \\ X^* D_{T_2} q^A d_* + Y^*(1-q^A)d_* \end{pmatrix}$$

Since $q^A d_* \in \mathcal{D}_{T_2}$ ($= \ker T_2$), $T_2 q^A d_* = o$ and $D_{T_2} q^A d_* = q^A d_*$. Hence

$$X^* D_{T_2} q^A d_* + Y^*(1-q^A)d_* = X^* q^A d_* + Y^*(1-q^A)d_* = (1-q_A)(\sigma_A^* p^A +$$

$+ \Gamma_1^*(1-p^A))d_* = (1-q_A)\Gamma_1^* d_*$ and taking into account that Γ_1 is an isometry , $\Gamma_1^* d_* = o$.

Hence $q^A d_* \oplus (1-q^A)d_* \in \ker T' = \mathcal{D}_{T'}$.

<u>Proof of Theorem 2.2.</u>

Let $\Gamma_1 : \mathcal{R}_A \longrightarrow \mathcal{R}^A$ be an isometry and $T' \in \mathcal{L}(H_2 \oplus H_1)$ as in Lemma 2.3.

Let H be the renormed Hilbert space of $H_1 \oplus H_2$ by A , $T = F^* T' F$, where F is the natural unitary operator-acting between H and $H_2 \oplus H_1$

$$F : H \longrightarrow H_2 \oplus H_1 \ , \ F(h_1 + h_2) = (h_2 + Ah_1) \oplus D_A h_1 \ , \ h_i \in H_i \ , \ i=1,2$$

By Proposition 1.2., T is a solution of Problem A, hence the $\mathcal{L}(H)$ valued function $\{F_n\}_{n \in Z}$ defined by : $F_n = T^n$, $n \geq 0$, $F_n = T^{*|n|}$, $n \leq 0$ is a solution of Problem B .

If $U \in \mathcal{L}(K)$ is the minimal isometric dilation of T, then by Theorem 1.1., $B = P^K_{K_2}|K_1$ is an intertwining dilation of A. (K_1 and K_2 are as in the proof of Theorem 1.1.).

We shall prove that in this case B is an isometry.

Taking into account Remark 2.1., we have to prove that :

$$K = K_2 = \bigvee_{n \geq 0} U^n H_2$$

Choose as minimal isometric dilation of T' :

$$K' = (H_2 \oplus H_1) \oplus \mathcal{D}_{T'} \oplus \mathcal{D}_{T'} \oplus \dots$$

$$(2.2) \qquad U' = \begin{pmatrix} T' & 0 & 0 & 0 & 0 & 0 \\ D_{T'} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_{\mathcal{D}_{T'}} & 0 & 0 & 0 & 0 \\ \cdot & \cdot & 0 & & & \\ \cdot & \cdot & 0 & & & \end{pmatrix}$$

Since T and T' are unitarily equivalent and $Fh_2 = h_2 \oplus 0$, $h_2 \in H_2$, we have to prove that :

$$K' = \bigvee_{n \geq 0} U'^n (H_2 \oplus \{0\} \oplus \dots)$$

Let $k' = h \oplus d_1 \oplus d_2 \oplus \dots$, such that $k' \perp \bigvee_{n \geq 0} U'^n (H_2 \oplus \{0\} \oplus \dots)$ where $h = h_2 \oplus h_1 \in H_2 \oplus H_1$.

Since $k' \perp H_2 \oplus \{0\} \oplus \{0\} \oplus \dots$, it follows that $h_2 = 0$.

By Lemma 2.3. ii), there exists $d_{*1} \in \mathcal{D}_{\Gamma^*_1}$ such that :

$d_1 = q^A d_{*1} \oplus (1-q^A) d_{*1}$. Using (2.2) one obtains :

$$U'^* k' = \left(T'^* \begin{pmatrix} 0 \\ h_1 \end{pmatrix} + d_1\right) \oplus d_2 \oplus \dots = ((Xh_1 + q^A d_{*1}) \oplus (Yh_1 + (1-q^A) d_{*1})) \oplus d_2 \oplus \dots ,$$

$h_1 = o$, $d_k = o$, $k \geqslant 1$, therefore $k' = o$, which completes the proof.

REMARK 2.4. With the notations of Theorem 2.2., from Corollary 1.4, it follows that if $B \in CID(A)$ is given by an A-choice sequence $\{\Gamma_n\}_{n \in Z}$ such that Γ_1 is an isometry (therefore $\Gamma_n = o$, $n \geqslant 2$), then B is an isometry.

Acknowledgement. The author expresses his gratitude to Professor Ion Suciu for many helpful and stimulating discutions.

REFERENCES

1. V.M.ADAMIAN , D.Z.AROV , M.G.KREĬN , Infinite Hankel matrices and generalized problems of Carathéodory-Fejer and I.Schur (Russian) , Funck.Anal. i Prilozen., 2:4(1968),1-17 .

2. V.M.ADAMIAN , D.Z.AROV , M.G.KREĬN, Infinite Hankel blockmatrices and related continuation problems, (Russian) Izv. Akad.Nauk Armijan.SSR , Matem., 6(1971), 87 - 112 .

3. GR.ARSENE , Z.CEAUŞESCU , C.FOIAŞ , On intertwining dilations VII Proc.Coll.Complex Analysis , Joensuu , Lecture Notes in Math.,Springer , 747(1979) , 24 - 45 .

4. GR.ARSENE , Z.CEAUŞESCU , C.FOIAŞ , On intertwining dilation VIII J.Operator Theory , 4(198o) , 55 - 91 .

5. Z.CEAUŞESCU , C.FOIAŞ , On intertwining dilation VI.Acta Sci.Math. Szeged , 4o(1978) , 9 - 32 .

6. B.SZ.-NAGY , C.FOIAŞ , Forme triangulaire d'une contraction et factorisation de la fonction caractéristique,Acta Sci. Math., 28(1967) , 2o1 - 212 .

7. B.SZ.-NAGY , C.FOIAŞ , Harmonic Analysis of Operators on Hilbert space , Amsterdam-Budapest.,197o .