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(new version)

Mihai PUTINAR

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(new version)

by

Mihai PUTINAR*)

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*) The National Institute for Scientific and Technical Creation, Department of Mathematics, Bd.Pacii 220, 79622 Bucharest, Romania

BASE CHANGE AND THE FREDHOLM INDEX

Mihai Putinar

Introduction

The present paper deals with a relationship between J.L.Taylor's functional calculus and the multidimensional Fredholm index, both refering to commutative n-tuples of operators. We develop an enlarged frame, with the purpose of making the Fredholm objects more flexible under analytic transformations. In particular two base change results, Theorem 3.1 and Theorem 4.2, are obtained in this paper.

Let us recall for the beginning some terminology and facts from multidimensional spectral theory. Let H be a complex Hilbert space and let $T=(T_1,\dots,T_n)$ be a commutative n-tuple of linear bounded operators on H. J.L.Taylor defined in [19] the joint spectrum Sp(T,H) of the n-tuple T on H as the subset of those points $\lambda\in\mathbb{C}^n$, with the property that Koszul's complex $K.(T-\lambda,H)$ is not exact. In other terms, $\lambda\notin Sp(T,H)$ iff the condition

(1) $$H_q(T-\lambda,H)=0\ ,\ q\geq 0,$$

is fulfiled, where we have denoted by $H.(T-\lambda,H)$ the homology spaces of the complex $K.(T-\lambda,H)$. This joint spectrum has many of the expected properties, when compared with the spectrum of a single linear operator, among which we recall only Taylor's theorem [20, Theorem 4.3] about the existence of a continuous functional calculus $f\longmapsto f(T)$, with analytic functions f defined in neighbourhoods of Sp(T,H).

Motivated by Taylor's ideas, several authors developed (in convergent directions) the Fredholm theory for commutative n-tuples of operators. Thus it is unanimously accepted that the point $\lambda\in\mathbb{C}^n$ doesn't belong to the <u>essential joint spectrum</u> $Sp_e(T,H)$ if the condition

(2) $$h_q=\dim H_q(T-\lambda,H)<\infty\quad ,\ q\geq 0,$$

holds true. Then the Fredholm index of T is the integer

$$\operatorname{ind}(T) = \sum_{q=0}^{n} (-1)^q h_q .$$

These definitions extend the corresponding notions for a single linear operator and the expected stability results hold (see [22, Chap. III. § 7]). However, in the multidimensional case new phenomena appear (see for instance [6]) and tha aim of this paper is to investigate one of them.

More precisely, the computation of the index, or of some finer invariants as the numbers h_q, of the m-tuple $f(T)$, where $f: U \longrightarrow \mathbb{C}^m$ is an analytic map defined on an open neighbourhood U of Sp(T,H), is the motivation of this paper. It turns out that, instead of working on $\mathbb{C}^m$ with the above usual definitions it is more convenient to lift the objects to U, to investigate there a Tor-intersection condition and to compute the corresponding intersection number between the analytic fibre $f^{-1}(0)$ and the n-tuple T. Let us explain this procedure in more detail.

The Fredholm condition (2) can be interpreted as follows. Let $\mathcal{L}(H)$ denote the algebra of bounded linear operators on H, with its two sided closed ideal $\mathcal{K}(H)$ of compact operators. Then condition (2) is equivalent by [5, §3] with

(2)' $\qquad H_q(T-\lambda, \mathcal{L}(H)/\mathcal{K}(H)) = 0 , \quad q \geq 0,$

where the operators $T_i - \lambda_i$ act by left multiplication on the Calkin algebra $\mathcal{L}(H)/\mathcal{K}(H)$.

For another interpretation consider $\mathcal{O}(\mathbb{C}^n)$ the algebra of holomorphic functions on $\mathbb{C}^n$ and let m_λ be its maximal ideal associated to λ :

$$m_\lambda = \left\{ f \in \mathcal{O}(\mathbb{C}^n) \mid f(\lambda)=0 \right\} .$$

The quotient $\mathcal{O}(\mathbb{C}^n)$-module $\mathcal{O}(\mathbb{C}^n)/m_\lambda$ admits a canonical free resolution given by the Koszul complex associated to the system $z-\lambda$, where $z=(z_1,\dots,z_n)$ stands for the coordinate n-tuple in $\mathbb{C}^n$:

$$K.(z-\lambda, \mathcal{O}(\mathbb{C}^n)) \longrightarrow \mathcal{O}(\mathbb{C}^n)/m_\lambda \longrightarrow 0.$$

Note that the quotient algebra $\mathcal{O}(\mathbb{C}^n)/m_\lambda$ is canonically isomorphic with $\mathbb{C}$. Then regarding the n-tuple T as the topological representation

$$\rho : \mathcal{O}(\mathbb{C}^n) \longrightarrow \mathcal{L}(H)/\mathcal{K}(H)$$

which associates T_i to z_i, the condition (2)' can be reformulated independently of the coordinates and of the Koszul resolution, as follows:

$$(2)'' \qquad \mathrm{Tor}_q^{\mathcal{O}(\mathbb{C}^n)}(\mathcal{O}(\mathbb{C}^n)/m_\lambda, \mathcal{L}(H)/\mathcal{K}(H)) = 0, \quad q \geq 0.$$

This equivalent definition for the Fredholm property of the n-tuple $T-\lambda$ can be easily extended to more general objects, a fact which will turn out to be extremely useful.

Thus, let $\mathcal{F}$ be an analytic coherent sheaf on $\mathbb{C}^n$, which admits a finite globally free resolution with finite type $\mathcal{O}$-modules, like the sheaf $\mathcal{O}/m_\lambda$ above. For explanations concerning such sheaves, see §1. Throughout this paper $\mathcal{O}$ denotes the sheaf of holomorphic functions on $\mathbb{C}^n$. Then the vanishing condition

$$(3) \qquad \mathrm{Tor}_q^{\mathcal{O}(\mathbb{C}^n)}(\mathcal{F}(\mathbb{C}^n), \mathcal{L}(H)/\mathcal{K}(H)) = 0, \quad q \geq 0,$$

will be interpreted as a generalized Fredholm "incidence" relation between the sheaf $\mathcal{F}$ and the n-tuple T. A natural K-theoretic difference construction will provide under the assumption (3) an integer

$$\mathrm{ind}^{\mathcal{O}(\mathbb{C}^n)}(\mathcal{F}, \mathcal{L}(H)/\mathcal{K}(H)) \in \mathbb{Z},$$

which generalizes the Fredholm index.

Let us illustrate these notions with two examples.

<u>Example</u> 1. Consider the n-tuple T and an integer k, $1 \leq k \leq n$. Let us denote by $\mathcal{J}$ the ideal of the sheaf $\mathcal{O}$ generated by $z_1, \ldots, z_k$. Then the sheaf $\mathcal{F} = \mathcal{O}/\mathcal{J}$, which is supported on the complex (n-k)-dimensional linear subspace $\{z_1 = z_2 = \ldots = z_k = 0\}$ of $\mathbb{C}^n$, has a finite free resolution given by the partial Koszul's complex $K.(z_1, \ldots, z_k; \mathcal{O})$. In this case condition (3) is equivalent with

the fredholmness of the k-tuple $(T_1,\dots,T_k)$ and

$$\operatorname{ind}^{\mathcal{O}(\mathbb{C}^n)}(\mathcal{O}/(z_1,\dots,z_k),\mathcal{L}(H)/\mathcal{K}(H)) = \operatorname{ind}(T_1,\dots,T_k).$$

Remark that the n-tuple T is Fredholm if (3) holds,but the n-dimensional index ind(T)vanishes when $k<n$.

Example 2. Let f be a non-constant analytic function on $\mathbb{C}^n$ and let $\mathcal{F}$ be the quotient sheaf $\mathcal{O}/(f)$.It has a finite free resolution,$0\to\mathcal{O}\xrightarrow{f}\mathcal{O}\to\mathcal{F}\to 0$, and it is supported on the hypersurface $\{f=0\}$.Then (3) means exactly that the operator f(T) is Fredholm,and in this case

$$\operatorname{ind}^{\mathcal{O}(\mathbb{C}^n)}(\mathcal{O}/(f),\mathcal{L}(H)/\mathcal{K}(H)) = \operatorname{ind} f(T).$$

Other examples will be presented in §3 and §5.

The present paper is an amplified version of an earlier paper,with the same title circulated as INCREST preprint (April,1981).Meanwhile two recent complementary papers dealing with Fredholm theory for n-tuples have appeared.R.Carey and J.Pincus have announced in [4] an interesting analytical method of checking, in our terminology,condition (3) and of computing the corresponding index for some special classes of n-tuples of operators and certain coherent sheaves with support of complex codimension 1.Secondly,R.Levi announced in [14] a K-theoretic framework for multioperatorial Fredholm theory.It is possible that this framework may be used for another approach to Theorem 3.1 of the present paper. Anyway the analytic computations in our proof of Theorem 3.1 would be still necessary.

The present paper has five sections.

The first section contains some preliminaries on analytic coherent sheaves on Stein spaces.

In the second section we define the notion of a Fredholm sheaf with respect to a Stein algebra representation into a Banach algebra with a distinguished two-sided closed ideal.Some stability results for Fredholm sheaves and for the associated index under geometrical and algebraic operations are then presented.

In the third section the notions and the results of §2 are applied to the case of commutative n-tuples of operators on Banach spaces. The main result is Theorem 3.1.

The fourth section is expository and contains another base change result, Theorem 4.2, which refers finally also to commutative n-tuples of operators.

In the fifth section a non-operatorial example is presented, which illustrates rather intuitively the general construction of the index.

§1. PRELIMINARIES

A class of analytic sheaves, which is stable under some algebraic and geometric transformations, is described. The reader is refered to the recent book [13] for an excellent introduction to analytic geometry.

Let X be a Stein space with structure sheaf $\mathcal{O}_X$. We denote by FR(X) the class of analytic coherent sheaves $\mathcal{F}$ on X, which admit a finite globally free resolution with finite type $\mathcal{O}_X$-modules:

$$(4) \qquad 0 \longrightarrow \mathcal{L}_n \longrightarrow \mathcal{L}_{n-1} \longrightarrow \dots \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{L}_i = \mathcal{O}_X^{m_i}$ for some non-negative integers m_i, $0 \leq i \leq n$.

In view of the equivalence between Stein modules and coherent sheaves on X established in [11], an element $\mathcal{F} \in FR(X)$ is completely determined by the Fréchet $\mathcal{O}(X)$-module $\mathcal{F}(X)$. Similarly, the exact sequence (4) can be derived from the free $\mathcal{O}(X)$-resolution

$$(4)' \qquad 0 \longrightarrow L_n \longrightarrow L_{n-1} \longrightarrow \dots \longrightarrow L_0 \longrightarrow \mathcal{F}(X) \longrightarrow 0,$$

where $L_i = \mathcal{O}(X)^{m_i}$. Thus we shall use, depending on the context, one of these two equivalent descriptions of the same object.

For the sake of completeness we sketch the proof of the following.

LEMMA 1.1 Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be a short exact sequence of analytic coherent sheaves on a Stein space X. If two of the sheaves $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ belong to FR(X), then the third also belongs to FR(X).

Proof. Let M_i denote the finite type $\mathcal{O}(X)$-module $\mathcal{F}_i(X)$. If two of the modules M_1, M_2, M_3, which are related by a short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

belong to FR(X), then the third admits by [3, Proposition III.6.3] a finite resolution with finite type projective modules.

Because each projective module is a direct summand of a free module, the projective resolution can be modified up to a special form:

$$(5) \qquad 0 \to P_n \to L_{n-1} \to \dots \to L_0 \to 0,$$

where P_n is projective and L_j, $0 \leq j \leq n-1$, are finite type free $\mathcal{O}(X)$-modules. Then by comparing the three resolutions (of M_1, M_2 and M_3) through the short exact sequence, one gets that P_n is stably free. Therefore the complex (5) is equivalent to a free resolution.

We describe now some examples of elements in FR(X).

LEMMA 1.2, [11, Proposition 4.6] Let X be a Stein manifold and let $x \in X$. Then $\mathcal{O}_X/m_x \in FR(X)$.

We have already remarked in the introduction that this lemma is valid when $X = \mathbb{C}^n$, because of the existence of a canonical resolution of $\mathcal{O}_X/m_x$. We mention also the following example. Let Z be a locally complete intersection in a contractible Stein manifold. Then the structure sheaf $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{J}(Z)$ belongs to FR(X).

LEMMA 1.3 Let $f: X \to Y$ be a flat morphism of Stein spaces and let $\mathcal{F}$ be a coherent sheaf on Y. Then $f^*\mathcal{F} \in FR(X)$ whenever $\mathcal{F} \in FR(Y)$.

Recall that the morphism $f:X \longrightarrow Y$ is said to be flat if the $\mathcal{O}_{Y,f(x)}$-module $\mathcal{O}_{X,x}$ is flat for any point $x \in X$.

The proof of Lemma 1.3 is immeuiate because a flat morphism preserves by pull-back the exactness of a resolution.

LEMMA 1.4 <u>If $\mathcal{F}$ and $\mathcal{G}$ belong to $FR(X)$, then $\mathcal{F} \otimes \mathcal{G}$ has the same property, provided that $\mathcal{F}$ is flat over $\mathcal{G}$</u>.

<u>Proof.</u> Let $\mathcal{L}^{\prime}_{\cdot}$ and $\mathcal{L}^{\prime\prime}_{\cdot}$ be finite free resolution of $\mathcal{F}$, respectively $\mathcal{G}$. Then the simple complex associated to the bicomplex $\mathcal{L}^{\prime}_{\cdot} \otimes \mathcal{L}^{\prime\prime}_{\cdot}$ is a finite free resolution for the sheaf $\mathcal{F} \otimes \mathcal{G}$.

§2. THE MAIN NOTIONS

In this section a generalized Fredholm theory for coherent sheaves of $FR(X)$ and relative to a Banach algebra representation of $\mathcal{O}(X)$, is described.

Let A be an unital, not necessarely commutative, Banach algebra and let I be a two-sided closed ideal of A. The basic facts concerning the K-theory of Banach algebras can be found in [20], where the commutativity is inessential. Thus the boundary operator

$$\partial : K_1(A/I) \longrightarrow K_0(I)$$

in the long exact sequence of K-theory for the pair (A,I) can be interpreted as a generalized index. We recall the classical situation $A = \mathcal{L}(H)$, $I = \mathcal{K}(H)$, where H is a Hilbert space. Then the group $K_0(\mathcal{K}(H))$ coincides with $\mathbb{Z}$ and ∂ is, after this identification, the Fredholm index map.

DEFINITION 2.1 Let X be a Stein space, let $\mathcal{F} \in FR(X)$ and let $\rho: \mathcal{O}(X) \longrightarrow A$ be an unital representation of the Stein algebra $\mathcal{O}(X)$ into a Banach algebra A with a distinguished two-sided closed ideal I.

The sheaf $\mathcal{F}$ is said to be <u>Fredholm relative to</u> ρ <u>and to</u> I, if

(6) $\quad \mathrm{Tor}_q^{\mathcal{O}(X)}(\mathcal{F}(X), A/I) = 0, \ q \quad 0.$

Formula (6) should be compared with relation (3) in the introduction.

Let us assume (6) holds and let $L.$ be a finite free resolution of $\mathcal{F}(X)$ with finite type $\mathcal{O}(X)$-modules. Then the complex of free (A/I)-modules $L. \otimes_{\mathcal{O}(X)} A/I$ is exact, hence homotopically trivial, that is, if the boundary operator d of $L.$ has degree -1, then there exists an operator ε of degree $+1$ on $L. \otimes_{\mathcal{O}(X)} A/I$, such that $(d\otimes 1)\varepsilon + \varepsilon(d\otimes 1) = \mathrm{id}$. The A/I-linear map

$$d\otimes 1_{A/I} + \varepsilon : L_e \otimes_{\mathcal{O}(X)} A/I \qquad\qquad L_o \otimes_{\mathcal{O}(X)} A/I$$

is invertible, where

$$L_e = \bigoplus_{p \geq 0} L_{2p} \qquad \text{and} \qquad L_o = \bigoplus_{p \geq 0} L_{2p+1} \ .$$

The free $\mathcal{O}(X)$-modules L_e and L_o have the same rank, and, after choosing bases in these two $\mathcal{O}(X)$-modules, the above map gives an element $[d\otimes 1_{A/I} + \varepsilon]$ in the group $K_1(A/I)$. We set

(7) $\quad \mathrm{ind}^{\mathcal{O}(X)}(\mathcal{F}, A/I) = \partial\,[d\otimes 1_{A/I} + \varepsilon] \ ,$

where $\partial : K_1(A/I) \longrightarrow K_0(I)$ is the generalized index.

PROPOSITION 2.2 <u>The definition of the index is independent of the choice of the resolution</u> $L.$, <u>the homotopy</u> ε <u>and the basis in</u> $L.$.

<u>Proof</u>. If ε' is another homotopy of the complex $L. \otimes_{\mathcal{O}(X)} A/I$ between the null morphism and the identity, then for each $t \in [0,1]$, $t\varepsilon + (1-t)\varepsilon'$ re - mains a good homotopy, thus the classes $[d\otimes 1_{A/I} + \varepsilon]$ and $[d\otimes 1_{A/I} + \varepsilon']$ coincide.

Changing the bases in the components of the complex $L.$ the element $[d\otimes 1_{A/I} + \varepsilon]$ becomes $[u\otimes 1_{A/I}][d\otimes 1_{A/I} + \varepsilon][v\otimes 1_{A/I}]$, where $u \in \mathrm{Aut}_{\mathcal{O}(X)}(L_e)$ and $v \in \mathrm{Aut}_{\mathcal{O}(X)}(L_o)$. If we denote by $\pi : A \longrightarrow A/I$ the natural projection, then $[u\otimes 1_{A/I}] = \pi_*[u\otimes 1_A]$, therefore $\partial\,[u\otimes 1_{A/I}] = 0$ and analogously $\partial\,[v\otimes 1_{A/I}] = 0$.

Then the additivity of ∂ implies the invariance of (7) with respect to the choice of bases.

Let $L'_.$ and $L''_.$ be two finite resolutions of $\mathcal{F}(X)$ with finite type free $\mathcal{O}(X)$-modules. There is a morphism of complexes of $\mathcal{O}(X)$-modules $h: L'_. \longrightarrow L''_.$ which is a quasi-isomorphism (i.e. it induces isomorphisms in homology). Let $L_.$ be the cone of h, that is the complex with components $L_p = L'_p \oplus L''_{p-1}$ and boundaries

$$d_p = \begin{pmatrix} d'_p & 0 \\ (-1)^p h_p & d''_{p-1} \end{pmatrix} .$$

The complex of free $\mathcal{O}(X)$-modules $L_.$ is exact by [3, Corollary I.6.6], hence it is homotopically trivial via a homotopy e and consequently

$$\mathrm{ind}^{\mathcal{O}(X)}(L_., A/I) := \partial\left[d \otimes 1_{A/I} + e \otimes 1_{A/I}\right] = \partial \pi_* \left[d \otimes 1_A + e \otimes 1_A\right] = 0.$$

Thus we have reduced the proof of Proposition 2.2 to the following lemma.

LEMMA 2.3 <u>Let</u> $0 \longrightarrow L''_. \longrightarrow L_. \longrightarrow L'_. \longrightarrow 0$ <u>be a short exact sequence of finite complexes of finite type free</u> $\mathcal{O}(X)$<u>-modules. If the three complexes are exact over</u> A/I, <u>then</u>

$$\mathrm{ind}^{\mathcal{O}(X)}(L''_., A/I) + \mathrm{ind}^{\mathcal{O}(X)}(L'_., A/I) = \mathrm{ind}^{\mathcal{O}(X)}(L_., A/I).$$

<u>Proof</u>. The boundary operator d of $L_.$ is of the form

$$d = \begin{pmatrix} d'' & h \\ 0 & d' \end{pmatrix},$$

where $(-1)^p h_p : L'_p \longrightarrow L''_p$ is a morphism of complexes.

Consider the complex of continuous functions $C([0,1], L_. \otimes_{\mathcal{O}(X)} A)$ with the boundary operator

$$d_t = \begin{pmatrix} d'' & th \\ 0 & d' \end{pmatrix} \otimes 1_A \quad , \ t \in [0,1] .$$

By [18, Theorem 2.1] the complex $C([0,1], L. \otimes_{\mathcal{O}(X)} A/I)$ is exact, and being a complex of free $C([0,1], A/I)$-modules, there is a homotopy e_t which depends continuously on t and which trivializes it. Then the elements $d_0 \otimes 1_{A/I} + e_0$ and $d_1 \otimes 1_{A/I} + e_1$ define the same class in $K_1(A/I)$, and

$$\text{ind}^{\mathcal{O}(X)}(L.,A/I) = \partial[d_1 \otimes 1_{A/I} + e_1] = \partial[d_0 \otimes 1_{A/I} + e_0] =$$

$$\text{ind}^{\mathcal{O}(X)}(L'_. \oplus L''_.,A/I) = \text{ind}^{\mathcal{O}(X)}(L'_.,A/I) + \text{ind}^{\mathcal{O}(X)}(L''_.,A/I) \ , \text{ q.e.d.}$$

COROLLARY 2.4 <u>Let</u> $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ <u>be a short exact sequence of</u> $\mathcal{O}_X$<u>-modules of</u> FR(X)<u>. If two of the sheaves</u> $\mathcal{F}'$, $\mathcal{F}$ <u>and</u> $\mathcal{F}''$ <u>are Fredholm relative to</u> ρ <u>and</u> I, <u>then the third has the same property and</u>

$$\text{ind}^{\mathcal{O}(X)}(\mathcal{F},A/I) = \text{ind}^{\mathcal{O}(X)}(\mathcal{F}',A/I) + \text{ind}^{\mathcal{O}(X)}(\mathcal{F}'',A/I) .$$

<u>Proof.</u> The first assertion follows from a long exact sequence of tor's, while the second assertion was proved in Lemma 2.3.

THEOREM 2.5 <u>Let</u> $f: X \to Y$ <u>be a flat morphism of Stein spaces and let</u> $\rho: \mathcal{O}(X) \to A$ <u>be an unital representation, where</u> A <u>is a Banach algebra with a distinguished two-sided closed ideal.</u>

<u>If</u> $\mathcal{F} \in FR(Y)$ <u>is Fredholm relative to</u> $\rho \circ f^*$ <u>and</u> I, <u>then</u> $f^*\mathcal{F}$ <u>is Fredholm relative to</u> ρ <u>and</u> I, <u>and then</u>

(8) $\quad \text{ind}^{\mathcal{O}(X)}(f^*\mathcal{F},A/I) = \text{ind}^{\mathcal{O}(Y)}(\mathcal{F},A/I).$

<u>Proof.</u> Let us assume $\mathcal{F} \in FR(Y)$ is Fredholm relative to $\rho \circ f^*$ and I, and let L. denote a finite free resolution of $\mathcal{F}(Y)$ by finite type free $\mathcal{O}(Y)$-modules. Then $f^*\mathcal{F} \in FR(X)$ by Lemma 1.3 and $L. \otimes_{\mathcal{O}(Y)} \mathcal{O}(X)$ is a resolution of $(f^*\mathcal{F})(X)$ by free $\mathcal{O}(X)$-modules. The sheaf $\mathcal{F}$ is Fredholm relative to ρ and I

iff the complex $L. \otimes_{\mathcal{O}(Y)} A/I = (L. \otimes_{\mathcal{O}(Y)} \mathcal{O}(X)) \otimes_{\mathcal{O}(X)} A/I$ is exact, hence iff $f^*\mathcal{F}$ is Fredholm.

The complexes which appear in the computation of the two sides of (8) are isomorphic, so that the proof is complete.

§3. AN APPLICATION TO COMMUTATIVE N-TUPLES OF OPERATORS

Let E be a complex Banach space and let $T=(T_1,\ldots,T_n)$ be a commutative n-tuple of linear bounded operators on E. First of all we explain the definitions of the previous section in the particular case of the representation

$$\rho: \mathcal{O}(\mathbb{C}^n) \longrightarrow \mathcal{L}(E) \quad , \quad \rho(\sum_{\alpha \geq 0} a_\alpha z^\alpha) = \sum_{\alpha \geq 0} a_\alpha T^\alpha,$$

and relative to the ideal $\mathcal{K}(E)$ of compact operators.

Let $\lambda \in \mathbb{C}^n$ be fixed. Then the sheaf $\mathcal{O}/m_\lambda$, which belongs to $FR(\mathbb{C}^n)$, is Fredholm relative to ρ and $\mathcal{K}(E)$, iff Koszul's complex $K.(T-\lambda, \mathcal{L}(E)/\mathcal{K}(E))$ is exact, because $K.(z-\lambda, \mathcal{O})$ provides a finite free resolution of $\mathcal{O}/m_\lambda$. This definition of a Fredholm n-tuple on a Banach space is similar with (2)'. However, it is not known as yet if condition (2) is equivalent with (2)' on an arbitrary Banach space.

As for the abstract index defined in the preceding section, we shall replace the morphism ∂ with the usual Fredholm index map ind: $K_1(\mathcal{L}(E)/\mathcal{K}(E)) \longrightarrow \mathbb{Z}$, in order to obtain numerical results. It is important to notice that this morphism of groups still factorizes through $\text{Coker}(K_1(\mathcal{L}(E)) \longrightarrow K_1(\mathcal{L}(E)/\mathcal{K}(E)))$.

In order to compute the index of $T-\lambda$, let ε be a homotopy which splits the exact complex $K.(T-\lambda, \mathcal{L}(E)/\mathcal{K}(E))$, that is $d\varepsilon + \varepsilon d = \text{id}$. It turns out that, denoting by e a lifting for ε with coefficients in $\mathcal{L}(E)$, the operator

$$d + e : \bigoplus_{p \geq 0} K_{2p}(T-\lambda, E) \longrightarrow \bigoplus_{p \geq 0} K_{2p+1}(T-\lambda, E)$$

is Fredholm, and consequently

$$\mathrm{ind}^{\mathcal{O}(\mathbb{C}^n)}(\mathcal{O}/m_\lambda, \mathcal{L}(E)/\mathcal{K}(E)) = \mathrm{ind}(d + e).$$

An induction argument, e.g. [15, Proposition 2.6], shows then that

$$\mathrm{ind}(d + e) = \sum_{p \geq 0} (-1)^p \dim H_p(T-\lambda, E).$$

This integer will be by definition the index of the n-tuple $T-\lambda$, and it will be denoted $\mathrm{ind}(T-\lambda)$.

The invariance theorems under small norm or compact perturbations of this multidimensional index are proved for instance in [15, § 2].

Adopting the weaker definition (2) for n-tuples on Banach spaces and defining the index by the same Euler characteristic all the stability results remain true, except for the invariance of the index under compact perturbations which is still an open question (see [10] and [21] for details).

The joint essential spectrum is by definition the set:

$$Sp_e(T, E) = \left\{ \lambda \in \mathbb{C}^n \;\middle|\; T-\lambda \text{ is not Fredholm} \right\}.$$

In view of the above interpretation of fredholmness, the joint essential spectrum coincides with Taylor's joint spectrum with coefficients in the quotient algebra $\mathcal{L}(E)/\mathcal{K}(E)$:

$$Sp_e(T, E) = Sp(T, \mathcal{L}(E)/\mathcal{K}(E)).$$

This equality implies, as in the Hilbert space case, some properties of the joint essential spectrum, like for instance the spectral mapping property.

The Fredholm index vanishes on the unbounded connected component of the open set $\mathbb{C}^n \setminus Sp_e(T, E)$, so that Fredholm domains of T, corresponding to a non-vanishing index, contain only 0-dimensional and n-dimensional analytic subsets.

Let U be a connected Stein open neighbourhood of $Sp(T, E)$ and let $f: U \longrightarrow \mathbb{C}^m$ be a m-tuple of non-constant analytic functions defined on U. Then Taylor's functional calculus produces a m-tuple of commuting operators $f(T)=(f_1(T),\dots,f_m(T))$, with joint spectra related to those of T by the spectral mapping relations:

$$Sp(f(T),E) = f(Sp(T,E)) \ , \ Sp_e(f(T),E) = f(Sp_e(T,E)).$$

We distinguish three cases:

1). $m<n$. Then $f(Sp_e(T,E)^c) = Sp(f(T),E)$, where K^c denotes the union of the compact K and of the bounded connected components of $\mathbb{C}^n \setminus K$. In particular $f(\lambda) \in Sp_e(f(T),E)$ whenever $\lambda \notin Sp_e(T,E)$ but $ind(T-\lambda)\neq 0$.

One might say that in this case the Fredholm behaviour of the m-tuple f(T) can be read on the analytic fibres of the map f. Indeed, let us assume in order to apply the general scheme of §2, that f is a flat map. If $\mu\in\mathbb{C}^m\setminus Sp_e(f(T),E)$, then by Theorem 2.5, $f^*(\mathcal{O}/m_\mu)\in FR(U)$ is Fredholm relative to T and

$$ind(f(T)-\mu) = ind^{\mathcal{O}(U)}(f^*(\mathcal{O}/m_\mu), \mathcal{L}(E)/\mathcal{K}(E)).$$

The analytic sheaf $f^*(\mathcal{O}/m_\mu)$ is concentrated on the analytic fibre $f^{-1}(\mu)$, which, as we already remarked, is disjoint of $Sp_e(T,E)^c$.

2). $m>n$. Then the set f(U) has not interior points, hence Sp(f(T),E) doesn't contain open sets. Therefore $ind(f(T)-\mu)=0$ for any $\mu\in\mathbb{C}^m\setminus Sp_e(f(T),E)$.

3). $m=n$. This is the case when numerical relations between the indices of T and f(T) are expected to hold. We prove furtheron such a formula in a particular but generic case.

THEOREM 3.1 <u>Let</u> T <u>be a commutative n-tuple of linear bounded operators on a Banach space</u> E <u>and let</u> $f:U\longrightarrow\mathbb{C}^n$ <u>be an analytic flat map with finite fibre at</u> $0\in\mathbb{C}^n$, <u>where</u> U <u>is a Stein open neighbourhood of</u> Sp(T,E).

<u>If the</u> n-<u>tuple</u> f(T) <u>is Fredholm, then the</u> n-<u>tuples</u> $T-\lambda$, $f(\lambda)=0$, <u>are Fredholm and</u>

$$(9) \qquad \operatorname{ind} f(T) = \sum_{f(\lambda)=0} \nu_\lambda(f) \operatorname{ind}(T-\lambda),$$

where $\nu_\lambda(f)$ *denotes the multiplicity of* f *at* λ.

Proof. We are under the hypotheses of Theorem 2.5, with $\mathcal{F} = \mathcal{O}_{\mathbb{C}^m}/m_0$ and the functional calculus representation $\rho : \mathcal{O}(U) \longrightarrow \mathcal{L}(E)$ associated to T. Hence the $\mathcal{O}_U$-module $f^*\mathcal{F}$ is Fredholm relative to ρ, and

$$\operatorname{ind} f(T) = \operatorname{ind}^{\mathcal{O}(U)}(f^*\mathcal{F}, \mathcal{L}(E)/\mathcal{K}(E)).$$

Thus we have to analyse the right term of this equality.

The sheaf $f^*\mathcal{F}$ is concentrated on the zeroes set of f. More precisely, at the level of global sections, there are isomorphisms

$$(f^*\mathcal{F})(U) \cong \mathcal{O}(U)/(f_1,\ldots,f_m) \cong \bigoplus_{f(\lambda)=0} \mathcal{O}_\lambda/f^* m_0.$$

The multiplicity $\nu_\lambda(f)$ is by definition the integer $\dim_{\mathcal{O}_\lambda/m_\lambda}(\mathcal{O}_\lambda/f^* m_0)$ and it is finite by the assumption on finite fiber at 0.

By induction on $N = \sum \nu_\lambda(f) = l(f^*\mathcal{F})$, the length of the $\mathcal{O}_U$-module $f^*\mathcal{F}$, we shall prove the equality

$$(10) \qquad \operatorname{ind}^{\mathcal{O}(U)}(f^*\mathcal{F}, \mathcal{L}(E)/\mathcal{K}(E)) = \sum_{f(\lambda)=0} \nu_\lambda(f) \operatorname{ind}^{\mathcal{O}(U)}(\mathcal{O}/m_\lambda, \mathcal{L}(E)/\mathcal{K}(E)),$$

and the fredholmness of the n-tuples $T-\lambda$, $f(\lambda)=0$.

If N=1, then $f^*\mathcal{F} = \mathcal{O}/m_\lambda$ and (10) is true. Moreover we claim that the sheaf $\mathcal{O}/m_\mu$ is Fredholm relative to ρ iff the n-tuple $T-\mu$ is Fredholm, $\mu \in U$. Indeed because Koszul's complex $K.(z-\mu, \mathcal{O}_U)$ provides a finite free resoluiton of $\mathcal{O}_U/m_\mu$ and because ρ is a functional calculus representation for T, both conditions are equivalent with the exactness of the complex $K.(T-\mu, \mathcal{L}(E)/\mathcal{K}(E))$.

Let us aasume relation (10) holds for $N = l(f^*\mathcal{F})-1$. There exists an exact sequence of $\mathcal{O}_U$-modules $0 \to \mathcal{G} \to f^*\mathcal{F} \to \mathcal{O}_U/m_{\lambda_0} \to 0$, where $f(\lambda_0)=0$. By the additivity of the length, $l(\mathcal{G})=l(f^*\mathcal{F})-1$ and by Lemma 1.1, $\mathcal{G} \in FR(U)$. Then (10) follows from Corollary 2.4 and the n-tuple $T-\lambda_0$ is Fredholm by the above remark.

The proof is complete after noticing that

$$\operatorname{ind}^{\mathcal{O}(U)}(\mathcal{O}/m_\lambda, \mathcal{L}(E)/\mathcal{K}(E)) = \operatorname{ind}(T-\lambda).$$

Let us illustrate the theorem with some examples of analytic flat and finite maps. With the above notations the n-tuple T is Fredholm whenever f(T) is Fredholm, if the analytic map $f: U \to \mathbb{C}^n$, $f(0)=0$, belongs to one of the following classes:

- f is one to one,
- $f(z)=(P_1(z_1),\dots,P_n(z_n))$ with $P_1,\dots,P_n$ non-constant polynomials,
- $f(z)=(S_1,\dots,S_n)$, denoting by S_j the elementary symmetric polynomials in $z_1,\dots,z_n$.

In each of the three cases the multiplicity of f at its zeroes can be easily computed.

REMARKS 3.2.a). Contrary to the case 1) above, in formula (9) only points λ contained in non-vanishing index Fredholm domains of T occur.

b). The theorem still holds true when U is allowed to be an arbitrary open neighbourhood of Sp(T,E), but then the flatness assumption must be replaced with the following condition

$$\text{(11)} \quad \operatorname{Tor}_q^{\mathcal{O}(\mathbb{C}^n)}(\mathcal{O}(\mathbb{C}^n)/m_0, \mathcal{O}(U)) = 0, \ q \geq 1.$$

This means that the complex $K.(f, \mathcal{O}(U))$ is exact in positive dimensions.

In this case the proof consists in passing to the envelope of holomorphy $\tilde{U}$ of U, and to make use there of coordinateless arguments, in the spirit of [16]. Namely, let $\tilde{f}: \tilde{U} \to \mathbb{C}^n$ be the analytic extension of f. Then the flatness of f is replaced by (11), while the finitness assumption which is not necessarily valid for f, is corrected in formula (9) by the vanishing relations

$$\operatorname{ind}^{\mathcal{O}(\tilde{U})}(\mathcal{O}_{\tilde{U}}/m_\lambda, \mathcal{L}(E)/\mathcal{K}(E)) = 0, \ \lambda \in \tilde{U} \setminus U.$$

These are in turn consequences of the inclusions of the coordinateless spectra in the generating part U of $\tilde{U}$:

$$\sigma(\tilde{U}, \mathcal{L}(E)/\mathcal{K}(E)) \cup \sigma(\tilde{U},E) \subset U,$$

see [16] for details. Then the proof of Theorem 3.1 can be adapted to this more general situation.

§4. BASE CHANGE FOR FREDHOLM COMPLEXES

A central role in the last section was played by the family of complexes $K.(T-\lambda, E)$, parametrized on $\lambda \in \mathbb{C}^n$. In order to analyse separately the behaviour of the homology spaces $H_q(T-\lambda, E)$ under analytic transformations we shall reduce this problem to a finite dimensional one and there we shall use the methods of analytic gemetry.

Let us recall for the beginning some terminology. A complex $(L., d.)$ of Banach spaces and linear operators

$$\cdots \rightarrow L_{q+1} \xrightarrow{d_{q+1}} L_q \xrightarrow{d_q} L_{q-1} \rightarrow \cdots$$

is said after G. Segal [17] to be Fredholm if the homology spaces are finite dimensional, $\dim H_q(L) < \infty$, and if every boundary operator d_q is direct, that is $\mathrm{Ker}(d_q)$ and $\mathrm{Im}(d_q)$ are closed complemented subspaces in L_q, respectively in L_{q-1}.

With the notations of the preceding section, the n-tuple T is Fredholm iff the complex $K.(T,E)$ is Fredholm in the above sense.

Let X be an analytic space with structure sheaf $\mathcal{O}_X$. The (topological free) $\mathcal{O}_X$-module associated to the presheaf $U \longmapsto \mathcal{O}_X(U) \otimes F$, where F is a Banach space will be denoted by $\mathcal{O}_X \hat{\otimes} F$. An analytically parametrized complex of Banach spaces is by definition a sequence $(L_q)_{q \in \mathbb{Z}}$ of Banach spaces and of elements

$d_q \in \Gamma(X, \mathcal{O}_X \hat{\otimes} \mathcal{L}(L_q, L_{q-1}))$, such that $d_q \circ d_{q+1}=0$, $q \in \mathbb{Z}$. The image of d_q through the natural projection

$$\Gamma(X, \mathcal{O}_X \hat{\otimes} \mathcal{L}(L_q, L_{q-1})) \longrightarrow \mathcal{O}_{X,x} \hat{\otimes} \mathcal{L}(L_q, L_{q-1})/m_x \hat{\otimes} \mathcal{L}(L_q, L_{q-1}) \cong \mathcal{L}(L_q, L_{q-1})$$

is a linear bounded operator $d_q(x) \in \mathcal{L}(L_q, L_{q-1})$. Thus one gets for every $x \in X$ a complex $(L., d.(x))$ of Banach spaces whose boundaries depend on x. This complex will be denoted in short by $L.(x)$ and its homology spaces by $H_q(L.(x))$. On the other hand for an analytically parametrized complex there is a corresponding complex of sheaves $(\mathcal{O}_X \hat{\otimes} L., d.)$ which will be denoted by $\mathcal{L}.$, and its homology sheaves by $\mathcal{H}_q(\mathcal{L}.)$. When X is a complex manifold, the above definiton of an analytically parametrized complex means nothing more than that the operators $d_q(x)$ depend analytically in the norm topology.

Finally, by a <u>right bounded complex</u> $L.$ we mean a complex with vanishing negative terms, $L_q=0$ for $q < 0$.

There are many stability results for analytically parametrized complexes of Banach spaces ([18, Lemma 2.2], [21], [22, Chap. III]). As a byproduct of them one obtains rather directly by a descending procedure the following result (see also [14, Proposition 1]).

PROPOSITION 4.1 <u>Let</u> $\mathcal{L}.$ <u>be a right bounded complex of Banach spaces, analytically parametrized on a reduced analytic space</u> Y, <u>and let</u> N <u>be a non-negative integer.</u>

<u>Suppose that the complexes</u> $L.(y)$ <u>are Fredholm for</u> $y \in Y$. <u>Then there exists locally on</u> Y <u>a right bounded complex</u> $\mathcal{P}.$ <u>of finite type free</u> $\mathcal{O}_Y$<u>-modules and a morphism of complexes</u>

$$\varphi : \mathcal{P}. \longrightarrow \mathcal{L}.$$

<u>which induces the isomorphisms</u>

$$\mathcal{H}_q(\mathcal{P}.) \cong \mathcal{H}_q(\mathcal{L}.) \quad , H_q(P.(y)) \cong H_q(L.(y))$$

<u>for</u> $0 \leq q \leq N$ <u>and locally on</u> $y \in Y$.

This proposition reduces many statements of the infinite dimensional analytic Fredholm theory to the finite dimensional case. In order to illustrate this principle we present without proof the following infinite dimensional version of Grauert's continuity theorem [2, Theorem 3.3.4].

THEOREM 4.2 Let $\mathcal{L}.$ be a right bounded complex of Banach spaces, analytically parametrized on a reduced analytic space Y and let q be a non-negative integer.

Suppose that the complexes $L.(y)$ are Fredholm for every $y \in Y$. Then the following assertions are equivalent:

a). The function $y \mapsto \dim H_q(L.(y))$ is locally constant on Y.

b). The families of subspaces $\{\mathrm{Im}\, d_{q+1}(y)\}$ and $\{\mathrm{Ker}\, d_q(y)\}$ form analytic Banach subbundles of the trivial bundle $Y \times L_q$.

c). If $f: X \longrightarrow Y$ is a morphism of analytic spaces, then the natural map

$$f^* \mathcal{H}_q(\mathcal{L}.) \longrightarrow \mathcal{H}_q(f^* \mathcal{L}.)$$

is an isomorphism.

The proof of the theorem reduces, via Proposition 4.1, the statement to the finite dimensional case and then uses Theorem 3.3.4 from [2]. We point out only the importance of the splitting assumption in the definition of a Frdholm complex, in order to reduce the property c) to the finite dimensional case.

As a continuation of the preceding section we derive some applications of Theorem 4.2 to multioperatorial Fredholm theory.

Let T be a commutative n-tuple of linear bounded operators on a Banach space E and let $D \subset \mathbb{C}^n$ be a Fredholm domain of T. Then $K.(T-\lambda, E)$ is an analytically parametrized family of Fredholm complexes on $\lambda \in D$. Let q be a fixed non-negative integer.

Then Proposition 4.1 implies that the jumping points of the function

$$\lambda \longmapsto \dim H_q(T-\lambda, E) \quad , \lambda \in D,$$

form a thin analytic subset S of D.

Applying Theorem 4.2 to the complex $K.(T-\lambda,E)$, one gets that the homology spaces $H_q(T-\lambda,E)$ form in a natural way an analytic vector bundle on $D\setminus S$. Indeed, let X be the simple point λ and let $f:X\longrightarrow D\setminus S$ be the inclusion map. Then $\mathcal{H}_q(\mathcal{L}.)/m_\lambda\mathcal{H}_q(\mathcal{L}.)=H_q(L.(\lambda))$ by c), hence $\mathcal{H}_q(\mathcal{L}.)$ is a locally free $\mathcal{O}_{D\setminus S}$-module.

We remark finally that formula (9) in Theorem 3.1 may be obtained also from Theorem 4.2.c).

5. A FINAL EXAMPLE

We exemplify the general notions of §2 by representations of Stein algebras into commutative C^*-algebras.

Let X be a Stein manifold of complex dimension n and let x be a fixed point of X. Consider a pair (A,B) of compact CW-complexes, $B\subset A$, and let $\varphi:A\longrightarrow X$ be a continuous map. We investigate the fredholmness of the sheaf $\mathcal{O}/m_x$ relative to the representation

$$\rho:\mathcal{O}(X)\longrightarrow C(A),\qquad \rho(f)=\varphi^*(f)=f\circ\varphi\quad, f\in\mathcal{O}(X),$$

and relative to the ideal $I=V(B)=\{g\in C(A)\mid g|B=0\}$. We have denoted as usually by C(K) the commutative C^*-algebra of continuous functions on the compact K.

By [11, Proposition 4.6] the simple point x is a complete intersection in X, hence there exists a n-tuple $f=(f_1,\dots,f_n)$ of holomorphic functions on X, such that the augmented Koszul complex

$$K.(f,\mathcal{O}_X)\longrightarrow\mathcal{O}_X/m_x\longrightarrow 0$$

is exact.

Thus the sheaf $\mathcal{O}_X/m_x$ is Fredholm relative to ρ and I iff the complex:

$$K.(f\circ\varphi\,,C(A)/I)$$

is exact. But the C^*-algebra $C(A)/I$ is naturally isomorphic with $C(B)$, therefore we have proved the following fact:

A). <u>The sheaf</u> $\mathcal{O}_X/m_x$ <u>is Fredholm relative to</u> ρ <u>and</u> $I=V(B)$ <u>iff</u> $x \notin \varphi(B)$.

In order to compute the index we identify the groups $K_1(C(A)/I)$ with $K^1(B)$ and $K_0(I)$ with $K^0(A,B)$. Then the boundary operator $\partial : K_1(C(A)/I) \longrightarrow K_0(I)$ becomes the usual coboundary $\partial : K^1(B) \longrightarrow K^0(A,B)$ in topological K-theory, see [20].

By its definition in §2, the index $\operatorname{ind}^{\mathcal{O}(X)}(\mathcal{O}/m_x, C(B))$ is the class in $K^0(A,B)$ of the complex of trivial vector bundles $K.(f\circ\varphi, A\times\mathbb{C})$, which is exact on B, in short

$$\operatorname{ind}^{\mathcal{O}(X)}(\mathcal{O}/m_x, C(B)) = \chi(K_{.}(f\circ\varphi, A\times\mathbb{C})),$$

where $\chi : \mathcal{C}_n(A,B) \longrightarrow K^0(A,B)$ stands for the generalized Euler-characteristic map, [1, §2.6].

Let U be an open neighbourhood of x in X, such that U is diffeomorphic equivalent by f with a ball $B(0,\delta)$ of radius $\delta > 0$. We may assume that the n-tuple $f-\lambda=(f_1-\lambda_1,\dots,f_n-\lambda_n)$ is still a regular sequence in $\mathcal{O}_X$ for every $\lambda\in B(0,\delta)$. Let ε be chosen such that $0<\varepsilon<\delta$. Then $\varphi(B)\cap\overline{V}=\emptyset$, where $V=f^{-1}B(0,\varepsilon)$.

Let $i:(\varphi(A),\varphi(B)) \longrightarrow (X, X\setminus V)$ denote the inclusion map and let $\Phi : K^0(\overline{U},\overline{U}\setminus V) \longrightarrow K^0(A,B)$ be the composition of the morphisms

$$\Phi : K^0(\overline{U},\overline{U}\setminus V) \cong K^0(X,X\setminus V) \xrightarrow{i^*} K^0(\varphi(A),\varphi(B)) \xrightarrow{\varphi^*} K^0(A,B),$$

where the first one is the excision isomorphism [1, §2.4]. Then for any $\lambda\in B(0,\varepsilon)$ the sequence $f-\lambda$ is still regular, hence

$$\operatorname{ind}^{\mathcal{O}(X)}(\mathcal{O}/m_y, C(B)) = \Phi\chi(K.(f-f(y),\overline{U}\times\mathbb{C})) \quad , y\in V.$$

But $K^0(\bar{U},\bar{U}\setminus V) \cong K^0(B^{2n},\partial B^{2n}) \cong \tilde{K}^0(S^{2n}) \cong \mathbb{Z}$,and $\chi(K.(f-f(y),\bar{U}\times\mathbb{C}))$ is the positive generator of this group,for every $y\in V$, [1,pg.115] .Concluding we can state the following assertion.

B). <u>The function</u> $x\mapsto \mathrm{ind}^{\mathcal{O}(X)}(\mathcal{O}/m_x,C(B)) \in K^0(A,B)$ <u>is locally constant on</u> $X\setminus\varphi(B)$ <u>and equals</u> $\varphi^*(\tau_x)$, <u>where</u> τ_x <u>stands for the positive generator of the group</u> $K^0(X,X\setminus\{x\}) = \mathbb{Z}$,<u>inherited from the complex structure of</u> X.

Notice that $\mathrm{ind}^{\mathcal{O}(X)}(\mathcal{O}/m_x,C(B)) = 0$ if $x\notin\varphi(A)$.Indeed,in this case the complex $K.(f\circ\varphi,A\times\mathbb{C})$ is exact.Therefore the index vanishes identically when $X\setminus\varphi(B)$ is a connected subset of X.

Let us conclude with an example of non-vanishing index.Let $X=\mathbb{C}$, $A=\{|z|\leq 1\}$ and $B=\{|z|=1\}$ .If $\varphi:A\longrightarrow\mathbb{C}$,is a continuous map,then the sheaf $\mathcal{O}/m_0$ is by A) Fredholm relative to the representation $\rho=\varphi^*$ and the ideal I=V(B) iff φ doesn't vanish on B'.The group $K^0(A,B)$ coincides with $\mathbb{Z}$,and in this case assertion B) has the following numerical interpretation:

$$\mathrm{ind}^{\mathcal{O}(\mathbb{C})}(\mathcal{O}/m_0,C(B)) = \deg(\varphi|B).$$

Notice the formal analogy with the classical theory of Toeplitz operators.

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Mihai Putinar

The National Institute for Scientific
and Technical Creation,INCREST
Department of Mathematics
B-dul Păcii 220, 79622 BUCHAREST
ROMANIA

REFERENCES

[1] ATIYAH,M.F., K-theory, Benjamin,New-York,Amsterdam,1967.

[2] BĂNICĂ,C.; STĂNĂŞILĂ,O., Méthodes algébriques dans la théorie globale des espaces complexes, Gauthier-Villars,Paris,1977.

[3] BASS,H., Algebraic K-theory,Benjamin,New-York,Amsterdam,1968.

[4] CAREY,R.; PINCUS,J., Operator theory and boundaries of complex curves, preprint 1982.

[5] CURTO,R.E., Fredholm and invertible tuples of operators.The deformation problem, Trans.Amer.Math.Soc. 266:1(1981),129-159.

[6] CURTO,R.E.; MUHLY,P., C_*-algebras of multiplication operators on Bergman spaces,preprint 1983.

[7] CURTO,R.E.; SALINAS,N., Generalized Bergman kernels and the Cowen-Douglas theory,preprint 1982.

[8] DOUGLAS,R.G.; VOICULESCU,D., On the smoothness of sphere extensions, J. Operator Theory 6(1981),103-111.

[9] FAINSTEIN,A.S.; SHULMAN,V.S., On Fredholm complexes of Banach spaces (in russian),Funct.Analysis and Appl. 14:4(1980),87-88.

[10] FAINSTEIN,A.S.; SHULMAN,V.S., Stability of the index of a short Fredholm complex of Banach spaces under small perturbations in the non-compactness measure,(in Russian),in the volume "Spectral theory of operators",4,Baku 1982.

[11] FORSTER,O., Zur Theorie des Steinschen Algebren und Moduln,Math.Z. 97(1967),376-405.

[12] KATO,T., Perturbation theory for linear operators,Springer,Berlin-Heidelberg-New York,1963.

[13] KAUP,L.; KAUP.B., Holomorphic functions of several variables.An introduction on the fundamental theory,Walter de Gruyter ed.,Berlin,1983.

[14] LEVY,R., Cohomological invariants for essentially commuting systems of operators (in Russian),Funct.Analysis and Appl. 17:3(1983),79-80.

[15] PUTINAR,M., Some invariants for semi-Fredholm systems of essentially commuting operators,J.Operator Theory 8(1982),65-90.

[16] PUTINAR,M., Uniqueness of Taylor's functional calculus,Proc.Amer.Math.Soc. 89:4(1983),647-650.

[17] SEGAL,G., Fredholm complexes, Quart J.Math.Oxford Ser. 21(1970),385-402.

[18] TAYLOR,J.L., A joint spectrum of several commuting operators,J.Funct. Analysis 6(1970),172-191.

[19] TAYLOR,J.L., Analytic functional calculus for several commuting operators Acta Math. 125(1970),1-38.

[20] TAYLOR,J.L., Banach algebras and topology,in the volume "Algebras in Analysis",Academic Press,1975,118-186.

[21] VASILESCU,F.-H., Stability of the index of a complex of Banach spaces, J.Operator Theory 1(1979),187-205.

[22] VASILESCU,F.-H., Analytic functional calculus and spectral decompositions Ed.Academiei and D.Reidel Co.,Bucharest and Dordrecht,1982.